On Minimizers of the Hamiltonian System $u'' = \nabla W(u)$ and on the Existence of Heteroclinic, Homoclinic and Periodic Orbits

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ABSTRACT. In the first part of the paper, we establish two necessary conditions for the existence of bounded one-dimensional minimizers u: the potential W must have a global minimum supposed to be 0 without loss of generality, and $W(u(x)) \to 0$ as $|x| \to \infty$. Furthermore, non-constant minimizers connect at $\pm\infty$ two distinct components of the set $\{W = 0\}$. In the second part, we prove (when the previous assumptions are satisfied) the existence of nontrivial minimizers. We also show the existence of heteroclinic, homoclinic, and periodic orbits in analogy with the scalar case. Finally, we study the asymptotic convergence of these solutions.

1. INTRODUCTION

Note that the existence of heteroclinic, homoclinic, and periodic orbits for the scalar O.D.E.

$$u'' = W'(u), \ u : \mathbb{R} \to \mathbb{R}, \ W \in C^2(\mathbb{R}, \mathbb{R}), \tag{1.1}$$

is textbook material. We recall that if $W > 0$ in the interval (a^-, a^+) and $W(a^\pm) = 0$, then we have the following:

(i) When $W'(a^\pm) = 0$, there exists a solution $u : \mathbb{R} \to (a^-, a^+)$ to (1.1) such that $\lim_{x\to\pm\infty} u(x) = a^\pm$. It is the heteroclinic connection that is unique up to translations.

(ii) When $W'(a^-) = 0$ and $W'(a^+) \neq 0$, there exists a unique even solution $u : \mathbb{R} \to (a^-, a^+]$ to (1.1) such that $\lim_{x\to\pm\infty} u(x) = a^-$ and $u(0) = a^+$. This is the homoclinic connection.

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(iii) When $W'(a^-) \neq 0$ and $W'(a^+) \neq 0$, there exists a periodic solution $u : \mathbb{R} \to [a^-, a^+]$ to (1.1) such that $u(0) = a^-$, $u(T/2) = a^+$ and $\forall\, x \in \mathbb{R}$: $u(x+T) = u(x)$, $u(x+T/2) = u(-x+T/2)$, for some $T > 0$.

In [1], one can find a variational proof of the existence of a heteroclinic connection for the scalar equation (1.1). The first existence proof of a heteroclinic connection in the vector case for a double-well potential was given in [11] and [12]. In this work, the Jacobi principle is used, but somewhat restrictive hypotheses on the behavior of W are imposed at the minima. In [3], the heteroclinic problem for the O.D.E. system

$$u'' = \nabla W(u),\ u : \mathbb{R} \to \mathbb{R}^m,\ W \in C^2(\mathbb{R}^m, \mathbb{R}), \tag{1.2}$$

has also been studied. Under the assumption that the potential W is nonnegative, vanishes only at two points a^+ and a^-, and satisfies a monotonicity assumption in a neighborhood of $a^\pm$, a solution to system (1.2) connecting a^- and a^+ at $\pm\infty$ was constructed, namely,

$$\lim_{x \to \pm\infty} u(x) = a^\pm. \tag{1.3}$$

The approach in [3] is variational, and consists of showing that the heteroclinic connection is a minimizer of the *Action* functional (cf. (3.1)) in the class of maps satisfying the constraint (1.3).

The scope of this paper is to study systematically the connection problem, and extend the results in [3], where only potential possessing several global minima were considered. In our more general setup (cf. Section 2), by *heteroclinic connection* we mean a solution to (1.2) taking its values in a connected component Ω of the set $\{u \in \mathbb{R}^m \mid W(u) > 0\}$, and approaching at $\pm\infty$ two distinct portions of $\partial\Omega$ where $\nabla W(u) = 0$. By *homoclinic connection* we mean an even solution u to (1.2) approaching at $\pm\infty$ the portion of $\partial\Omega$ where $\nabla W(u) = 0$, and such that $u(0) \in \partial\Omega$, $\nabla W(u(0)) \neq 0$. Recall that by the uniqueness result for O.D.E., $\nabla W(u(0)) = 0$ is excluded. Indeed, since u is even and $u'(0) = 0$, note that $\nabla W(u(0)) = 0$ would imply u is constant. According to these definitions, the shape of a connecting orbit u can be very complicated. However, we will give sufficient conditions on W and $\partial\Omega$ to ensure that the limits of u at $\pm\infty$ exist (cf. Section 6). In this case, we obtain the usual notion of a heteroclinic orbit converging at $\pm\infty$ to two distinct points $a^\pm \in \partial\Omega$ where $\nabla W(a^\pm) = 0$.

Assuming that $\partial\Omega$ is partitioned into two compact subsets $A^\pm$, we establish in one step the following, by particularizing an abstract theorem (cf. Section 4):

(i) The existence of a heteroclinic orbit connecting $A^\pm$ (when $\nabla W(u) = 0$ on $A^\pm$),

(ii) The existence of a homoclinic orbit connecting $A^\pm$ (when $\nabla W(u) = 0$ on A^-, and $\nabla W(u) \neq 0$ on A^+),

(iii) The existence of a periodic orbit connecting $A^\pm$ (when $\nabla W(u) \neq 0$ on $A^\pm$).

As far as we know, (ii) and (iii) have not appeared in the literature. We can see that the conditions that for system (1.2) ensure the existence of connecting orbits are similar to the corresponding ones for the scalar equation (1.1). Despite this analogy, the vector case is much more complex, and we mention in Section 2 some new phenomena that may occur.

We also emphasize the relation between local minimizers and heteroclinic connections (cf. Section 3). We show that bounded local minimizers exist only for potentials possessing a global minimum, supposed to be 0 without loss of generality. In addition, nontrivial local minimizers take their values in a connected component Ω of the set $\{u : \mathbb{R}^m \mid W(u) > 0\}$, and approach at $\pm\infty$ two distinct portions of $\partial\Omega$. Thus, they are heteroclinic connections according to our definition.

The plan of the remaining sections is as follows. In Section 4, we prove our main theorem. Assuming that $\partial\Omega$ is partitioned into two compact subsets $A^\pm$, we construct a minimizer of the Action in the class of maps satisfying a constraint similar to (1.3). We follow the approach in [3], and utilize a comparison argument in [10] (cf. Lemma 4.3) to remove the monotonicity assumption in [3]. In Section 5, we particularize the previous result, according to which of the two hypotheses $\nabla W(u) = 0$ or $\nabla W(u) \neq 0$ holds on $A^\pm$. Then, the existence of the aforementioned connecting orbits is straightforward. Finally, in Section 6, we study the asymptotic convergence of these solutions, and establish an exponential estimate under a convexity assumption on W. From this estimate, it follows that the limits of the heteroclinic and homoclinic connections exist at $\pm\infty$. As a consequence, in many standard situations, these orbits connect two points of $\partial\Omega$.

We point out that phase transition problems for potentials vanishing on submanifolds of $\mathbb{R}^m$ have recently been examined in the literature (cf. [4] and [7]). In particular, Section 2 of [7] is dedicated to minimal connecting orbits of (1.2). However, in their setup, the authors focus on potentials depending on the distance from the set $\{W = 0\}$.

2. Preliminaries

Let $W \in C^2(\mathbb{R}^m, \mathbb{R})$ be a general potential, and let $\Omega \neq \mathbb{R}^m$ be a connected component of the set $\{u \in \mathbb{R}^m \mid W(u) > 0\}$. Clearly, $W = 0$ on $\partial\Omega$. We also consider the sets

$$\partial\Omega_0 := \{u \in \partial\Omega \mid \nabla W(u) = 0\},$$
$$\partial\Omega_{\neq} := \{u \in \partial\Omega \mid \nabla W(u) \neq 0\},$$
$$Z := \{u \in \mathbb{R}^m \mid W(u) = 0\},$$

and denote by d the Euclidean distance in $\mathbb{R}^m$, by $|\cdot|$ the Euclidean norm, by $\cdot$ the Euclidean inner product, and by u' or $\dot{u}$ the first derivative of a map

$u : \mathbb{R} \to \mathbb{R}^m$. In analogy with the scalar case, we give the following definitions of the heteroclinic, homoclinic, and periodic orbits.

Definition 2.1. Assuming that $K^\pm$ are two closed subsets of $\partial\Omega_0$ with $K^+ \cap K^- = \emptyset$, we say that a bounded solution $u \in C^2(\mathbb{R};\Omega)$ to system (1.2) such that $d(u(x), K^\pm) \to 0$ as $x \to \pm\infty$ is a heteroclinic orbit connecting $K^\pm$.

In particular, if $W \geq 0$ and $Z := \{a_1, a_2, \dots, a_N\}$, then $\Omega = \mathbb{R}^m \setminus Z$ and $\partial\Omega = \partial\Omega_0 = Z$. By taking, for instance, $K^- = \{a_1\}$ and $K^+ = \{a_2\}$, we obtain the usual notion of a heteroclinic orbit connecting a_1 and a_2.

Definition 2.2. We use *homoclinic orbit* to refer to every bounded solution $u \in C^2(\mathbb{R};\bar{\Omega})$ to system (1.2) that is even, and such that the following hold:

- $u(0) \in \partial\Omega_{\neq}$.
- $u(x) \in \Omega \Leftrightarrow x \neq 0$.
- $d(u(x), \partial\Omega_0) \to 0$ as $|x| \to \infty$.

Definition 2.3. Assuming that $a^\pm \in \partial\Omega_{\neq}$, $a^+ \neq a^-$, we consider a solution $u \in C^2(\mathbb{R};\bar{\Omega})$ to system (1.2) such that the following hold:

- $u(0) = a^-$, $u(T/2) = a^+$.
- $\forall\, x \in \mathbb{R}$: $u(x+T) = u(x)$, $u(x+T/2) = u(-x+T/2)$, for some $T > 0$.
- $u(x) \in \Omega \Leftrightarrow x \notin (T/2)\mathbb{Z}$.

Here, then, this solution is a periodic orbit connecting $a^\pm$.

The most typical situations allowing the existence of homoclinic and periodic orbits are represented in Figure 2.1 (see also Section 5).

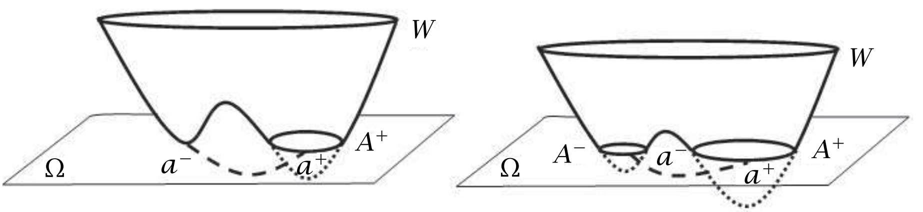

Figure 2.1. On the left is a homoclinic orbit connecting a^-, a local minimum of W, and $a^+ = u(0) \in A^+ := \partial\Omega_{\neq}$. Note that in this picture $A^- := \partial\Omega_0 = \{a^-\}$. On the right is a periodic orbit connecting a^- and a^+. Here, $\partial\Omega = \partial\Omega_{\neq} = A^- \cup A^+$.

We recall that the *Hamiltonian* $H := \frac{1}{2}|u'(x)|^2 - W(u(x))$ of a solution u to (1.2) is a constant. Clearly, $H = 0$ for homoclinic and periodic orbits, since $u'(0) = 0$ by symmetry, and $W(u(0)) = 0$. Now, we show that heteroclinic orbits also satisfy the equipartition relation (2.1).

Proposition 2.4. *Let $u \in C^2(\mathbb{R}; \bar{\Omega})$ be a bounded solution to system (1.2) such that $d(u(x), \partial\Omega_0) \to 0$ as $x \to -\infty$. Then,*

$$\frac{1}{2}|u'(x)|^2 = W(u(x)), \quad \forall\, x \in \mathbb{R}. \tag{2.1}$$

As a consequence of this relation, $u(x) \in \Omega \cup \partial\Omega_{\neq}$, $\forall\, x \in \mathbb{R}$, if the solution u is not constant.

Proof. Since $W(u(x)) \to 0$ as $x \to -\infty$, we immediately see that $H \geq 0$. Now, suppose by contradiction that $H > 0$, and define the function $\varphi(x) := |u(x)|^2$. By differentiating φ, we obtain

$$\varphi''(x) = 2|u'(x)|^2 + 2u(x) \cdot \nabla W(u(x)) = 4H + 4W(u(x)) + 2u(x) \cdot \nabla W(u(x)).$$

Since by assumption $\nabla W(u(x)) \to 0$ as $x \to -\infty$, we deduce that $\varphi'' \geq 2H$ in some interval $(-\infty, \alpha)$. As a consequence, $\varphi'(x) \to -\infty$ and $\varphi(x) \to \infty$ as $x \to -\infty$, which is impossible. Thus, the equipartition relation is proved. Finally, if for some $x_0 \in \mathbb{R}$ we have $W(u(x_0)) = 0$ and $\nabla W(u(x_0)) = 0$, it follows that $u'(x_0) = 0$, and by the uniqueness result for O.D.E., u is constant. ◻

Remark 2.5. As we mentioned in the Introduction, new kinds of connecting orbits may appear in the vector case. The most surprising is the construction in [9] of a periodic solution to (1.2) connecting the two zeros $a^\pm$ of a nonnegative potential $W : \mathbb{R}^2 \to \mathbb{R}$ at finite time (i.e., $u(kT) = a^+$ and $u(kT + T/2) = a^-$, $\forall\, k \in \mathbb{Z}$) and for some period $T > 0$. This is possible since the inequality $|u'(x)|^2 \leq 2W(u(x))$ does not hold in general for bounded solutions to system (1.2) (cf. [8], [9]), and consequently it may happen that $u'(0) \neq 0$ even if $W(u(0)) = 0$. Otherwise, if $u'(0) = 0$ and $u(0) = a^+$ with $W(a^+) = 0$, the uniqueness result for O.D.E. implies that u is constant.

Remark 2.6. In a similar way, one can construct a nonnegative potential $W : \mathbb{R}^2 \to \mathbb{R}$ possessing a unique zero a, and a nontrivial solution u to (1.2) such that $u(x) \to a$, as $|x| \to \infty$. Thus, the condition $\lim_{|x|\to\pm\infty} d(u(x), \Omega_0) \to 0$ does not guarantee that the limits of u at $\pm\infty$ are different. This is the reason why in the definition of the heteroclinic orbit, we imposed the convergence at $\pm\infty$ to two distinct portions $K^\pm$ of $\partial\Omega_0$. In the scalar case, a nontrivial solution satisfying $\lim_{x\to-\infty} u(x) = a$ (with $W(a) = 0$ and $W(u) > 0$, $\forall u \neq a$) is strictly monotonous because of the equipartition relation.

Remark 2.7. The conclusion of Proposition 2.4 does not remain true if we weaken the hypothesis $\lim_{x\to-\infty} d(u(x), \partial\Omega_0) \to 0$ and assume only that

$$\lim_{x\to-\infty} d(u(x), \partial\Omega) \to 0. \tag{2.2}$$

Indeed, we construct below a bounded solution $u \in C^2(\mathbb{R}; \Omega)$ to (1.2) satisfying (2.2), and such that $H > 0$. Let us consider the Hamiltonian system

$$u'' = (|u|^2 - 1)u, \tag{2.3}$$

corresponding to the potential $G : \mathbb{R}^2 \to \mathbb{R}$, $G(u) = \frac{1}{4}(|u|^2 - 1)^2$. Its solutions $u : \mathbb{R} \to \mathbb{R}^2 \sim \mathbb{C}$ of the form $u(x) = r(x)e^{i\theta(x)}$, with $r : \mathbb{R} \to (0, \infty)$ and $\theta : \mathbb{R} \to \mathbb{R}$, are easy to describe since they satisfy

$$r'' = F_c'(r), \quad \text{with } F_c(r) = \frac{r^4}{4} - \frac{r^2}{2} - \frac{c^2}{2r^2}, \tag{2.4a}$$

$$\theta' = \frac{c}{r^2}, \tag{2.4b}$$

for a constant $c \in \mathbb{R}$. In (2.4a), we recognize the scalar equation (1.1). After studying the variations of the function F_c, we can deduce the existence of a solution $u(x) = r(x)e^{i\theta(x)}$ to (2.3) such that the following hold:

- r is even, and strictly increasing in the interval $[0, \infty)$.
- $r(0) > 0$ and $\lim_{x\to\infty} r(x) = \rho$, with $\rho < 1$ close to 1.
- $\lim_{x\to\pm\infty} \frac{1}{2}|u'(x)|^2 = \frac{1}{2}(\rho^2 - \rho^4) > 0$.

Finally, we set $W(u) = G(u) - \frac{1}{4}(\rho^2 - 1)^2$, and check that the ball $\Omega = \{u \in \mathbb{R}^2 : |u| < \rho\}$ is a connected component of the set $\{W > 0\}$. Clearly, u is a solution to $u'' = \nabla W(u)$ satisfying (2.2), and such that $H = \frac{1}{2}|u'(x)|^2 - W(u(x)) = \frac{1}{2}(\rho^2 - \rho^4) > 0$.

3. NECESSARY CONDITIONS FOR THE EXISTENCE OF BOUNDED LOCAL MINIMIZERS

We recall that the solutions $u \in C^2(\mathbb{R}; \mathbb{R}^m)$ to system (1.2) are the critical points of the *Action* functional:

$$J_{[\alpha,\beta]}(v) := \int_\alpha^\beta \left\{ \frac{1}{2}|v'(x)|^2 + W(v(x)) \right\} \mathrm{d}x \quad (\text{with } \alpha < \beta), \tag{3.1}$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0} J_{[\alpha,\beta]}(u + \lambda\xi) = \int_\alpha^\beta u'(x)\xi'(x) + \nabla W(u(x))\xi(x)\,\mathrm{d}x = 0,$$

$\forall\, \xi \in W_0^{1,2}([\alpha, \beta]; \mathbb{R}^m)$. This is the weak formulation of (1.2).

Local minimizers of (1.2) are solutions satisfying the stronger condition:

$$J_{[\alpha,\beta]}(u) \le J_{[\alpha,\beta]}(u + \xi), \quad \forall\, \xi \in W_0^{1,2}([\alpha, \beta]; \mathbb{R}^m),\ \forall\, \alpha < \beta.$$

In what follows, we establish necessary conditions for the existence of nontrivial bounded local minimizers.[1]

Proposition 3.1. *If there exists a local minimizer $u \in L^\infty(\mathbb{R}; \mathbb{R}^m)$ for system (1.2), then the potential W has a global minimum that is supposed to be 0 without loss of generality. In addition, $J_{\mathbb{R}}(u) < \infty$, $\lim_{|x|\to\infty} W(u(x)) = 0$, and $\lim_{|x|\to\infty} d(u(x), Z) = 0$, where $Z := \{u \in \mathbb{R}^m \mid W(u) = 0\}$.*

[1] Sufficient conditions will be given in Theorem 5.2 below.

Proof. Since u is bounded, there exists a sequence $x_n \to \infty$ ($n \in \mathbb{N}$) such that $u(x_n) \to b \in \mathbb{R}^m$. Suppose by contradiction that W does not have a global minimum. This implies there exists $a \in \mathbb{R}^m$ such that $W(a) + \varepsilon \leq \min\{W(v) : |v| \leq \|u\|_{L^\infty(\mathbb{R};\mathbb{R}^m)}\}$, for some $\varepsilon > 0$. Next, define

(3.2)
$$v_n(x) = \begin{cases} (1 + x_0 - x)u(x_0) + (x - x_0)a, & \text{for } x_0 \leq x \leq x_0 + 1, \\ a, & \text{for } x_0 + 1 \leq x \leq x_n - 1, \\ (x_n - x)a + (x - x_n + 1)u(x_n), & \text{for } x_n - 1 \leq x \leq x_n. \end{cases}$$

On the one hand, we have

$$J_{[x_0,x_n]}(v_n) \leq W(a)(x_n - x_0 - 2) + M,$$

for some constant M independent of n, and on the other hand

$$J_{[x_0,x_n]}(u) \geq (W(a) + \varepsilon)(x_n - x_0).$$

Since by construction $u(x_0) = v_n(x_0)$ and $u(x_n) = v_n(x_n)$, we deduce (thanks to the minimality of u) that

$$\begin{aligned} (W(a) + \varepsilon)(x_n - x_0) &\leq W(a)(x_n - x_0 - 2) + M \\ \Longrightarrow\ \varepsilon(x_n - x_0) &\leq -2W(a) + M, \end{aligned}$$

which is impossible. This proves the first statement of the proposition. Next, assuming that $\min_{\mathbb{R}^m} W = 0$, we are going to show that $J_{\mathbb{R}}(u) < \infty$. To see this, consider again the sequence defined in (3.2), with $a \in \mathbb{R}^m$ such that $W(a) = 0$. Since $J_{[x_0,x_n]}(u) \leq J_{[x_0,x_n]}(v_n) \leq M$, it is immediate that $J_{[x_0,\infty)}(u) < \infty$, and by a similar argument at $-\infty$ it follows that $J_{\mathbb{R}}(u) < \infty$. Furthermore, u is uniformly continuous. Indeed, for every $x \leq y$ we have

$$|u(y) - u(x)| \leq \int_x^y |\dot{u}(t)|\,\mathrm{d}t \leq (2J_{\mathbb{R}}(u))^{1/2}|y - x|^{1/2}.$$

Finally, if $W(u(x))$ or $d(u(x), Z)$ do not converge to 0 as $|x| \to \infty$, there exists a sequence $x_n \to \pm\infty$ such that $u(x_n) \to b \in \mathbb{R}^m$ with $W(b) > 0$. Thanks to the uniform continuity of u, we can also see that

$$\forall\, n \geq N, \quad \forall\, x \in [x_n - \delta, x_n + \delta] : W(u(x)) \geq \frac{W(b)}{2},$$

for some $\delta > 0$ independent of n. Therefore, we have

$$\forall\, n \geq N : J_{[x_n - \delta, x_n + \delta]}(u) \geq \delta W(b).$$

Since, by passing to a subsequence if necessary, we can assume that the intervals $[x_n - \delta, x_n + \delta]$, $n \geq N$ are disjoint, this contradicts $J_{\mathbb{R}}(u) < \infty$. ☐

Remark 3.2. If $W \geq 0$, every solution u to (1.2) such that $J_{\mathbb{R}}(u) < \infty$ satisfies the equipartion relation (2.1). Indeed, if $H = \frac{1}{2}|u'(x)|^2 - W(u(x))$ is a non-zero constant, writing

$$J_{\mathbb{R}}(u) = \int_{\mathbb{R}} \{2W(u(x)) + H\}\, \mathrm{d}x = \int_{\mathbb{R}} \{|u'(x)|^2 - H\}\, \mathrm{d}x,$$

we immediately see that $J_{\mathbb{R}}(u)$ is not finite.

Now, we assume that 0 is the global minimum of W. Let us then define on $Z := \{u \in \mathbb{R}^m \mid W(u) = 0\}$ the equivalence relation $u \sim v$, if and only if there exists a path $\gamma \in W^{1,2}([\alpha, \beta]; \mathbb{R}^m)$ such that $\gamma([\alpha, \beta]) \subset Z$, and $\gamma(\alpha) = u$, $\gamma(\beta) = v$. According to the following proposition, if a local minimizer u connects at $-\infty$ and $+\infty$ the same equivalence class of Z, then it is constant.

Proposition 3.3. *Let W be a potential so that* $\min_{\mathbb{R}^m} W = 0$. *If $u \in L^\infty(\mathbb{R}; \mathbb{R}^m)$ is a local minimizer for system* (1.2)*, and if there exist two sequences $x_n \to -\infty$ and $y_n \to +\infty$ such that $u(x_n) \to a^-$ and $u(y_n) \to a^+$, with $a^\pm \in Z$, $a^- \sim a^+$, then u is constant.*

Proof. Let $\gamma \in W^{1,2}([0, \ell]; \mathbb{R}^m)$ be a path connecting $a^- = \gamma(0)$ and $a^+ = \gamma(\ell)$ in Z. We define
(3.3)

$$v_n(x) = \begin{cases} (1 + x_n - x)u(x_n) + (x - x_n)a^-, & \text{for } x_n \leq x \leq x_n + 1, \\ \gamma\left(\dfrac{\ell(x - x_n - 1)}{y_n - x_n - 2}\right), & \text{for } x_n + 1 \leq x \leq y_n - 1, \\ (y_n - x)a^+ + (x - y_n + 1)u(y_n), & \text{for } y_n - 1 \leq x \leq y_n, \end{cases}$$

and compute

$$\begin{aligned} J_{[x_n, y_n]}(v_n) &= o(1) + \frac{\ell^2}{2(y_n - x_n - 2)^2} \int_{x_n+1}^{y_n-1} \left| \dot{\gamma}\left(\frac{\ell(x - x_n - 1)}{y_n - x_n - 2}\right) \right|^2 \mathrm{d}x \\ &= o(1) + \frac{\ell}{2(y_n - x_n - 2)} \int_0^\ell |\dot{\gamma}(y)|^2\, \mathrm{d}y = o(1). \end{aligned}$$

Since by construction $u(x_n) = v_n(x_n)$ and $u(y_n) = v_n(y_n)$, we deduce by the minimality of u that $J_{[x_n, y_n]}(u) = o(1)$. As a consequence, $J_{\mathbb{R}}(u) = 0$ and u is constant. □

Corollary 3.4. *If W is a potential such that* $\min_{\mathbb{R}^m} W = 0$*, then every nontrivial, bounded, local minimizer for system* (1.2) *is a heteroclinic connection in the sense of Definition 2.1.*

Proof. Let u be a nontrivial, bounded, local minimizer. Because of the equipartition relation (cf. Remark 3.2), we know that u takes its values in a connected component Ω of the set $\{W > 0\}$. Let $K^\pm$ be the sets of limit points of u at

$\pm\infty$, which are compact since u is bounded. By Proposition 3.1, $K^{\pm} \subset \partial\Omega$. In addition, since u is not constant, the sets $K^{\pm}$ are disjoint (cf. Proposition 3.3). Finally, by the definition of $K^{\pm}$, $d(u(x), K^{\pm}) \to 0$ as $x \to \pm\infty$, and thus u is a heteroclinic connection. □

Remark 3.5. For $m \geq 2$, the 0 level set of the Ginzburg-Landau potential $W(u) = \frac{1}{4}(|u|^2 - 1)^2$ is the unit sphere that is obviously path connected. By Proposition 3.3, it follows that, for this potential, the only one-dimensional local minimizers are the constants of modulus 1. In particular, we see that the solution $u : \mathbb{R} \to \mathbb{R}^m$, $u(x) = (\tanh(x/\sqrt{2}), 0, \dots, 0)$ that connects at $\pm\infty$ the points $(\pm 1, 0, \dots, 0)$ is minimal only when $m = 1$.

Remark 3.6. According to Propositions 3.1 and 3.3, nontrivial bounded local minimizers u connect at $\pm\infty$ distinct components of the zero set of W. The converse, however, is not true: heteroclinic connections are not always minimal solutions. In what follows, we explain how to construct such a counterexample for a nonnegative potential $H \in C^{\infty}(\mathbb{R}^2, \mathbb{R})$ vanishing only at the points $a^{\pm} = (\pm 1, 0)$, and such that $D^2H(a^{\pm})$ is a positive definite matrix. We therefore consider again the Ginzburg-Landau potential $W(u) = \frac{1}{4}(|u|^2 - 1)^2$ for $u \in \mathbb{R}^2 \sim \mathbb{C}$. Next, we compute the Action of the solution $u(x) = (\tanh(x/\sqrt{2}), 0)$ in the interval $[-R, R]$:

$$J_{[-R,R]}(u) = \sqrt{2}\left[u(R) - \frac{(u(R))^3}{3}\right] \to \frac{2\sqrt{2}}{3} \quad \text{as } R \to \infty.$$

We also define the map $v(x) = -u(R)e^{i\pi(x+R)/(2R)}$ for $x \in [-R, R]$. By construction, $v(\pm R) = u(\pm R)$, and we can see that

$$J_{[-R,R]}(v) = 2RW(u(R)) + |u(R)|^2 \frac{\pi^2}{4R} \to 0 \quad \text{as } R \to \infty.$$

Thus, for R big enough, $J_{[-R,R]}(u) > J_{[-R,R]}(v)$, and to complete the proof, we just have to modify W outside the closed ball of radius $u(R) < 1$ centered at the origin. We set $H(u_1, u_2) := W(u) + u_2^2 \varphi(|u|^2)$, where $\varphi \in C^{\infty}(\mathbb{R}, [0, \infty))$ is such that

$$\varphi(t) = \begin{cases} 0 & \text{for } t \leq |u(R)|^2 + \varepsilon, \\ 1 & \text{for } 1 - \varepsilon \leq t, \end{cases}$$

and $\varepsilon > 0$ is small enough. Since $H(u_1, 0) = W(u_1, 0)$ and $(\partial H/\partial u_2)(u_1, 0) = 0$, we can check that H has all the desired properties, and that $u'' = \nabla H(u)$. Clearly, u is not a minimal solution of $u'' = \nabla H(u)$, since its action over $[-R, R]$ is bigger than the action of the competitor v.

4. The Main Theorem

Recall that $\Omega \neq \mathbb{R}^m$ is a connected component of the set $\{u \in \mathbb{R}^m \mid W(u) > 0\}$, and assume the following:

(H1) The potential $W \in C^2(\mathbb{R}^m, \mathbb{R})$[2] is such that $\partial\Omega$ is partitioned into two disjoint compact subsets A^- and A^+. In addition, $\mathbb{R}^m \setminus \Omega$ is partitioned into two disjoint closed sets $F^\pm$, with $\partial F^\pm = A^\pm$.
(H2) $\liminf_{u \in \Omega, |u| \to +\infty} W(u) > 0$, if Ω is not bounded.

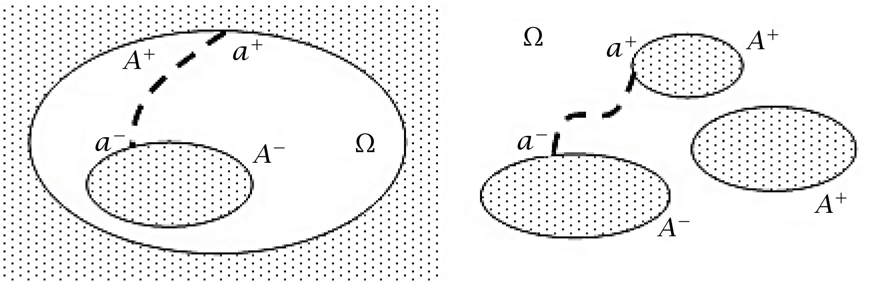

FIGURE 4.1. The sets Ω and $A^\pm$, and the trajectory of the minimizer $\bar{u}$. For the sake of simplicity, we assumed that the limits of $\bar{u}$ exist at $\pm\infty$.

Let $\bar{q} \in (0, d(A^-, A^+)/2)$, where d denotes the Euclidean distance, and let $\mathcal{A}$ be defined by

$$\mathcal{A} = \Big\{u \in W^{1,2}_{\text{loc}}(\mathbb{R}; \bar{\Omega}) \mid d(u(x), A^-) \leq \bar{q}, \text{ for } x \leq x_u^-, \\ d(u(x), A^+) \leq \bar{q}, \text{ for } x \geq x_u^+, \text{ for some } x_u^- < x_u^+\Big\}.$$

Remark 4.1. Note that in the definition of $\mathcal{A}$ no limitation is imposed on the numbers $x_u^- < x_u^+$ that may largely depend on u.

We are going to prove the existence of a connecting minimizer in the class $\mathcal{A}$.

Theorem 4.2. *Assume* $W : \mathbb{R}^m \to \mathbb{R}$ *satisfies* (H1), (H2). *Then,* $J_{\mathbb{R}}(u)$ *admits a minimizer* $\bar{u} \in \mathcal{A}$*:*

$$J_{\mathbb{R}}(\bar{u}) = \min_{u \in \mathcal{A}} J_{\mathbb{R}}(u) < +\infty.$$

Moreover, the result is that $\lim_{x \to \pm\infty} d(\bar{u}(x), A^\pm) = 0$.

Proof.
Step 1. There exists $u_0 \in \mathcal{A}$ that satisfies

$$J_{\mathbb{R}}(u_0) < +\infty. \tag{4.1}$$

[2]Note that the C^2 smoothness of W is only used in the proof of Theorems 5.2, 5.4, and 5.5. To prove Theorem 4.2 (respectively, Proposition 5.1) it is sufficient to assume that W is continuous (respectively, C^1).

Indeed, let $a^\pm \in A^\pm$ be such that $d(a^-, a^+) = d(A^-, A^+)$. We can check that the line segment $[a^-, a^+]$ is included in $\bar{\Omega}$. Next, we set

$$u_0(x) = \begin{cases} a^-, & \text{for } x \leq 0, \\ a^- + x(a^+ - a^-), & \text{for } 0 \leq x \leq 1, \\ a^+, & \text{for } 1 \leq x. \end{cases}$$

From (4.1), it follows that

$$\inf_{u \in \mathcal{A}} J_{\mathbb{R}}(u) = \inf_{u \in \mathcal{A}_b} J_{\mathbb{R}}(u) < +\infty,$$

where

$$\mathcal{A}_b = \mathcal{A} \cap \{J_{\mathbb{R}}(u) \leq J_{\mathbb{R}}(u_0)\}.$$

Step 2. Given $A^* = A^+$ or A^-, and $0 < q' < q/2 < q \leq \bar{q}$, we let $\mathcal{U}_q^{q'}$ be the set of $W^{1,2}$ maps $u : [\alpha, \beta] \to \bar{\Omega}$ that satisfy

$$d(u(\alpha), A^*) \geq q, \quad 0 < d(u(\beta), A^*) \leq q',$$

for some $\alpha < \beta$ that may depend on u. For each $u \in \mathcal{U}_q^{q'}$, we shall define $v_u : [\beta - 1, \beta] \to \bar{\Omega}$ by setting

$$v_u(x) = a^* + (x - \beta + 1)(u(\beta) - a^*), \tag{4.2}$$

where $a^* \in A^*$ and $d(u(\beta), A^*) = d(u(\beta), a^*)$.

Lemma 4.3. *For each $q \in (0, \bar{q}]$, there exists $q' \in (0, q/2)$ such that*

$$J_{[\alpha,\beta]}(u) \geq J_{[\beta-1,\beta]}(v_u), \quad \text{for } u \in \mathcal{U}_q^{q'},\ A^* = A^\pm.$$

Proof. Define

$$\varphi(q) = \min\{W(u) \mid u \in \Omega,\ q \leq d(u, A^*) \leq \bar{q}\}, \quad q \in (0, \bar{q}],$$
$$\Phi(q) = \max\{W(u) \mid u \in \Omega,\ d(u, A^*) \leq q\}, \quad q \in (0, \bar{q}].$$

From these definitions, it follows that

$$J_{[\alpha,\beta]}(u) \geq \int_\alpha^\beta \sqrt{2W(u)}|\dot{u}|\,\mathrm{d}x \geq \sqrt{2\varphi\left(\frac{q}{2}\right)}\frac{q}{2},$$
$$J_{[\beta-1,\beta]}(v_u) \leq \Phi(q') + \frac{1}{2}|u(\beta) - a^*|^2 \leq \Phi(q') + \frac{1}{2}(q')^2.$$

Therefore, to conclude the proof it suffices to observe that, given $q \in (0, \bar{q}]$, for $q' \in (0, q/2)$ sufficiently small, we have the inequality

$$\Phi(q') + \frac{1}{2}(q')^2 < \sqrt{2\varphi\left(\frac{q}{2}\right)}\frac{q}{2}.$$

□

Step 3. Given $q \in (0, \bar{q}]$ let $\ell_q = 2J_{\mathbb{R}}(u_0)/\varphi(q')$ where q' is as in Lemma 4.3. Then, for each $u \in \mathcal{A}_b$ we have the implication

$$(x_1, x_2) \subset (-\infty, x_u^-) \text{ and } x_2 - x_1 \geq \ell_q \implies d(u(x_0), A^-) < q',$$

for some $x_0 \in (x_1, x_2)$. This follows from the fact that

$$\min\{d(u(x), A^-) \mid x \in (x_1, x_2)\} \geq q'$$

implies

$$\varphi(q')(x_2 - x_1) \leq J_{(x_1,x_2)}(u) \leq J_{\mathbb{R}}(u_0).$$

Similarly, for each $u \in \mathcal{A}_b$ we have the implication

$$(x_1, x_2) \subset (x_u^+, +\infty) \text{ and } x_2 - x_1 \geq \ell_q \implies d(u(x_0), A^+) < q',$$

for some $x_0 \in (x_1, x_2)$.

Step 4. There is $M > 0$ such that, for each $u \in \mathcal{A}_b$, there is $\hat{u} \in \mathcal{A}_b$ with the properties

$$\|\hat{u}\|_{L^\infty(\mathbb{R};\mathbb{R}^m)} \leq M, \tag{4.3a}$$

$$J_{\mathbb{R}}(\hat{u}) \leq J_{\mathbb{R}}(u). \tag{4.3b}$$

Let $\bar{q}'$ be the number given by Lemma 4.3 in correspondence to $\bar{q}$. From *Step 3*, for each $u \in \mathcal{A}_b$ there is $x_0 \in (-\infty, x_u^-)$ such that $d(u(x_0), A^-) < \bar{q}'$, and therefore there exists

$$\bar{x}_u^- = \max\{x \mid d(u(x), A^-) \leq \bar{q}'\}.$$

Assume there exists $\alpha \in (-\infty, \bar{x}_u^-)$ such that $d(u(\alpha), A^-) = \bar{q}$, and define $\hat{u}$ by setting

$$\begin{cases} \hat{u}(x) = a^-, & \text{for } x < \bar{x}_u^- - 1, \\ \hat{u}(x) = v_u(x), & \text{for } [\bar{x}_u^- - 1, \bar{x}_u^-], \\ \hat{u}(x) = u(x), & \text{for } x > \bar{x}_u^-, \end{cases} \tag{4.4}$$

where $v_u : [\bar{x}_u^- - 1, \bar{x}_u^-] \to \bar{\Omega}$ is the map associated by (4.2) with the restriction of u to the interval $[\alpha, \bar{x}_u^-]$, and $a^- \in A^-$ is such that $d(u(\bar{x}_u^-), A^-) = d(u(\bar{x}_u^-), a^-)$. From (4.4) and Lemma 4.3, it follows that

$$|\hat{u}(x) - a^-| \leq \bar{q}' < \bar{q}, \quad \text{for } x \in (-\infty, \bar{x}_u^-],$$
$$J_{(-\infty,\bar{x}_u^-]}(\hat{u}) = J_{[\bar{x}_u^- - 1,\bar{x}_u^-]}(v_u) \leq J_{[\alpha,\bar{x}_u^-]}(u) \leq J_{(-\infty,\bar{x}_u^-]}(u).$$

By similar arguments, one proves the existence of

$$\bar{x}_u^+ = \min\{x \mid d(u(x), A^+) \le \bar{q}'\},$$

and concludes that $\hat{u}$ can be constructed so that there holds

$$d(\hat{u}(x), A^+) \le \bar{q}, \quad \text{for } x \in [\bar{x}_u^+, \infty),$$

together with (4.3b).

To complete the proof of (4.3a), we show that

$$\bar{x}_u^+ - \bar{x}_u^- \le \frac{J_{\mathbb{R}}(u_0)}{w}, \tag{4.5}$$

where $w = \min\{W(u) \mid u \in \Omega,\ d(u, \partial\Omega) \ge \bar{q}'\}$. This follows from the definition of $\bar{x}_u^\pm$ and

$$w(\bar{x}_u^+ - \bar{x}_u^-) \le J_{(\bar{x}_u^-, \bar{x}_u^+)}(u) \le J_{\mathbb{R}}(u_0).$$

From (4.5) we obtain

$$|u(x) - u(\bar{x}_u^-)| \le (x - \bar{x}_u^-)^{1/2} \left(\int_{\bar{x}_u^-}^{x} |\dot{u}|^2 \,\mathrm{d}x \right)^{1/2} \le \left(\frac{2}{w} \right)^{1/2} J_{\mathbb{R}}(u_0),$$

for $x \in [\bar{x}_u^-, \bar{x}_u^+]$, which completes the proof of (4.3a).

Step 5. For each $u \in \mathcal{A}_b$, we have

$$\lim_{x \to \pm\infty} d(u(x), A^\pm) = 0.$$

Suppose there exists a sequence $x_k \to +\infty$ and $q_0 \in (0, \bar{q})$ such that we have $d(u(x_k), A^+) \ge q_0$ for $k = 1, 2, \ldots$. Then, since $u \in \mathcal{A}_b$ implies u is uniformly continuous, we have

$$d(u(x), A^+) \ge \frac{q_0}{2}, \quad \text{for } x \in (x_k - \delta, x_k + \delta),$$

for some $\delta > 0$ independent of k. Therefore, we have

$$J_{(x_k - \delta, x_k + \delta)}(u) \ge 2\delta\varphi\left(\frac{q_0}{2}\right), \quad k = 1, 2, \ldots .$$

Since, by passing to a subsequence if necessary, we can assume that the intervals $(x_k - \delta, x_k + \delta)$, $k = 1, 2, \ldots$ are disjoint, this contradicts $J_{\mathbb{R}}(u) \le J_{\mathbb{R}}(u_0)$.

Step 6. From

$$|u(x_1)-u(x_2)| \le |x_1-x_2|^{1/2}\Big(\int_{x_1}^{x_2}|\dot u|^2\,\mathrm{d}x\Big)^{1/2} \le |x_1-x_2|^{1/2}\sqrt{2J_{\mathbb{R}}(u_0)},$$

it follows that $\mathcal{A}_b$ is an equicontinuous set. From *Step 4*, if $\{u_k\}\subset\mathcal{A}_b$ is a minimizing sequence, then we can also assume that $\{u_k\}\subset L^\infty(\mathbb{R},\mathbb{R}^m)$ is uniformly bounded.

Step 7: Conclusion. Let $\{u_k\}\subset\mathcal{A}_b$ be a minimizing sequence. By *Step 6*, we can assume $\{u_k\}$ is equibounded and equicontinuous. By (4.5) in *Step 4*, and the translation invariance of $J_{\mathbb{R}}$ and $\mathcal{A}$, we can assume

$$\bar x^-_{u_k}=0,\ \bar x^+_{u_k}\le \frac{J_{\mathbb{R}}(u_0)}{w},\qquad k=1,\dots .$$

By passing to subsequences if necessary, we can also assume the following:

(i) The sequence u_k converges to a continuous map $\bar u$ uniformly in compact intervals, and from the embedding

$$\|u\|_{L^2([-K,K];\mathbb{R}^m)}\le\sqrt{2K}\|u\|_{C([-K,K];\mathbb{R}^m)},\ \text{also in } L^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m).$$

This follows from the Ascoli-Arzelá theorem via a diagonal argument.

(ii) The sequence $\dot u_k$ converges weakly in $L^2(\mathbb{R};\mathbb{R}^m)$: $\dot u_k\rightharpoonup v$, for some $v\in L^2(\mathbb{R};\mathbb{R}^m)$. This is a consequence of the bound $\int_{\mathbb{B}}|\dot u_k|^2\le 2J_{\mathbb{R}}(u_0)$.

Here, (i) and (ii) imply that $(u_k,\dot u_k)$ converges weakly in $(L^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m))^2$ to $(\bar u, v)$. This and the fact that the derivative operator is weakly closed yield $v=\dot{\bar u}$, and therefore we conclude that $\bar u\in W^{1,2}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m)$.

From the lower semicontinuity of the L^2 norm, we have

$$\liminf_{k\to+\infty}\int_{\mathbb{B}}|\dot u_k|^2\ge\int_{\mathbb{B}}|\dot{\bar u}|^2,$$

and since, from (i), u_k converges pointwise to $\bar u$, we can apply Fatou's lemma to the sequence of nonnegative functions $\{W(u_k)\}$ to conclude that

$$\liminf_{k\to+\infty}\int_{\mathbb{B}}W(u_k)\ge\int_{\mathbb{B}}W(\bar u).$$

These inequalities imply that

$$\begin{aligned}J_{\mathbb{R}}(\bar u)&\le\liminf_{k\to+\infty}\int_{\mathbb{B}}\frac12|\dot u_k|^2+\liminf_{k\to+\infty}\int_{\mathbb{B}}W(u_k)\\&\le\lim_{k\to+\infty}J_{\mathbb{R}}(u_k)=\inf_{u\in\mathcal{A}_b}J_{\mathbb{R}}(u)\le J_{\mathbb{R}}(u_0).\end{aligned}\tag{4.6}$$

That is, $\bar{u}$ is a minimizer. Moreover, (4.6), and the uniform convergence in (i) imply $\bar{u} \in \mathcal{A}_b$; thus, from *Step 5* we have $\lim_{x\to\pm\infty} d(\bar{u}(x), A^{\pm}) = 0$. This concludes the proof. ◻

5. Heteroclinic, Homoclinic and Periodic Orbits

We first establish that the minimizer $\bar{u}$ from Theorem 4.2 satisfies the Euler-Lagrange equation and the equipartition relation on the interval where $W(\bar{u}) > 0$.

Proposition 5.1. *There exist L^- and L^+, $-\infty \le L^- < 0 < L^+ \le +\infty$ such that $(L^-, L^+) = \{x \in \mathbb{R} \mid \bar{u}(x) \in \Omega\}$, and if $L^- \in \mathbb{R}$ (respectively, $L^+ \in \mathbb{R}$), then we have $x \le L^- \implies \bar{u}(x) = \bar{u}(L^-) \in A^-$ (respectively, $x \ge L^+ \implies \bar{u}(x) = \bar{u}(L^+) \in A^+$). In addition, on the interval (L^-, L^+), $\bar{u}$ satisfies the Euler-Lagrange equation*

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}x^2} = \nabla W(\bar{u}),$$

and the equipartion relation

$$\frac{1}{2}\left|\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}(x)\right|^2 = W(\bar{u}(x)).$$

Proof. Since the Action functional J is translation invariant, we assume without loss of generality that $\bar{u}(0) \in \Omega$, and define

$$L^- = \inf\{x < 0 \mid \bar{u}((x,0]) \subset \Omega\}, \quad L^+ = \sup\{x > 0 \mid \bar{u}([0,x)) \subset \Omega\}.$$

We can see that if $L^- \in \mathbb{R}$ (respectively, $L^+ \in \mathbb{R}$), then $\bar{u}(L^-) \in A^-$ (respectively, $\bar{u}(L^+) \in A^+$), and by construction, $\bar{u}$ is constant on the interval $(-\infty, L^-]$ (respectively, $[L^+, \infty)$). In addition, $\bar{u}$ satisfies the Euler-Lagrange equation on (L^-, L^+). Now, let us prove the equipartion relation. We recall that the Hamiltonian of the solution $\bar{u}$

$$H := \frac{1}{2}\left|\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}(x)\right|^2 - W(\bar{u}(x))$$

is constant on the interval (L^-, L^+). Since $\lim_{x\to L^{\pm}} W(\bar{u}(x)) = 0$, we immediately see that $H \ge 0$. It remains to show that $H \le 0$. Indeed, we have

$$J_{(0,+\infty)}(\bar{u}) = \int_0^{L^+} \left(\frac{1}{2}\left|\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}(x)\right|^2 + W(\bar{u}(x))\right)\mathrm{d}x = \int_0^{L^+} (H + 2W(\bar{u}(x)))\,\mathrm{d}x,$$

and since $J_{(0,+\infty)}(\bar{u}) \in [0,\infty)$, it is clear that $L^+ = +\infty \Rightarrow H = 0$. Now, suppose $L^+ < +\infty$, and define

$$v(x) = \begin{cases} \bar{u}(x), & \text{for } x \le 0, \\ \bar{u}\left(\dfrac{x}{k}\right), & \text{for } x \ge 0, \end{cases}$$

where $k > 1$. We compute

$$\begin{aligned}
J_{(0,+\infty)}(v) &= \int_0^{kL^+} \left(\frac{1}{2k^2} \left| \frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\left(\frac{x}{k}\right) \right|^2 + W\left(\bar{u}\left(\frac{x}{k}\right)\right) \right) \mathrm{d}x \\
&= \int_0^{L^+} \left(\frac{1}{2k} \left| \frac{\mathrm{d}\bar{u}}{\mathrm{d}x}(t) \right|^2 + kW(\bar{u}(t)) \right) \mathrm{d}t \\
&= \int_0^{L^+} \left(\frac{H}{k} + \frac{k^2+1}{k} W(\bar{u}(t)) \right) \mathrm{d}t,
\end{aligned}$$

and note that

$$J_{(0,+\infty)}(\bar{u}) \le J_{(0,+\infty)}(v) \implies HL^+ \le (k-1)\int_0^{L^+} W(\bar{u}(x))\,\mathrm{d}x.$$

Letting $k \to 1^+$, we deduce that $H \le 0$. □

Next, given $A^* = A^+$ or A^-, we assume that one of the following is true:

(He) $\nabla W(u) = 0$, $\forall\, u \in A^*$.

(Ho) $\nabla W(u) \ne 0$, $\forall\, u \in A^*$.

According to which of the hypotheses (He) and (Ho) holds on A^- and A^+, we distinguish the following cases, and prove the existence of the heteroclinic, homoclinic, and periodic orbits.

Theorem 5.2. *If (He) holds on A^- and A^+, then the minimizer $\bar{u}$ constructed in Theorem 4.2 is a heteroclinic connection:*

$$\begin{gathered}
\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}x^2} = \nabla W(\bar{u}), \quad \text{on } \mathbb{R}, \\
\lim_{x\to\pm\infty} d(\bar{u}(x), A^\pm) = 0, \text{ and } \bar{u}(x) \in \Omega, \quad \forall\, x \in \mathbb{R}.
\end{gathered}$$

In particular, if $W \ge 0$, then[3] *$\bar{u}$ is also a local minimizer.*

Proof. Suppose for instance that $L^+ < +\infty$. By the equipartition relation and (He), we see that $\bar{u}$ is C^2 smooth and solves the Euler-Lagrange equation on the interval $(0, +\infty)$. Since $(\mathrm{d}\bar{u}/\mathrm{d}x)(L^+) = 0$ and $\nabla W(\bar{u}(L^+)) = 0$, we deduce by the uniqueness result for O.D.E. that $\bar{u}$ coincides with the constant solution $v(x) \equiv \bar{u}(L^+)$, which is a contradiction. Thus, $L^\pm = \pm\infty$, and $\bar{u}$ solves the Euler-Lagrange equation on all $\mathbb{R}$. When $W \ge 0$, let $z \in W^{1,2}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m)$ be a map coinciding with $\bar{u}$ outside a compact interval $[\alpha,\beta]$. If $z(\mathbb{R}) \subset \bar{\Omega}$, it is clear by the definition of $\bar{u}$ that $J_{\mathbb{R}}(\bar{u}) \le J_{\mathbb{R}}(z)$. Otherwise, there exists $s \in \mathbb{R}$ such that $z(s) \notin \bar{\Omega}$. Without loss of generality, we suppose that $z(s) \in F^-$, and define $s^- = \max\{s \in \mathbb{R} \mid z(s) \in F^-\}$. Now, if there exists $t > s^-$ such that $z(t) \in F^+$,

[3]Clearly, $W \ge 0$ implies that (He) holds on $A^\pm$.

we set $s^+ = \min\{t > s^- \mid z(t) \in F^+\}$. Since $s^- < s^+$, $z(s^\pm) \in \partial F^\pm = A^\pm$, and $z([s^-, s^+]) \subset \bar{\Omega}$, we deduce that $J_{\mathbb{R}}(\bar{u}) \leq J_{[s^-,s^+]}(z) \leq J_{\mathbb{R}}(z)$. Finally, if $z(t) \in \bar{\Omega}$, $\forall\, t \geq s^-$, then we have again $J_{\mathbb{R}}(\bar{u}) \leq J_{[s^-,+\infty)}(z) \leq J_{\mathbb{R}}(z)$. Thus, for every choice of z, $J_{\mathbb{R}}(\bar{u}) \leq J_{\mathbb{R}}(z)$ holds, and therefore $\bar{u}$ is a local minimizer. □

Corollary 5.3. *Assume $W : \mathbb{R}^m \to [0, \infty)$ has N zeros $a_1, \dots, a_N$, and satisfies* (H2). *Then, for every $i \in \{1, \dots, N\}$, there exists $j \in \{1, \dots, N\}$, $j \neq i$, and a minimal solution $\bar{u}$ of*

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}x^2} = \nabla W(\bar{u}), \quad \text{on } \mathbb{R},$$

such that

$$\lim_{x\to-\infty} |\bar{u}(x) - a_i| = 0, \text{ and } \lim_{x\to+\infty} |\bar{u}(x) - a_j| = 0.$$

Proof. Take $\Omega = \mathbb{R}^m \setminus \{a_1, \dots, a_N\}$, $A^- = \{a_i\}$, and $A^+ = \{a_1, \dots, a_N\} \setminus \{a_i\}$. □

Theorem 5.4. *Assume W satisfies* (H1), (H2), (He) *on A^-, and* (Ho) *on A^+; then, there exists a homoclinic connection v:*

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \nabla W(v) \text{ on } \mathbb{R},\ v(x) = v(-x) \quad \forall\, x \in \mathbb{R},$$
$$\lim_{x\to\pm\infty} d(v(x), A^-) = 0 \text{ and } v(x) \in A^+ \iff x = 0,\ v(x) \in \Omega,\ \forall\, x \neq 0.$$

Moreover, v is a minimizer of the Action $J_{\mathbb{R}}$ in the class

$$\mathcal{A}_{(\mathrm{Ho})} = \Big\{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; \bar{\Omega}) \mid d(u(x), A^-) \leq \bar{q}, \text{ for } |x| \geq x_u, \text{ for some } x_u,\ u(0) \in A^+\Big\}.$$

Proof. Let $\bar{u}$ be the minimizer given by Theorem 4.2. As in the proof of Theorem 5.2, we show that $L^- = -\infty$. We now prove that $L^+ < +\infty$. Suppose by contradiction that $L^+ = +\infty$. Then, $\bar{u}$ satisfies the equipartition relation $\frac{1}{2}|\bar{u}_x(x)|^2 = W(\bar{u}(x))$, $\forall\, x \in \mathbb{R}$, and we can ensure that for $x > M$ big enough,

$$\frac{\mathrm{d}^2 W(\bar{u})}{\mathrm{d}x^2}(x) = |\nabla W(\bar{u})|^2 + \mathrm{D}^2 W(\bar{u}(x))(\bar{u}'(x), \bar{u}'(x)) \geq \varepsilon > 0.$$

As a consequence, $W(\bar{u}(x)) \to +\infty$ as $x \to +\infty$, which is a contradiction. Thus, $L^+ < +\infty$. Now, define

$$v(x) = \begin{cases} \bar{u}(x + L^+), & \text{for } x \leq 0, \\ \bar{u}(-x + L^+), & \text{for } x \geq 0. \end{cases}$$

Clearly, $v \in \mathcal{A}_{(\mathrm{Ho})}$, and again by the equipartition relation, we see that the derivative of v at $x = 0$ exists and vanishes. By symmetry, we also check that v is C^2

smooth and satisfies the Euler-Lagrange equation on all $\mathbb{R}$. Finally, it is obvious that $J_{\mathbb{R}}(v) = 2J_{\mathbb{R}}(\bar{u})$, and that v minimizes the Action in the class $\mathcal{A}_{(\text{Ho})}$. ◻

Theorem 5.5. *Assume W satisfies* (H1), (H2), *and* (Ho) *on $A^{\pm}$; then, there exists a periodic solution $\bar{v}$ of the Euler-Lagrange equation*

$$\frac{\mathrm{d}^2\bar{v}}{\mathrm{d}x^2} = \nabla W(\bar{v}) \text{ on } \mathbb{R},\ \bar{v}(x) = \bar{v}(-x), \quad \forall\, x \in \mathbb{R},$$

such that for every $x \in \mathbb{R}$,

$$\bar{v}(x+T) = \bar{v}(x), \text{ and } \bar{v}\left(x + \frac{T}{2}\right) = \bar{v}\left(-x + \frac{T}{2}\right) \quad \text{for some } T > 0,$$

and

$$\bar{v}(0) \in A^-,\ \bar{v}\left(\frac{T}{2}\right) \in A^+,\ \bar{v}(x) \in \Omega \iff x \notin \frac{T}{2}\mathbb{Z}.$$

Moreover, $\bar{v}$ is characterized variationally as follows:

$$J_{[0,T/2]}(\bar{v}) = \min\{J_{[0,\ell]}(u) \mid u \in \mathcal{B}_\ell,\ \ell > 0\},$$

where $\mathcal{B}_\ell := \{u \in W^{1,2}([0,\ell];\bar{\Omega}) \mid u(0) \in A^-,\ u(\ell) \in A^+\}$.

Proof. Let $\bar{u}$ be the minimizer given by Theorem 4.2. Proceeding as in the proof of Theorem 5.4, we show that $L^{\pm} \in \mathbb{R}$. Next, we set $T := 2(L^+ - L^-)$, and define

$$\bar{v}(x) = \begin{cases} \bar{u}(x + L^-), & \text{for } 0 \le x \le \dfrac{T}{2}, \\ \bar{u}(-x + 2L^+ - L^-), & \text{for } \dfrac{T}{2} \le x \le T. \end{cases}$$

Since $\bar{v}(0) = \bar{v}(T)$, $\bar{v}$ can be extended periodically on all $\mathbb{R}$. By the equipartition relation, we see that the derivative of $\bar{v}$ exists and vanishes at the points $x = 0$ and $x = T/2$. By symmetry, we also check that $\bar{v}$ is C^2 smooth and satisfies the Euler-Lagrange equation on all $\mathbb{R}$. Finally, the variational characterization of $\bar{v}$ is straightforward. ◻

Remark 5.6. From Proposition 5.1, we have the following:

- When $L^- = -\infty$ and $L^+ = +\infty$, $\bar{u}$ is a heteroclinic orbit connecting $A^{\pm}$. Indeed, since $\bar{u}'(x) \to 0$ and $\bar{u}'''(x) = \mathrm{D}^2W(\bar{u}(x))\bar{u}'(x) \to 0$ as $x \to \pm\infty$, we can see that $\bar{u}''(x) = \nabla W(\bar{u}(x)) \to 0$ as $x \to \pm\infty$ (cf. Section 3.4 in [6]). Thus, $d(\bar{u}(x), A^{\pm} \cap \partial\Omega_0) \to 0$ as $x \to \pm\infty$, in accordance with Definition 2.1.
- When $L^- = -\infty$ and $L^+ \in \mathbb{R}$ (respectively, when $L^{\pm} \in \mathbb{R}$), there exists a homoclinic (respectively, a periodic) orbit (see the proofs of Theorems 5.4 and 5.5).

The uniform conditions (He) and (Ho) have only been imposed to determine when $L^{\pm}$ is finite, and to ensure the existence of the corresponding orbits. Under the assumptions of Theorem 4.2, there always exists either a heteroclinic, a homoclinic, or a periodic orbit connecting $A^{\pm}$ at $\pm\infty$.

6. Asymptotic Convergence

A natural question arises in the case of Theorems 5.2 and 5.4. Does the solution $\bar{u}$ (respectively, v) converge to a point of $A^{\pm}$ (respectively, A^{-}) at $\pm\infty$? Before answering this question, we establish the following exponential estimate.

Proposition 6.1. *Assume that $A^{-} \subset \mathbb{R}^m$ is a C^2 compact orientable surface with unit normal $\mathbf{n}$, and that W satisfies* (He) *and $(\partial^2 W/\partial \mathbf{n}^2)(u) > 0$, $\forall\, u \in A^{-}$. Then, the heteroclinic connection $\bar{u}$ of Theorem 5.2 satisfies $d(\bar{u}(x), A^{-}) \le Ke^{kx}$, $\forall\, x \le 0$, for some constants $K, k > 0$.*[4]

Proof. First, we consider a tubular neighborhood $\mathcal{V}$ of A^{-} (cf. [5]) such that

$$md^2(u, A^{-}) \le W(u) \le Md^2(u, A^{-}), \quad \forall\, u \in \mathcal{V}, \tag{6.1}$$

and for some constants $0 < m < M$. Let x_0 be such that $\bar{u}(x) \in \mathcal{V}$ for all $x \le x_0$. For $x \le x_0$ fixed, let $a^{-} \in A^{-}$ be the point such that $d(\bar{u}(x), a^{-}) = d(\bar{u}(x), A^{-})$, and define the map

$$z(t) = \begin{cases} \bar{u}(x) + (x-t)(a^{-} - \bar{u}(x)), & \text{for } x-1 \le t \le x, \\ a^{-}, & \text{for } t \le x-1. \end{cases} \tag{6.2}$$

By the variational characterization of $\bar{u}$ and (6.1), we can see that

$$\begin{aligned} m\int_{-\infty}^{x} d^2(\bar{u}(t), A^{-})\,\mathrm{d}t &\le J_{(-\infty,x]}(\bar{u}) \le J_{[x-1,x]}(z) \\ &\le (M+1)d^2(\bar{u}(x), A^{-}). \end{aligned} \tag{6.3}$$

Setting

$$\theta(x) := \int -\infty x d^2(\bar{u}(t), A^{-})\,\mathrm{d}t,$$

we deduce that $c\theta(x) \le \theta'(x)$, $\forall\, x \le x_0$, and for some constant $c > 0$. Integrating this inequality, it follows that

$$\theta(x) \le \theta(x_0)e^{c(x-x_0)}. \tag{6.4}$$

[4]Clearly, the homoclinic connection v of Theorem 5.4 satisfies the same estimate at $-\infty$.

Now, we note that by (6.1), the constant q' in Lemma 4.3 can be chosen to be proportional to q; that is, $q' = \lambda q$ for some $\lambda > 0$. As a consequence,

$$t \in [x-1, x] \implies \lambda d(\bar{u}(x-1), A^-) \leq d(\bar{u}(t), A^-), \tag{6.5}$$

and

$$\lambda^2 d^2(\bar{u}(x-1), A^-) \leq \int_{x-1}^{x} d^2(\bar{u}(t), A^-)\,\mathrm{d}t \leq \theta(x). \tag{6.6}$$

From (6.4) and (6.6), we conclude that $d^2(\bar{u}(x), A^-) = O(e^{cx})$. ◻

Remark 6.2. Here, the exponential estimate of Proposition 6.1 also holds for bounded local minimizers $\bar{u}$ such that $d(\bar{u}(x), A^-) \to 0$ as $x \to -\infty$. Some adjustments have to be done in the proof. Instead of (6.2), define a comparison map as in (3.3). Then, (6.3) remains true at the limit. To prove (6.5), a similar argument is used.

Corollary 6.3. *Under the assumptions of Proposition 6.1, there exists $\ell \in A^-$ such that $\bar{u}(x) \to \ell$, as $x \to -\infty$.*

Proof. From the exponential estimate of Proposition 6.1, and by (6.1) and the equipartition relation, we also have

$$W(\bar{u}(x)) = O(e^{2kx}) \quad \text{and} \quad \left| \left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\right)(x) \right| = O(e^{kx}).$$

In particular, $|(\mathrm{d}\bar{u}/\mathrm{d}x)(x)|$ is integrable in a neighborhood of $-\infty$. Suppose by contradiction that $\bar{u}(x)$ does not have a limit at $-\infty$. Then, there exist two sequences $x_n \to -\infty$ and $y_n \to -\infty$ such that $\lim_{n\to\infty} \bar{u}(x_n) = \ell_1 \in A^-$, $\lim_{n\to\infty} \bar{u}(y_n) = \ell_2 \in A^-$, and $\ell_1 \neq \ell_2$. This implies that the length of the curve defined by $\bar{u}$ is infinite, and thus $|(\mathrm{d}\bar{u}/\mathrm{d}x)(x)|$ is not integrable in a neighborhood of $-\infty$. ◻

Remark 6.4. The shape of minimal connecting orbits can be very complicated (cf. [12]). In general, the limit of $\bar{u}$ at $-\infty$ cannot be determined.[5]. Indeed, the presence of an "obstacle" where W is big near a portion of A^- may prevent the minimizer $\bar{u}$ from approaching certain points of A^-. Recall (cf. [3]) that the curve described by $\bar{u}$ minimizes the functional

$$L(\Gamma) = \int_I \sqrt{2W(\gamma(t))}\,|\gamma'(t)|\,\mathrm{d}t$$

[5]We would, however, like to mention two interesting particular cases. For the class of potentials considered in [7], which are functions of distance to the target manifolds, the connecting orbits are straight line segments. For potentials $W : \mathbb{R}^2 \sim \mathbb{C} \to \mathbb{R}$ such that $W(u) = |f(u)|^2$ with $f = g'$ holomorphic, it is shown in [2] that the image under g of the trajectories of heteroclinics are also straight line segments.

in the class of curves $\Gamma : I \ni t \to \gamma(t) \in \Omega$ connecting A^- and A^+. Therefore, for certain potentials that do not satisfy the convexity assumption of Proposition 6.1, a curve describing a spiral around A^- may also minimize the functional L.

Now, assuming that the surface A^- bounds a convex set S, we prove that the results of Proposition 6.1 and Corollary 6.3 remain true, for every solution to (1.2) approaching A^- in the complement of S.

Proposition 6.5. *Let S be a compact and convex set, with C^2 boundary and outer unit normal $\mathbf{n}$, and let $u : (-\infty, \alpha) \to \mathbb{R}^m \setminus S$ be a solution to (1.2) such that $\delta(x) := d(u(x), S) \to 0$ as $x \to -\infty$. Assume that W satisfies $W(u) = 0$, $\nabla W(u) = 0$, and $(\partial^2 W/\partial \mathbf{n}^2)(u) > 0$, $\forall\, u \in \partial S$. Then, in a neighborhood of $-\infty$, the function $x \to \delta(x)$ is increasing, and moreover, $\delta(x) = O(e^{kx})$ for some constant $k > 0$* [6]*. As a consequence, the limit of u at $-\infty$ exists.*

Proof. For every $u \in \mathbb{R}^m \setminus S$, let p be the projection of u onto S, and $\mathbf{n}$ the outer unit normal at p. Then, we have $u = p + d(u, S)\mathbf{n}$ and $d(u, S) = (u - p) \cdot \mathbf{n}$. By differentiating twice the function δ, we obtain after some easy simplifications

$$\dot{\delta} = \dot{u} \cdot \mathbf{n},$$
$$\ddot{\delta} = \ddot{u} \cdot \mathbf{n} - \ddot{p} \cdot \mathbf{n} + \delta |\dot{\mathbf{n}}|^2.$$

Furthermore, since S is convex, we have $\ddot{p} \cdot \mathbf{n} \le 0$ and $\ddot{\delta} \ge (\partial W/\partial \mathbf{n})(u)$. Next, using the properties of W, we can see that δ is convex in a neighborhood of $-\infty$, and since $\lim_{x \to -\infty} \dot{\delta}(x) = 0$ by the equipartition relation (cf. Proposition 2.4), δ is increasing in a neighborhood of $-\infty$. Finally, the convexity assumption on W implies that $\ddot{\delta} \ge (\partial W/\partial \mathbf{n})(u) \ge k^2 \delta$ in a neighborhood of $-\infty$, for a constant $k > 0$. Integrating this inequality, and taking into account the limits of δ and $\dot{\delta}$ at $-\infty$, it follows that $\dot{\delta} \ge k\delta$ and $\delta(x) = O(e^{kx})$, for x in a neighborhood of $-\infty$. The convergence of u to a point of ∂S is established as in Corollary 6.3. □

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[6]To prove that δ is increasing, it is sufficient to assume that $u \in \partial S \Rightarrow W(u) = 0$, $\nabla W(u) = 0$, and $(\partial W/\partial \mathbf{n})(u) \ge 0$ for $0 < d(u, S) \le \varepsilon$, with $\varepsilon > 0$ small. We point out that the statement of the proposition also holds if S is reduced to a point.

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