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Gradient theory of domain walls in thin, nematic liquid crystals films

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In this paper, we describe domain walls appearing in a thin, nematic liquid crystal sample subject to an external field with intensity close to the Fréedericksz transition threshold. Using the gradient theory of the phase transition adapted to this situation, we show that depending on the parameters of the system, domain walls occur in the bistable region or at the border between the bistable and the monostable region.

Keywords: Domain walls; phase transition; liquid crystals; minimizer.

Mathematics Subject Classification 2020: 35J20, 35B25, 35B05

1. Introduction

Inhomogeneous initial conditions caused by, e.g., inherent fluctuations of macroscopic systems, generate the emergence of equilibria in different parts of space, which are usually identified as spatial domains. These domains are separated by domain walls or interface between the equilibria. A classic example of this phenomena is the magnetic domains and walls [11]. Depending on the configuration of the magnetization these walls are usually denominated as Ising, Bloch, and Neel. Likewise, similar walls have been observed in liquid crystals, when a liquid crystal film is subjected to magnetic or electric fields [15]. In particular, nematic liquid crystals with planar anchoring exhibit Ising walls [3]. Close to the reorientation instability of the molecules, Fréedericksz transition, this system is well described by the Allen-Cahn equation. Besides, using a photosensitive wall, it is possible to induce a molecular reorientation in a thin liquid crystal film [8]. This type of device is called a liquid crystal light valve (see [17] and references therein). Due to the inhomogeneous illumination generated by light on the liquid crystal layer, the dynamics of molecular reorientation is described by

$$\partial_t u(x_1, x_2, t) = \epsilon^2 \Delta u + \mu(x_1, x_2)u - u^3 + a\epsilon x_1 f(x_1, x_2), \tag{1.1}$$

where $u(x_1, x_2, t)$ accounts for the average rotational amplitude of the molecules, t, x_1, and x_2, respectively, stand for time and the transverse coordinates of the liquid crystal layer, x_1 is the direction in which the molecules are anchored, $f(x_1, x_2) = -\frac{1}{2}\partial_{x_1}\mu(x_1, x_2)$, and non-dimensional parameters ϵ, a are positive. Equation (1.1) is the amplitude equation and can be obtained formally from the Oseen–Frank model of liquid crystals (see [5, 9]). Because in general the size of a liquid crystal sample is large compared with the size of a defect (domain wall in this particular case), to describe it, it is reasonable to consider (1.1) in the whole space.

The function

$$\mu(x_1, x_2) = \mu_0 + I_o e^{-\frac{x_1^2 + x_2^2}{w^2}}, \tag{1.2}$$

which accounts the forcing given by the external electric field and the effect of the illuminated photo-sensitive wall characterized by the light intensity $I_o > 0$, is typically sign changing i.e. $-I_0 < \mu_0 < 0$. This last condition describes the situation when the electrical voltage applied to the liquid crystal sample is less than the Fréedericksz voltage. The level set $\{\mu(x_1, x_2) = 0\}$ separates two disjoint regions where μ is of constant sign. For any $x \in \{\mu > 0\}$ the potential

$$U(z, x) = -\mu(x)\frac{z^2}{2} + \frac{z^4}{4}$$

has precisely two non-degenerate minima of equal depth at $z = \pm\sqrt{\mu(x)}$, while in the region $\{\mu < 0\}$, U is nonnegative and its only minimum occurs at $z = 0$. Motivated by this we will call the set $\{\mu > 0\} \subset \mathbb{R}^2$, the bistable region and the set $\{\mu < 0\} \subset \mathbb{R}^2$ the monostable region. Note that with the choice of the function μ in (1.2) the bistable region is a disk and the monostable region is its complement in $\mathbb{R}^2$. The objective of this paper is to understand how the location of the domain walls, defined as the set of zeros of the solutions of (1.1), changes when the parameters ϵ and α vary. For this purpose we will restrict our attention to the time independent solutions, the idea being that the system quickly relaxes to its stationary state.

If one ignores the dependence on the transversal x_2 coordinate, the system exhibits two type of walls that separate domains that evanesce asymptotically [1, 5].

One corresponds to the extension of Ising wall, standard kink, in this inhomogeneous system, which is a symmetric solution and centered in the region of the maximal illumination i.e. $x = 0$ (since $\mu(x)$ attains its maximum in the origin). The other corresponds to a wall centered in the non-illuminated part, shadow kink [1, 5]. To understand the latter one can expand the solution around the point where $\mu(x) = 0$. In this limit the profile of the transition is described by the second Painlevé equation [4, 5, 20]. This paper is devoted to understand the physically relevant equilibrium situation when the dependence on the second coordinate is not neglected.

In this limit the stationary solutions of (1.1) can be characterized as the minima of the following energy functional

$$E(u) = \int_{\mathbb{R}^2} \frac{\epsilon}{2}|\nabla u|^2 - \frac{1}{2\epsilon}\mu(x)u^2 + \frac{1}{4\epsilon}u^4 - af_1(x)u, \tag{1.3}$$

where $u \in H^1(\mathbb{R}^2)$ and $\epsilon > 0$, $a \geq 0$ are real parameters. More generally as in (1.2) we suppose that $\mu \in C^\infty(\mathbb{R}^2)$ is radial i.e. $\mu(x) = \mu_{\mathrm{rad}}(|x|)$, with $\mu_{\mathrm{rad}} \in C^\infty(\mathbb{R})$ an even function. We take $f = (f_1, f_2) \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ also to be radial i.e. $f(x) = f_{\mathrm{rad}}(|x|)\frac{x}{|x|}$. Note that since f is smooth, f_{rad} has an odd extension $f_{\mathrm{rad}} \in C^\infty(\mathbb{R})$ to the whole real line. In addition we assume that

$$\begin{cases} \mu \in L^\infty(\mathbb{R}^2), \quad \mu'_{\mathrm{rad}} < 0 \quad \text{in } (0,\infty), \quad \text{and} \quad \mu_{\mathrm{rad}}(\rho) = 0 \text{ for a unique } \rho > 0, \\ f \in L^1(\mathbb{R}^2,\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2,\mathbb{R}^2), \quad \text{and} \quad f_{\mathrm{rad}} > 0 \quad \text{on } (0,\infty). \end{cases} \tag{1.4}$$

The Euler–Lagrange equation of E is

$$\epsilon^2\Delta u + \mu(x)u - u^3 + \epsilon a f_1(x) = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{1.5}$$

and we also write its weak formulation:

$$\int_{\mathbb{R}^2} -\epsilon^2 \nabla u \cdot \nabla\psi + \mu u\psi - u^3\psi + \epsilon a f_1 \psi = 0, \quad \forall\, \psi \in H^1(\mathbb{R}^2), \tag{1.6}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^2$. Note that due to the radial symmetry of μ and f, the energy (1.3) and Eq. (1.5) are invariant under the transformations $u(x_1, x_2) \mapsto -u(-x_1, x_2)$, and $u(x_1, x_2) \mapsto u(x_1, -x_2)$.

Our purpose in this paper is to study qualitative properties of the global minimizers of E as the parameters a and ϵ vary. Our focus will be mainly on the regime where $\epsilon > 0$ is small and $a \geq 0$ is fixed. For convergence problems in the singular limit using geometric measure theory we refer to the work of Modica-Mortola [14], Modica [12, 13], and Caffarelli-Córdoba [2]. In the case of the energy functionals $J_\epsilon(u) = \int_\Omega (\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u))\mathrm{d}x$, where W is a bistable potential i.e. $W(u) = \frac{1}{4}(u^2-1)^2$, Modica [13] proved using Γ convergence, that any sequence of minimizers (u_ϵ) of J_ϵ with uniformly bounded energy, converge to some $u_S = \chi_S - \chi_{\Omega\setminus S}$ in certain sense, where ∂S has minimal perimeter. Furthermore, it is established in [2] that the level sets $\{u_\epsilon = \lambda\}$ converge locally uniformly to the interface.

In our setup the potential associated to Eq. (1.5)

$$W(x,u)=\frac{1}{4\epsilon}u^4-\frac{\mu(x)}{2\epsilon}u^2+\epsilon a f_1(x)$$

(cf. also (1.3)) is not bistable, and this is a major difference with the situation described previously. Indeed, in the region where $|x|\geq\rho$, W is monostable, that is, it is a convex function of u with only one global minimum. On the other hand, when $|x|<\rho$ and $x_1\neq 0$, the potential W is unbalanced in the sense that as a function of u, it has one global minimum and one local minimum. In our previous work [5, 6], we examined respectively the cases of minimizers $v:\mathbb{R}\to\mathbb{R}$, and $v:\mathbb{R}^2\to\mathbb{R}^2$. In this paper, we follow the approach presented therein, and introduce several new ideas to address the specific issues occurring for minimizers $v:\mathbb{R}^2\to\mathbb{R}$. In particular, new variational arguments to determine the limit points of the zero level set of v, which is now a curve (cf. the conclusion of the proofs of Theorem 1.2(ii) and (iii)), and a computation of the energy that reduces to a one-dimensional problem, by using iterated integrals.

Proceeding as in [6], one can see that under the above assumptions there exists a global minimizer v of E in $H^1(\mathbb{R}^2)$, namely that $E(v)=\min_{H^1(\mathbb{R}^2)}E$. In addition, we show that v is a classical solution of (1.5), and v is even with respect to x_2 i.e. $v(x_1,x_2)=v(x_1,-x_2)$. In the sequel, we will always denote by v the global minimizer, and by u an arbitrary critical point of E in $H^1(\mathbb{R})$. Some basic properties are stated in:

Theorem 1.1. *For $\epsilon\ll 1$, and $a\geq 0$ bounded (possibly dependent on ϵ), let $v_{\epsilon,a}$ be a global minimizer of E, let $\rho>0$ be the zero of $\mu_{\rm rad}$ and let $\mu_1:=\mu'_{\rm rad}(\rho)<0$. The following statements hold:*

(i) *Let $\Omega\subset D(0;\rho)$ be an open set such that $v_{\epsilon,a}>0$ (respectively, $v_{\epsilon,a}<0$) on Ω, for every $\epsilon\ll 1$. Then $v_{\epsilon,a}\to\sqrt{\mu}$ (respectively, $v_{\epsilon,a}\to-\sqrt{\mu}$) in $C^0_{\rm loc}(\Omega)$.*

(ii) *For every $\xi=\rho e^{i\theta}$, we consider the local coordinates $s=(s_1,s_2)$ in the basis $(e^{i\theta},ie^{i\theta})$, and the rescaled minimizers:*

$$w_{\epsilon,a}(s)=2^{-1/2}(-\mu_1\epsilon)^{-1/3}v_{\epsilon,a}\left(\xi+\epsilon^{2/3}\frac{s}{(-\mu_1)^{1/3}}\right).$$

Assuming that $\lim_{\epsilon\to 0}a(\epsilon)=a_0$, then as $\epsilon\to 0$, the function $w_{\epsilon,a}$ converges in $C^2_{\rm loc}(\mathbb{R}^2)$ up to subsequence, to a function y bounded in $[s_0,\infty)\times\mathbb{R}$ for every $s_0\in\mathbb{R}$, which is a minimal solution of

$$\Delta y(s)-s_1y(s)-2y^3(s)-\alpha=0,\quad\forall\, s=(s_1,s_2)\in\mathbb{R}^2,\tag{1.7}$$

with $\alpha=\frac{a_0f_1(\xi)}{\sqrt{2}\mu_1}$.

(iii) *Assuming that $\lim_{\epsilon\to 0}a(\epsilon)=a_0$, then we have $\lim_{\epsilon\to 0}\frac{u_{\epsilon,a}(x)}{\epsilon}=-\frac{a_0}{\mu(x)}f_1(x)$ uniformly on compact subsets of $\{|x|>\rho\}$.*

Looking at the energy E it is evident that as $\epsilon \to 0$ the modulus of the global minimizer $|v_{\epsilon,a}|$ should approach a non-negative root of the polynomial

$$-\mu(x)u + u^3 - a\epsilon f_1(x) = 0,$$

or in other words, $|v_{\epsilon,a}| \to \sqrt{\mu^+}$ as $\epsilon \to 0$ in some, perhaps weak, sense. We observe for instance that as a corollary of Theorem 1.1(i) and Theorem 1.2(ii) below we obtain when $a < a_*$ the convergence in $C^0_{\mathrm{loc}}(D(0;\rho))$, thus $\Omega = D(0;\rho)$ in this case (cf. the conclusion of Theorem 1.2(ii) for more details).

Because of the analogy between the functional E and the Gross–Pitaevskii functional in theory of Bose–Einstein condensates we will call $\sqrt{\mu^+}$ the Thomas–Fermi limit of the global minimizer. Theorem 1.1 gives account on how non-smoothness of the limit of $v_{\epsilon,a}$ is mediated near the circumference $|x| = \rho$, where μ changes sign, through the solution of (1.7).

This equation is a natural generalization of the second Painlevé ODE

$$y'' - sy - 2y^3 - \alpha = 0, \quad s \in \mathbb{R}. \tag{1.8}$$

In [5] we showed that this last equation plays an analogous role in the one-dimensional, scalar version of the energy E:

$$E(u,\mathbb{R}) = \int_{\mathbb{R}} \frac{\epsilon}{2}|u_x|^2 - \frac{1}{2\epsilon}\mu(x)u^2 + \frac{1}{4\epsilon}|u|^4 - af(x)u$$

where μ and f are scalar functions satisfying similar hypothesis to those we have described above. In this case the Thomas–Fermi limit of the global minimizer is simply $\sqrt{\mu^+(x)}$, which is non-differentiable at the points $x = \pm\xi$ which are the zeros of the even function μ. Near these two points a rescaled version of the global minimizer approaches a solution of (1.8) similarly as it is described in Theorem 1.1(ii).

It is very important to realize that not every solution of (1.8) can serve as the limit of the global minimizer, since in our case the limiting solutions of (1.7) are necessarily minimal as well. To explain what this means, let

$$E_{\mathrm{P_{II}}}(u,A) = \int_A \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}s_1u^2 + \frac{1}{2}u^4 + \alpha u\right].$$

By definition a solution of (1.7) is minimal if

$$E_{\mathrm{P_{II}}}(y, \operatorname{supp}\phi) \le E_{\mathrm{P_{II}}}(y+\phi, \operatorname{supp}\phi) \tag{1.9}$$

for all $\phi \in C_0^\infty(\mathbb{R}^2)$. This notion of minimality is standard for many problems in which the energy of a localized solution is actually infinite due to non-compactness of the domain.

The study of minimal solutions of (1.8) was recently initiated in [5] where we showed that the Hastings–McLeod solutions h and $-h$, are the only minimal solutions of the homogeneous equation

$$y'' - sy - 2y^3 = 0, \quad s \in \mathbb{R}, \tag{1.10}$$

which are bounded at $+\infty$. We recall (cf. [10]) that $h : \mathbb{R} \to \mathbb{R}$ is positive, strictly decreasing ($h' < 0$) and such that

$$\begin{aligned} h(s) &\sim Ai(s), \quad s \to \infty, \\ h(s) &\sim \sqrt{\frac{|s|}{2}}, \quad s \to -\infty. \end{aligned} \tag{1.11}$$

On the other hand in [7] we considered when $a = 0$, the odd minimizer u of (1.3)[a] in the class $H^1_{\mathrm{odd}}(\mathbb{R}^2) := \{u \in H^1(\mathbb{R}^2) : u(x_1, x_2) = -u(-x_1, x_2)\}$ of odd functions with respect to x_1, and following Theorem 1.1(ii), we established the existence of a nontrivial solution y of the homogeneous equation (1.7). It has a form of a quadruple connection between the Airy function $Ai(x)$, the two one-dimensional Hastings–McLeod solutions $\pm h(x)$ and the heteroclinic orbit $\eta(x) = \tanh(x/\sqrt{2})$ of the ODE $\eta'' = \eta^3 - \eta$. Although we know (cf. [7, Theorem 2.1]) that Theorem 1.1(ii) applied to the global minimizer v in the homogeneous case $a = 0$, gives at the limit either $y(s_1, s_2) = h(s_1)$ or $y(s_1, s_2) = -h(s_1)$, we are not aware if in the nonhomogeneous case $a \neq 0$, Theorem 1.1(ii) produces a new kind of minimal solution. This goes beyond the scope of the present paper.

Finally, regarding Theorem 1.1(iii) we note that since the sign of the local limit of the rescaled global minimizer in $|x| > \rho$ is determined by the sign of f_1, one may expect that the zero level set of $v_{\epsilon,a}$ is a smooth curve (cf. Lemma 3.1) partitioning the plane. In Theorem 1.2 we will determine the limit of this level set according to the value of a, and discuss the dependence of the global minimizer on a, when $\epsilon \ll 1$.

Before stating our second result we recall that the heteroclinic orbit $\eta(x) = \tanh(x/\sqrt{2})$ ($\eta : \mathbb{R} \to (-1, 1)$) of the ODE $\eta'' = \eta^3 - \eta$, connecting the two minima ± 1 of the potential $W(u) = \frac{1}{4}(1 - u^2)^2$ ($W : \mathbb{R} \to [0, \infty)$) plays a crucial role in the study of minimal solutions of the Allen–Cahn equation

$$\Delta u = u^3 - u, u : \mathbb{R}^n \to \mathbb{R}. \tag{1.12}$$

Again, we say that u is a minimal solution of (1.12) if

$$E_{\mathrm{AC}}(u, \operatorname{supp} \phi) \leq E_{\mathrm{AC}}(u + \phi, \operatorname{supp} \phi),$$

for all $\phi \in C_0^\infty(\mathbb{R}^2)$, where

$$E_{\mathrm{AC}}(u, \Omega) := \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1 - u^2)^2$$

is the Allen–Cahn energy associated to (1.12). It is known [19] that in dimension $n \leq 7$, any minimal solution u of (1.12) is either trivial i.e. $u \equiv \pm 1$ or one-dimensional i.e. $u(x) = \eta((x - x_0) \cdot \nu)$, for some $x_0 \in \mathbb{R}^n$, and some unit vector $\nu \in \mathbb{R}^n$.

Theorem 1.2. *Let $v_{\epsilon,a}$ be a global minimizer of E, with $a \geq 0$ fixed (independent of ϵ), and let $Z = \{l \in \mathbb{R}^2$ is a limit point of the set of zeros of $v_{\epsilon,a}$ as $\epsilon \to 0\}$. The*

[a]Due to the symmetry of μ and f, u is also a critical point of (1.3) (cf. [16]), thus it solves (1.5).

following statements hold.

(i) *When* $a = 0$ *the global minimizer* v *is unique up to change of* v *by* $-v$*. It can be written as* $v(x) = v_{\mathrm{rad}}(|x|)$*, with* $v_{\mathrm{rad}} \in C^\infty(\mathbb{R})$*, positive and even.*

(ii) *There exists a constant* $a_* > 0$ *such that for all* $a \in (0, a_*)$*, we have up to change of* $v(x_1, x_2)$ *by* $-v(-x_1, x_2)$:

$$\{x_1 < 0, |x| = \rho\} \cup \{x_1 = 0, |x_2| \geq \rho\} \subset Z \subset \{|x| = \rho\} \cup \{x_1 = 0, |x_2| \geq \rho\},$$

and

$$\lim_{\epsilon \to 0} v(x + s\epsilon) = \sqrt{\mu^+(x)}, \quad \forall\, x \in \mathbb{R}^2, \tag{1.13}$$

in the $C^2_{\mathrm{loc}}(\mathbb{R})$ *sense. The above asymptotic formula holds as well when* $a = 0$.

(iii) *Suppose that* $f'_{\mathrm{rad}}(0) \neq 0$*, then there exists a constant* $a^* \geq a_*$ *such that for all* $a > a^*$ *we have* $Z = \{x_1 = 0\}$*, and the global minimizer* v *satisfies*

$$\lim_{\epsilon \to 0} v(x + \epsilon s) = \begin{cases} \sqrt{\mu^+(x)} & \text{for } x_1 > 0, \\ -\sqrt{\mu^+(x)} & \text{for } x_1 < 0, \end{cases} \tag{1.14}$$

in the $C^2_{\mathrm{loc}}(\mathbb{R})$ *sense. Next, if* $\bar{x}_{\epsilon,a} = (\bar{t}_{\epsilon,a}, x_2)$ *is a zero of* $v_{\epsilon,a}$ *with fixed ordinate* x_2*, then up to subsequence and for a.e.* $x_2 \in (-\rho, \rho)$ *we have*

$$\lim_{\epsilon \to 0} v(\bar{x} + \epsilon s) = \sqrt{\mu(0, x_2)} \tanh\left(s_1 \sqrt{\frac{\mu(0, x_2)}{2}}\right), \text{ in the } C^2_{\mathrm{loc}}(\mathbb{R}) \text{ sense.} \tag{1.15}$$

Finally, when $f = -\frac{1}{2}\nabla\mu$ *we have* $a_* = a^* = \sqrt{2}$.

Perhaps the most interesting and unexpected is the statement (ii) of the above theorem. It says that, at least in the limit $\epsilon \to 0$ the domain wall Z is located at the border between the monostable region $\{\mu < 0\}$ and the bistable region $\{\mu > 0\}$. Physically this means that as the intensity of the illumination, measured by a, is relatively small then no defect is visibly seen. For this reason and by analogy with [5, 6] we call it the shadow domain wall. As a increases the shadow domain wall penetrates the bistable region becoming the standard domain wall, as described in (iii).

It is natural to expect in Theorem 1.2(ii) that $Z = \{x_1 < 0, |x| = \rho\} \cup \{x_1 = 0, |x_2| \geq \rho\}$. However, the energy considerations presented in the proof of Theorem 1.2 do not exclude the existence of a limit point of the zeros of v in the half-circle $\{x_1 > 0, |x| = \rho\}$. Actually, the existence of such a limit point induces an infinitesimal variation of the total energy that makes it difficult to detect. For the same reason, the limit (1.15) in Theorem 1.2(iii) holds only for a.e. $x_2 \in (-\rho, \rho)$. We also point out that the assumption that f is radial, is essential to prove the existence of the constants a_* and a^* (cf. Lemma 3.2).

Before giving in the next sections the proofs of our results we explain the general strategy developed in this paper. The existence of a global minimizer $v_{\epsilon,a}$ for functional E is easily established by the direct method (cf. Lemma 2.1). To determine in Theorem 1.1, the singular limit of $v_{\epsilon,a}$ as $\epsilon \to 0$, we proceed as follows. We first rescale the minimizers according to the region we are studying. The appropriate rescaling is the one providing when $\epsilon \ll 1$, uniform bounds up to the second derivatives. Lemmas 2.2, 2.3 and 2.4 below, apply respectively to the regions where $\mu > 0$, $\mu \approx 0$, and $\mu < 0$, and we refer to the proof of Theorem 1.1(i)–(iii) to compare the corresponding rescalings. Next, in view of the theorem of Ascoli, we establish the convergence of the rescaled minimizers $\tilde{v}_{\epsilon,a}$ to a solution $\tilde{V}$ of the limiting equation. Since the convergence is in C^1_{loc}, the limit $\tilde{V}$ is minimal for perturbations with compact support. In the case where $\mu > 0$, the limiting equation is the Allen–Cahn PDE (1.12), and the limit $\tilde{V}$ is determined thanks to the result of Savin [19] mentioned previously. On the other hand, in the case where $\mu \approx 0$, the relevant limiting equation is the Painlevé PDE (1.7). Finally, in the region where $\mu < 0$, the limiting equation (3.10) is easy to study since it is associated to a convex potential.

The convergence of the set of zeros of $v_{\epsilon,a}$ is determined in view of the previous results and energy considerations. In Lemma 3.1, we prove that in the region where $\mu > 0$, the zero set is a smooth one-dimensional manifold. In addition, we establish estimates for the minimizer in a vicinity of this set. Next, in Lemma 3.2, we introduce the critical values a_* and a^* determining the two regimes described in Theorem 1.2(ii) and (iii). The proof of Theorem 1.2(ii) and (iii) follows from a computation of the contribution of each term appearing in the definition of the energy functional (1.3): cf. respectively Lemmas 3.5 and 3.6. These lemmas are obtained by first computing an upper bound of the energy in Lemma 3.3, and then in Lemma 3.4, a lower bound. While the upper bound follows only from the construction of two appropriate comparison functions, the computation of the lower bound is more involved, and is based on a reduction to the one-dimensional problem studied in [5].

2. General Results for Minimizers and Solutions

In this section, we gather general results for minimizers and solutions that are valid for any values of the parameters $\epsilon > 0$ and $a \geq 0$. We first prove the existence of global minimizers.

Lemma 2.1. *For every $\epsilon > 0$ and $a \geq 0$, there exists $v \in H^1(\mathbb{R}^2)$ such that $E(v) = \min_{H^1(\mathbb{R}^2)} E$. As a consequence, v is a C^∞ classical solution of* (1.5). *Moreover $v(x) \to 0$ as $|x| \to \infty$, and $v(x_1, x_2) = v(x_1, -x_2)$.*

Proof. We proceed as in [6, Lemma 2.1] to establish that the global minimizer exists and is a smooth solution of (1.5) converging to 0 as $|x| \to \infty$. It remains to show that $v(x_1, x_2) = v(x_1, -x_2)$. We first note that $E(v, \mathbb{R} \times [0, \infty)) = E(v, \mathbb{R} \times (-\infty, 0])$. Indeed, if we assume without loss of generality that $E(v, \mathbb{R} \times [0, \infty)) <$

$E(v, \mathbb{R} \times (-\infty, 0])$, the function

$$\tilde{v}(x_1, x_2) = \begin{cases} v(x_1, x_2) & \text{when } x_2 \geq 0, \\ v(x_1, -x_2) & \text{when } x_2 \leq 0, \end{cases} \tag{2.1}$$

has strictly less energy than v, which is a contradiction. Thus, $E(v, \mathbb{R} \times [0, \infty)) = E(v, \mathbb{R} \times (-\infty, 0])$, and as a consequence the function $\tilde{v}$ is also a global minimizer and a solution. It follows by unique continuation [18] that $\tilde{v} \equiv v$. $\square$

To study the limit of solutions as $\epsilon \to 0$, we need uniform bounds in the different regions considered in Theorem 1.1.

Lemma 2.2. *For ϵa belonging to a bounded interval, let $u_{\epsilon,a}$ be a solution of* (1.5) *converging to* 0 *as $|x| \to \infty$. Then, the solutions $u_{\epsilon,a}$ and the maps $\epsilon \nabla u_{\epsilon,a}$ are uniformly bounded.*

Proof. We drop the indexes and write $u := u_{\epsilon,a}$. Since $|f|$, μ, and ϵa are bounded, the roots of the cubic equation in the variable u

$$u^3 - \mu(x)u - \epsilon a f_1(x) = 0$$

belong to a bounded interval, for all values of x, ϵ, a. If u takes positive values, then it attains its maximum $0 \leq \max_{\mathbb{R}^2} u = u(x_0)$, at a point $x_0 \in \mathbb{R}^2$. In view of (1.5):

$$0 \geq \epsilon^2 \Delta u(x_0) = u^3(x_0) - \mu(x_0)u(x_0) - \epsilon a f_1(x_0),$$

thus it follows that $u(x_0)$ is uniformly bounded above. In the same way, we prove the uniform lower bound for u. The boundedness of $\epsilon \nabla u_{\epsilon,a}$ follows from (1.5), the uniform bound of $u_{\epsilon,a}$, and standard elliptic estimates. $\square$

Lemma 2.3. *For $\epsilon \ll 1$ and a belonging to a bounded interval, let $u_{\epsilon,a}$ be a solution of* (1.5) *converging to* 0 *as $|x| \to \infty$. Then, there exists a constant $K > 0$ such that*

$$|u_{\epsilon,a}(x)| \leq K(\sqrt{\max(\mu(x), 0)} + \epsilon^{1/3}), \quad \forall\, x \in \mathbb{R}^2. \tag{2.2}$$

As a consequence, if for every $\xi = \rho e^{i\theta}$ we consider the local coordinates $s = (s_1, s_2)$ in the basis $(e^{i\theta}, ie^{i\theta})$, then the rescaled functions $\tilde{u}_{\epsilon,a}(s) = \frac{u_{\epsilon,a}(\xi + s\epsilon^{2/3})}{\epsilon^{1/3}}$ are uniformly bounded on the half-planes $[s_0, \infty) \times \mathbb{R}$, $\forall\, s_0 \in \mathbb{R}$.

Proof. For the sake of simplicity we drop the indexes and write $u := u_{\epsilon,a}$. Let us define the following constants

- $M > 0$ is the uniform bound of $|u_{\epsilon,a}|$ (cf. Lemma 2.2),
- $\lambda > 0$ is such that $3\mu_{\text{rad}}(\rho - h) \leq 2\lambda h$, $\forall\, h \in [0, \rho]$,
- $F := \sup_{\mathbb{R}^2} |f_1|$,
- $\kappa > 0$ is such that $\kappa^3 \geq 3aF$, and $\kappa^4 \geq 6\lambda$.

Next, we construct the following comparison function

$$\chi(x) = \begin{cases} \lambda\Big(\rho - |x| + \dfrac{\epsilon^{2/3}}{2}\Big) & \text{for } |x| \le \rho, \\ \dfrac{\lambda}{2\epsilon^{2/3}}(|x| - \rho - \epsilon^{2/3})^2 & \text{for } \rho \le |x| \le \rho + \epsilon^{2/3}, \\ 0 & \text{for } |x| \ge \rho + \epsilon^{2/3}. \end{cases} \tag{2.3}$$

One can check that $\chi \in C^1(\mathbb{R}^2 \backslash \{0\}) \cap H^1(\mathbb{R}^2)$ satisfies $\Delta\chi \le \frac{2\lambda}{\epsilon^{2/3}}$ in $H^1(\mathbb{R}^2)$. Finally, we define the function $\psi := \frac{u^2}{2} - \chi - \kappa^2\epsilon^{2/3}$, and compute:

$$\begin{aligned} \epsilon^2\Delta\psi &= \epsilon^2(|\nabla u|^2 + u\Delta u - \Delta\chi) \\ &\ge -\mu u^2 + u^4 - \epsilon a f_1 u - \epsilon^2\Delta\chi \\ &\ge -\mu u^2 + u^4 - \epsilon a F|u| - 2\epsilon^{4/3}\lambda. \end{aligned} \tag{2.4}$$

Now, one can see that when $x \in \omega := \{x \in \mathbb{R}^2 : \psi(x) > 0\}$, we have $\frac{u^4}{3} - \mu u^2 \ge 0$, since

$$x \in \omega \cap \overline{D(0;\rho)} \Rightarrow \frac{u^4}{3} \ge \frac{2\lambda}{3}\Big(\rho - |x| + \frac{\epsilon^{2/3}}{2}\Big)u^2 \ge \mu u^2.$$

On the open set ω, we also have: $\frac{u^4}{3} \ge \frac{\kappa^4}{3}\epsilon^{4/3} \ge 2\epsilon^{4/3}\lambda$, and $\frac{u^4}{3} \ge \frac{\kappa^3}{3}\epsilon|u| \ge \epsilon aF|u|$. Thus $\Delta\psi \ge 0$ on ω in the H^1 sense. To conclude, we apply Kato's inequality that gives: $\Delta\psi^+ \ge 0$ on $\mathbb{R}^2$ in the H^1 sense. Since ψ^+ is subharmonic with compact support, we obtain by the maximum principle that $\psi^+ \equiv 0$ or equivalently $\psi \le 0$ on $\mathbb{R}^2$. The statement of the lemma follows by adjusting the constant K. □

Lemma 2.4. *Assume that a is bounded and let $u_{\epsilon,a}$ be solutions of* (1.5) *uniformly bounded. Then, the functions $\frac{u_{\epsilon,a}}{\epsilon}$ and the maps $\nabla u_{\epsilon,a}$ are uniformly bounded on the sets $\{x : |x| \ge \rho_1\}$ for every $\rho_1 > \rho$, provided that $\epsilon \ll 1$.*

Proof. We consider the sets $S := \{x : \ |x| \ge \rho_1\} \subset S' := \{x : \ |x| > \rho_1'\}$, with $\rho < \rho_1' < \rho_1$, and define the constants:

- $M > 0$ which is the uniform bound of $|u_{\epsilon,a}|$,
- $\mu_0 = -\mu_{\mathrm{rad}}(\rho_1') > 0$,
- $f_\infty = \|f_1\|_{L^\infty}$,
- $a^* := \sup a(\epsilon)$,
- $k = \frac{2a_* f_\infty}{\mu_0} > 0$.

Next we introduce the function $\psi(x) = \frac{1}{2}(u^2 - k^2\epsilon^2)$ satisfying:

$$\begin{aligned} \epsilon^2\Delta\psi = \epsilon^2\Delta\frac{u^2}{2} &\ge u^4 + \mu_0 u^2 - \epsilon a_* f_\infty |u|, \quad \forall\, x \in S', \\ &\ge \mu_0\psi, \quad \forall\, x \in S' \text{ such that } \psi(x) \ge 0. \end{aligned}$$

By Kato's inequality we have $\epsilon^2\Delta\psi^+ \ge \mu_0\psi^+$ on S', in the H^1 sense, and utilizing a standard comparison argument, we deduce that $\psi^+(x) \le M^2 e^{-\frac{c}{\epsilon}d(x,\partial S')}$, $\forall\, x \in S$, and $\forall\, \epsilon \ll 1$, where d stands for the Euclidean distance, and $c > 0$ is a constant. It is clear that

$$d(x, \partial S') > -\frac{\epsilon}{c} \ln\left(\frac{k^2\epsilon^2}{2M^2}\right) \Rightarrow M^2 e^{-\frac{c}{\epsilon} d(x,\partial S')} < \frac{k^2\epsilon^2}{2} \Rightarrow u^2 < 2k^2\epsilon^2.$$

Therefore, there exists ϵ_0 such that

$$\frac{|u_{\epsilon,a}(x)|}{\epsilon} \leq \sqrt{2}k, \quad \forall\, \epsilon < \epsilon_0, \ \ \forall\, x \in S. \tag{2.5}$$

The boundedness of $\nabla u_{\epsilon,a}$ follows from (1.5), the uniform bound (2.5), and standard elliptic estimates. □

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1(i). Without loss of generality we assume that $v_{\epsilon,a} > 0$ on Ω. Suppose by contradiction that v does not converge uniformly to $\sqrt{\mu}$ on a closed set $F \subset \Omega$. Then there exist a sequence $\epsilon_n \to 0$ and a sequence $\{x_n\} \subset F$ such that

$$|v_{\epsilon_n}(x_n) - \sqrt{\mu(x_n)}| \geq \delta, \quad \text{for some } \delta > 0. \tag{3.1}$$

In addition, we may assume that up to a subsequence $\lim_{n\to\infty} x_n = x_0 \in F$. Next, we consider the rescaled functions $\tilde{v}_n(s) = v_{\epsilon_n}(x_n + \epsilon_n s)$ that satisfy

$$\Delta\tilde{v}_n(s) + \mu(x_n + \epsilon_n s)\tilde{v}_n(s) - \tilde{v}_n^3(s) + \epsilon_n a f_1(x_n + \epsilon_n s) = 0, \quad \forall\, s \in \mathbb{R}^2. \tag{3.2}$$

In view of Lemma 2.2 and (3.2), $\tilde{v}_n$ and its first derivatives are uniformly bounded for $\epsilon \ll 1$. Moreover, by differentiating (3.2), one also obtains the boundedness of the second derivatives of $\tilde{v}_n$ on compact sets. Thus, we can apply the theorem of Ascoli via a diagonal argument, and show that for a subsequence still called $\tilde{v}_n$, $\tilde{v}_n$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to a function $\tilde{V}$, that we are now going to determine. For this purpose, we introduce the rescaled energy

$$\begin{aligned}\tilde{E}(\tilde{u}) &= \int_{\mathbb{R}^2} \left(\frac{1}{2}|\nabla\tilde{u}(s)|^2 - \frac{1}{2}\mu(x_n + \epsilon_n s)\tilde{u}^2(s) + \frac{1}{4}\tilde{u}^4(s) - \epsilon_n a f_1(x_n + \epsilon_n s)\tilde{u}(s)\right) \mathrm{d}s \\ &= \frac{1}{\epsilon_n} E(u),\end{aligned}$$

where we have set $\tilde{u}(s) = u_{\epsilon_n}(x_n + \epsilon_n s)$ i.e. $u_{\epsilon_n}(x) = \tilde{u}(\frac{x - x_n}{\epsilon_n})$. Let $\tilde{\xi}$ be a test function with support in the compact set K. We have $\tilde{E}(\tilde{v}_n + \tilde{\xi}, K) \geq \tilde{E}(\tilde{v}_n, K)$, and at the limit $G_0(\tilde{V} + \tilde{\xi}, K) \geq G_0(\tilde{V}, K)$, where

$$G_0(\psi, K) = \int_K \left[\frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\mu(x_0)\psi^2 + \frac{1}{4}\psi^4\right],$$

or equivalently $G(\tilde{V}+\tilde{\xi},K)\geq G(\tilde{V},K)$, where

$$G(\psi,K)=\int_K\left[\frac{1}{2}|\nabla\psi|^2-\frac{1}{2}\mu(x_0)\psi^2+\frac{1}{4}\psi^4+\frac{(\mu(x_0))^2}{4}\right]$$
$$=\int_K\left[\frac{1}{2}|\nabla\psi|^2+\frac{1}{4}(\psi^2-\mu(x_0))^2\right]. \tag{3.3}$$

Thus, we deduce that $\tilde{V}$ is a bounded minimal solution of the PDE associated to functional (3.3):

$$\Delta\tilde{V}(s)+(\mu(x_0)-\tilde{V}^2(s))\tilde{V}(s)=0. \tag{3.4}$$

If $\tilde{V}$ is the constant solution $\sqrt{\mu(x_0)}$, then we have $\lim_{n\to\infty}v_{\epsilon_n}(x_n)=\sqrt{\mu(x_0)}$ which is excluded by (3.1). Therefore we obtain $\tilde{V}(s)=\sqrt{\mu(x_0)}\tanh(\sqrt{\mu(x_0)/2}(s-s_0)\cdot\nu)$, for some unit vector $\nu\in\mathbb{R}^2$, and some $s_0\in\mathbb{R}^2$. This implies that v_n takes negative values in the open disc $D(x_n;2\epsilon_n|s_0|)$ for $\epsilon_n\ll 1$, which contradicts the fact that $v_\epsilon>0$ on Ω for $\epsilon\ll 1$. □

Proof of Theorem 1.1(ii). For every $\xi=\rho e^{i\theta}$ we consider the local coordinates $s=(s_1,s_2)$ in the basis $(e^{i\theta},ie^{i\theta})$, and we rescale the global minimizer v by setting $\tilde{v}_{\epsilon,a}(s)=\frac{v_{\epsilon,a}(\xi+s\epsilon^{2/3})}{\epsilon^{1/3}}$. Clearly $\Delta\tilde{v}(s)=\epsilon\Delta v(\xi+s\epsilon^{2/3})$, thus,

$$\Delta\tilde{v}(s)+\frac{\mu(\xi+s\epsilon^{2/3})}{\epsilon^{2/3}}\tilde{v}(s)-\tilde{v}^3(s)+af_1(\xi+s\epsilon^{2/3})=0,\quad\forall\, s\in\mathbb{R}^2.$$

Writing $\mu(\xi+h)=\mu_1h_1+h\cdot A(h)$, with $\mu_1:=\mu'_{\mathrm{rad}}(\rho)<0$, $A\in C^\infty(\mathbb{R}^2,\mathbb{R}^2)$, and $A(0)=0$, we obtain

$$\Delta\tilde{v}(s)+(\mu_1s_1+A(s\epsilon^{2/3})\cdot s)\tilde{v}(s)-\tilde{v}^3(s)+af_1(\xi+s\epsilon^{2/3})=0,\quad\forall\, s\in\mathbb{R}^2. \tag{3.5}$$

Next, we define the rescaled energy by

$$\tilde{E}(\tilde{u})=\int_{\mathbb{R}^2}\left(\frac{1}{2}|\nabla\tilde{u}(s)|^2-\frac{\mu(\xi+s\epsilon^{2/3})}{2\epsilon^{2/3}}\tilde{u}^2(s)+\frac{1}{4}\tilde{u}^4(s)-af_1(\xi+s\epsilon^{2/3})\tilde{u}(s)\right)\mathrm{d}s. \tag{3.6}$$

With this definition $\tilde{E}(\tilde{u})=\frac{1}{\epsilon^{5/3}}E(u)$. From Lemma 2.3 and (3.5), it follows that $\Delta\tilde{v}$, and also $\nabla\tilde{v}$, are uniformly bounded on compact sets. Moreover, by differentiating (3.5) we also obtain the boundedness of the second derivatives of $\tilde{v}$. Thanks to these uniform bounds, we can apply the theorem of Ascoli via a diagonal argument to obtain the convergence of $\tilde{v}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ (up to a subsequence) to a solution $\tilde{V}$ of the PDE.

$$\Delta\tilde{V}(s)+\mu_1 s_1\tilde{V}(s)-\tilde{V}^3(s)+a_0 f_1(\xi)=0,\quad \forall\, s\in\mathbb{R}^2,\ \text{with } a_0:=\lim_{\epsilon\to 0}a(\epsilon),\tag{3.7}$$

which is associated to the functional

$$\tilde{E}_0(\phi,J)=\int_J\left(\frac{1}{2}|\nabla\phi(s)|^2-\frac{\mu_1}{2}s_1\phi^2(s)+\frac{1}{4}\phi^4(s)-a_0 f_1(\xi)\phi(s)\right)\mathrm{d}s.\tag{3.8}$$

Setting $y(s):=\frac{1}{\sqrt{2}(-\mu_1)^{1/3}}\tilde{V}(\frac{s}{(-\mu_1)^{1/3}})$, (3.7) reduces to (1.7), that is, y solves (1.7) with $\alpha=\frac{a_0 f_1(\xi)}{\sqrt{2}\mu_1}$. Finally, we can see as in the previous proof that the limit $\tilde{V}$ obtained in (3.7), as well as the solution y of (1.7), is minimal in the sense of definition (1.9). □

Proof of Theorem 1.1(iii). For every $x_0\in\mathbb{R}^2$ such that $|x_0|>\rho$, we consider the rescaled minimizers $\tilde{v}_{\epsilon,a}(s)=\frac{v_{\epsilon,a}(x_0+\epsilon s)}{\epsilon}$, with $s=(s_1,s_2)$, satisfying

$$\Delta\tilde{v}(s)+\mu(x_0+\epsilon s)\tilde{v}(s)-\epsilon^2\tilde{v}(s)^3+af_1(x_0+\epsilon s)=0,\quad\forall\, s\in\mathbb{R}^2.\tag{3.9}$$

In view of the bound provided by Lemma 2.4 and (3.9), we can see that the first derivatives of $\tilde{v}_{\epsilon,a}$ are uniformly bounded on compact sets for $\epsilon\ll 1$. Moreover, by differentiating (3.9), one can also obtain the boundedness of the second derivatives of $\tilde{v}$ on compact sets. As a consequence, we conclude that $\lim_{\epsilon\to 0,a\to a_0}\tilde{v}_{\epsilon,a}(s)=\tilde{V}(s)$ in C^2_{loc}, where $\tilde{V}(s)\equiv-\frac{a_0}{\mu(x_0)}f_1(x_0)$ is the unique bounded solution of

$$\Delta\tilde{V}(s)+\mu(x_0)\tilde{V}(s)+a_0 f_1(x_0)=0,\quad\forall\, s\in\mathbb{R}^2.\tag{3.10}$$

Indeed, consider a smooth and bounded solution $\phi:\mathbb{R}^2\to\mathbb{R}$ of $\Delta\phi=W'(\phi)$ where the potential $W:\mathbb{R}\to\mathbb{R}$ is smooth and strictly convex. Then, we have $\Delta(W(\phi))=|W'(\phi)|^2+W''(\phi)|\nabla\phi|^2\geq 0$, and since $W(\phi)$ is bounded we deduce that $W(\phi)$ is constant. Therefore, $\phi\equiv\phi_0$ where $\phi_0\in\mathbb{R}$ is such that $W'(\phi_0)=0$. To prove the uniform convergence $\frac{v_{\epsilon,a}(x)}{\epsilon}\to-\frac{a_0}{\mu(x)}f_1(x)$ on compact subsets of $\{|x|>\rho\}$, we proceed by contradiction. Assuming that the uniform convergence does not hold, one can find a sequence $\epsilon_n\to 0$, a sequence $a_n\to a_0$, and a sequence $x_n\to x_0$, with $|x_0|>\rho$, such that $\left|\frac{v_{\epsilon_n,a_n}(x_n)}{\epsilon_n}+\frac{a_0}{\mu(x_n)}f_1(x_n)\right|\geq\delta$, for some $\delta>0$. However, by reproducing the previous arguments, it follows that the rescaled functions $\tilde{v}_n(s)=\frac{v_{\epsilon_n,a_n}(x_n+\epsilon_n s)}{\epsilon_n}$ converge in C^2_{loc} to the constant $\tilde{V}(s)\equiv-\frac{a_0}{\mu(x_0)}f_1(x_0)$. Thus, we have reached a contradiction. □

Proof of Theorem 1.2(i). We first notice that $v\not\equiv 0$ for $\epsilon\ll 1$. Indeed, by choosing a test function $\psi\not\equiv 0$ supported in $D(0;\rho)$, and such that $\psi^2<2\mu$, one can see that

$$E(\psi)=\frac{\epsilon}{2}\int_{\mathbb{R}^2}|\nabla\psi|^2+\frac{1}{4\epsilon}\int_{\mathbb{R}^2}\psi^2(\psi^2-2\mu)<0,\quad\epsilon\ll 1.$$

Let $x_0\in\mathbb{R}^2$ be such that $v(x_0)\neq 0$. Without loss of generality we may assume that $v(x_0)>0$. Next, consider $\tilde{v}=|v|$ which is another global minimizer and thus another solution. Clearly, in a neighborhood of x_0 we have $v = |v|$, and as a consequence of the unique continuation principle (cf. [18]) we deduce that $v \equiv \tilde{v} \geq 0$ on $\mathbb{R}^2$. Furthermore, the maximum principle implies that $v > 0$, since $v \not\equiv 0$. To prove that v is radial we consider the reflection with respect to the line $x_1 = 0$. We can check that $E(v, \{x_1 > 0\}) = E(v, \{x_1 < 0\})$, since otherwise by even reflection we can construct a map in H^1 with energy smaller than v. Thus, the map $\tilde{v}(x) = v(|x_1|, x_2)$ is also a minimizer, and since $\tilde{v} = v$ on $\{x_1 > 0\}$, it follows by unique continuation that $\tilde{v} \equiv v$ on $\mathbb{R}^2$. Repeating the same argument for any line of reflection, we deduce that v is radial. To complete the proof, it remains to show the uniqueness of v up to change of v by $-v$. Let $\tilde{v}$ be another global minimizer such that $\tilde{v} > 0$, and $\tilde{v} \not\equiv v$. Choosing $\psi = u$ in (1.6), we find for any solution $u \in H^1(\mathbb{R}^2)$ of (1.5) the following alternative expression of the energy:

$$E(u) = -\int_{\mathbb{R}^2} \frac{u^4}{4\epsilon}. \tag{3.11}$$

Formula (3.11) implies that v and $\tilde{v}$ intersect for $|x| = r > 0$. However, setting

$$w(x) = \begin{cases} v(x) & \text{for } |x| \leq r, \\ \tilde{v}(x) & \text{for } |x| \geq r, \end{cases}$$

we can see that w is another global minimizer, and again by the unique continuation principle we have $w \equiv v \equiv \tilde{v}$. $\square$

Proof of Theorem 1.2(ii), (iii). We first establish two lemmas.

Lemma 3.1. *Let $a > 0$ and $\rho_0 \in (0, \rho)$ be fixed, and set $l := \frac{\sqrt{\mu_{\mathrm{rad}}(\rho_0)}}{2\mu(0)}$, $\lambda := \frac{\sqrt{2}\tanh^{-1}(8/9)}{\sqrt{\mu_{\mathrm{rad}}(\rho_0)}}$, and $\lambda' := \frac{\mu_{\mathrm{rad}}(\rho_0)}{2\cosh^2(\lambda\sqrt{\mu(0)/2})}$. Then, there exist $\epsilon_0 > 0$ such that*

(i) *for every $\epsilon \in (0, \epsilon_0)$ the set $Z_\epsilon := \{\bar{x} \in D(0; \rho_0) : v_{\epsilon,a}(\bar{x}) = 0\}$ is a smooth one-dimensional manifold. Let $\nu(\bar{x})$ be a unit normal vector at $\bar{x} \in Z_\epsilon$.*
(ii) *for every $\epsilon \in (0, \epsilon_0)$, $\bar{x} \in Z_\epsilon$, and $|s| \leq l$, we have $|v(\bar{x} + \epsilon s)| \leq \frac{1}{2}\sqrt{\mu_{\mathrm{rad}}(\rho_0)}$.*
(iii) *for every $\epsilon \in (0, \epsilon_0)$, and $\bar{x} \in Z_\epsilon$, we have $|v(\bar{x} + \epsilon\lambda\nu)| \geq \frac{3}{4}\sqrt{\mu_{\mathrm{rad}}(\rho_0)}$,*
(iv) *for every $\epsilon \in (0, \epsilon_0)$, $\bar{x} \in Z_\epsilon$, and $t \in [-\lambda, \lambda]$ we have $\epsilon\left|\frac{\partial v}{\partial \nu}(\bar{x} + \epsilon t\nu)\right| \geq \lambda'$.*

Proof. To prove (i) it is sufficient to establish that there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ and $\bar{x} \in Z_\epsilon$, we have $\nabla v_{\epsilon,a}(\bar{x}) \neq 0$. Assuming by contradiction that this does not hold, we can find a sequence $\epsilon_n \to 0$, and a sequence $Z_{\epsilon_n} \ni \bar{x}_n \to x_0 \in \overline{D(0; \rho_0)}$ such that $\nabla v_{\epsilon_n,a}(\bar{x}_n) = 0$. However, by considering the rescaled functions $\tilde{v}_n(s) = v_{\epsilon_n,a}(\bar{x}_n + \epsilon_n s)$, it follows as in the proof of Theorem 1.1(i) that $\tilde{v}_n$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ (up to a subsequence) to $\tilde{V}(s) = \sqrt{\mu(x_0)}\tanh(\sqrt{\mu(x_0)/2}(s \cdot \nu))$, where $\nu \in \mathbb{R}^2$ is a unit vector. Since $\nabla\tilde{V}(0) \neq 0$, we have reached a contradiction. To prove (ii), we proceed again by contradiction, and assume that we can find a sequence $\epsilon_n \to 0$, a sequence $Z_{\epsilon_n} \ni \bar{x}_n \to x_0 \in \overline{D(0; \rho_0)}$, and a sequence $\overline{D(0; l)} \ni s_n \to s_0$ such that $|v(\bar{x}_n + \epsilon_n s_n)| > \sqrt{\mu_{\mathrm{rad}}(\rho_0)}/2$. As before, we obtain that

$\tilde{v}_n(s) = v_{\epsilon_n,a}(\bar{x}_n + \epsilon_n s)$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to $\tilde{V}(s) = \sqrt{\mu(x_0)}\tanh(\sqrt{\mu(x_0)/2}(s \cdot \nu))$. In particular, it follows that

$$\lim_{n\to\infty} |v_{\epsilon_n,a}(\bar{x}_n + \epsilon_n s_n)| = \sqrt{\mu(x_0)}|\tanh\left(\sqrt{\frac{\mu(x_0)}{2}}(s_0 \cdot \nu)\right)| \leq \frac{\mu(0)l}{\sqrt{2}} < \frac{\sqrt{\mu_{\mathrm{rad}}(\rho_0)}}{2},$$

which is a contradiction. The proofs of (iii) and (iv) are similar. □

Lemma 3.2. *Let*

$$a_* := \inf_{x_1 \leq 0, |x| < \rho} \frac{\sqrt{2}(\mu(x))^{3/2}}{3\displaystyle\int_{-\sqrt{\rho^2 - x_2^2}}^{x_1} |f_1(t, x_2)|\sqrt{\mu(t, x_2)}\mathrm{d}t}, \tag{3.12}$$

and

$$a^* := \sup_{x_1 < 0, |x| \leq \rho} \frac{\sqrt{2}((\mu(0, x_2))^{3/2} - (\mu(x))^{3/2})}{3\displaystyle\int_{x_1}^{0} |f_1(t, x_2)|\sqrt{\mu(t, x_2)}\mathrm{d}t}, \tag{3.13}$$

then we have $a_* \in (0, \infty)$ and

$$a_* \leq \frac{2\sqrt{2}\displaystyle\int_{|r|<\rho} (\mu_{\mathrm{rad}}(r))^{3/2}\mathrm{d}r}{3\displaystyle\int_{D(0;\rho)} |f_1|\sqrt{\mu}} \leq a^*. \tag{3.14}$$

Moreover, if $f'_{\mathrm{rad}}(0) > 0$, then $a^* < \infty$. Finally, if $f = -\frac{1}{2}\nabla\mu$, then $a_* = a^* = \sqrt{2}$.

Proof. We first check that $a_* \in (0, \infty)$ and $a^* \in [a_*, \infty]$. Let us define the auxiliary function

$$\begin{aligned} \{x \in \mathbb{R}^2 : x_1 \leq 0, |x| \leq \rho\} &\ni x \to \beta_*(x) \\ &= \frac{\sqrt{2}}{3}(\mu(x))^{3/2} - a\int_{-\sqrt{\rho^2 - x_2^2}}^{x_1} |f_1(t, x_2)|\sqrt{\mu(t, x_2)}\mathrm{d}t, \end{aligned}$$

and compute $\frac{\partial\beta_*}{\partial x_1}(x) = (\frac{\sqrt{2}}{2}\mu'_{\mathrm{rad}}(r) - af_{\mathrm{rad}}(r))\sqrt{\mu(x)}\cos\theta$, where $x = (r\cos\theta, r\sin\theta)$. It is clear that for sufficiently small $a_1 > 0$ and $\gamma > 0$, we have $\frac{\partial\beta_*}{\partial x_1}(x) > 0$ provided that $x_1 < 0$, $\rho - \gamma < |x| < \rho$, and $a \leq a_1$. Since $\beta_*(x) = 0$ for $|x| = \rho$, it follows that $\beta_*(x) \geq 0$ provided that $x_1 \leq 0$, $\rho - \gamma \leq |x| \leq \rho$, and $a \leq a_1$. There also exists $a_2 > 0$ such that for $a \leq a_2$, we have $\beta_* \geq 0$ on the set $\{x_1 \leq 0, |x| \leq \rho - \gamma\}$. Thus, we can see that $a_* \geq \min(a_1, a_2) > 0$. Furthermore, since the inequalities $a_* \leq \frac{2\sqrt{2}(\mu(0,x_2))^{3/2}}{3\int_{|t|<\sqrt{\rho^2-x_2^2}} |f_1(t,x_2)|\sqrt{\mu(t,x_2)}\mathrm{d}t} \leq a^*$ hold for every $x_2 \in (-\rho, \rho)$, we obtain after an integration (3.14). Next, we define a second auxiliary function

$$\begin{aligned}\{x\in\mathbb{R}^2: x_1\le 0, |x|\le\rho\}\ni x\to\beta^*(x)\\ =\frac{\sqrt{2}}{3}[(\mu(0,x_2))^{3/2}-(\mu(x))^{3/2}]-a\int_{x_1}^{0}|f_1(t,x_2)|\sqrt{\mu(t,x_2)}\mathrm{d}t,\end{aligned}$$

and compute $\frac{\partial\beta^*}{\partial x_1}(x)=\left(\frac{\sqrt{2}}{2}\mu'_{\mathrm{rad}}(r)+af_{\mathrm{rad}}(r)\right)\sqrt{\mu(x)}|\cos\theta|$, where $x=(r\cos\theta, r\sin\theta)$. Since $f'_{\mathrm{rad}}(0)>0$, one can see that $\frac{\sqrt{2}}{2}\mu''_{\mathrm{rad}}(r)+af'_{\mathrm{rad}}(r)>0$, provided that $r\in[0,\gamma]$ and $a\ge a_3$, with $\gamma>0$ sufficiently small, and $a_3>0$ sufficiently big. Thus, $\frac{\sqrt{2}}{2}\mu'_{\mathrm{rad}}(r)+af_{\mathrm{rad}}(r)\ge 0$, and $\frac{\partial\beta^*}{\partial x_1}(x)>0$, when $r=|x|\le\gamma$, $x_1<0$, and $a\ge a_3$. On the other hand it is clear that for sufficiently big $a_4>0$, we have $\frac{\partial\beta^*}{\partial x_1}(x)>0$ provided that $x_1<0$, $\gamma\le|x|<\rho$, and $a\ge a_4$. Since $\beta^*(x)=0$ for $x_1=0$, it follows that $\beta^*(x)\le 0$ provided that $x_1\le 0$, $|x|\le\rho$, and $a\ge\max(a_3,a_4)$. This proves that $a^*\le\max(a_3,a_4)$. Finally, one can check that $a_*=a^*=\sqrt{2}$ when $f=-\frac{1}{2}\nabla\mu\Rightarrow f_1=-\frac{1}{2}\frac{\partial\mu}{\partial x_1}$, by computing the integrals appearing in the denominators of (3.12), (3.13). □

The minimum of the energy defined in (1.3) is nonpositive and tends to $-\infty$ as $\epsilon\to 0$. Since we are interested in the behavior of the minimizers as $\epsilon\to 0$, it is useful to define a renormalized energy, which is obtained by adding to (1.3) a suitable term so that the result is tightly bounded from above. We define the renormalized energy as

$$\begin{aligned}\mathcal{E}(u):=E(u)+\int_{|x|<\rho}\frac{\mu^2}{4\epsilon}=\int_{\mathbb{R}^2}\frac{\epsilon}{2}|\nabla u|^2+\int_{|x|<\rho}\frac{(u^2-\mu)^2}{4\epsilon}\\ +\int_{|x|>\rho}\frac{u^2(u^2-2\mu)}{4\epsilon}-a\int_{\mathbb{R}^2}f_1u, \qquad (3.15)\end{aligned}$$

and claim the bound:

Lemma 3.3.

$$\limsup_{\epsilon\to 0}\mathcal{E}_{\epsilon,a}(v_{\epsilon,a})\le\min\left(0,\frac{2\sqrt{2}}{3}\int_{-\rho}^{\rho}(\mu_{\mathrm{rad}}(r))^{3/2}\mathrm{d}r-a\int_{D(0;\rho)}|f_1|\sqrt{\mu}\right),$$

for arbitrary fixed a.

(3.16)

Proof. Let us consider the C^1 piecewise function:

$$\psi_\epsilon(x)=\begin{cases}\sqrt{\mu(x)} & \text{for } |x|\le\rho-\epsilon^{2/3},\\ k_\epsilon\epsilon^{-1/3}(\rho-|x|) & \text{for } \rho-\epsilon^{2/3}\le|x|\le\rho,\\ 0 & \text{for } |x|\ge\rho\end{cases}$$

with k_ϵ defined by $k_\epsilon\epsilon^{1/3}=\sqrt{\mu_{\mathrm{rad}}(\rho-\epsilon^{2/3})}\Rightarrow k_\epsilon=|\mu'_{\mathrm{rad}}(\rho)|^{\frac{1}{2}}+o(1)$. Since $\psi\in H^1(\mathbb{R}^2)$, it is clear that $\mathcal{E}(v)\le\mathcal{E}(\psi)$. We check that $\mathcal{E}(\psi)=\frac{\pi|\mu_1|\rho}{6}|\epsilon\ln\epsilon|+\mathcal{O}(\epsilon)$, since it is the sum of the following integrals:

$$\int_{\rho-\epsilon^{2/3}<|x|<\rho} \frac{(\psi^2-\mu)^2}{4\epsilon} = \mathcal{O}(\epsilon), \quad \int_{|x|>\rho-\epsilon^{2/3}} \frac{\epsilon}{2}|\nabla\psi|^2 = \mathcal{O}(\epsilon),$$

$$\int_{|x|\leq\rho-\epsilon^{2/3}} \frac{\epsilon}{2}\frac{|\mu'_{\mathrm{rad}}(|x|)|^2}{4\mu} = \frac{\epsilon|\mu_1|}{8}\int_{|x|\leq\rho-\epsilon^{2/3}} \frac{1}{\rho-|x|} + \mathcal{O}(\epsilon) = \frac{\pi|\mu_1|\rho}{6}|\epsilon\ln\epsilon| + \mathcal{O}(\epsilon).$$

Thus, $\limsup_{\epsilon\to 0}\mathcal{E}_{\epsilon,a}(v_{\epsilon,a}) \leq \limsup_{\epsilon\to 0}\mathcal{E}_{\epsilon,a}(\psi_\epsilon) = 0$.

Next, we set $\zeta_\epsilon := \epsilon^{-\beta}$, with $\beta\in(\frac{1}{3},\frac{4}{9})$, and define the C^1 piecewise functions:

$$l_\epsilon(x_2) = \begin{cases} \dfrac{\psi_\epsilon(\epsilon\zeta_\epsilon, x_2)}{\tanh\left(\zeta_\epsilon\dfrac{\psi_\epsilon(0,x_2)}{\sqrt{2}}\right)} & \text{for } |x_2| \leq (\rho^2-\epsilon^2\zeta_\epsilon^2)^{1/2}, \\ 0 & \text{for } |x_2| \geq (\rho^2-\epsilon^2\zeta_\epsilon^2)^{1/2} \end{cases}$$

and

$$\chi_\epsilon(x) = \begin{cases} l_\epsilon(x_2)\tanh\left(\dfrac{x_1\psi_\epsilon(0,x_2)}{\sqrt{2}\epsilon}\right) & \text{for } |x_1|\leq\epsilon\zeta_\epsilon, \\ \psi_\epsilon(x) & \text{for } x_1\geq\epsilon\zeta_\epsilon, \\ -\psi_\epsilon(x) & \text{for } x_1\leq-\epsilon\zeta_\epsilon. \end{cases}$$

We also consider the sets

$$D^1_\epsilon := \{(x_1,x_2) : |x_1|\leq\epsilon\zeta_\epsilon, |x_2|\leq((\rho-\epsilon^{2/3})^2-\epsilon^2\zeta_\epsilon^2)^{1/2}\},$$
$$D^2_\epsilon := \{(x_1,x_2) : |x_1|\leq\epsilon\zeta_\epsilon, |x_2|\geq((\rho-\epsilon^{2/3})^2-\epsilon^2\zeta_\epsilon^2)^{1/2}, |x|\leq\rho\},$$

and

$$D^3_\epsilon := \{(x_1,x_2) : |x_1|\geq\epsilon\zeta_\epsilon, |x|\leq\rho\}.$$

One the one hand, it is clear that

$$\lim_{\epsilon\to 0} -a\int_{\mathbb{R}^2} f_1\chi_\epsilon = -a\int_{D(0;\rho)} |f_1|\sqrt{\mu},$$

and

$$\lim_{\epsilon\to 0}\int_{D^3_\epsilon}\left(\frac{\epsilon}{2}|\nabla\chi_\epsilon|^2 + \frac{(\chi_\epsilon^2-\mu)^2}{4\epsilon}\right) = 0.$$

In addition, it is a simple calculation to verify that

$$\lim_{\epsilon\to 0}\int_{D^2_\epsilon}\left(\frac{\epsilon}{2}|\nabla\chi_\epsilon|^2 + \frac{(\chi_\epsilon^2-\mu)^2}{4\epsilon}\right) = 0,$$

since the Lebesgue measure of D^2_ϵ is of order $\mathcal{O}(\zeta_\epsilon\epsilon^{5/3})$, while $|\nabla\chi_\epsilon| = \mathcal{O}(\zeta_\epsilon\epsilon^{-2/3})$ on D^2_ϵ. On the other hand when $|x_2| \leq ((\rho-\epsilon^{2/3})^2-\epsilon^2\zeta_\epsilon^2)^{1/2} =: \tau_\epsilon$, we have $l^2_\epsilon(x_2) = \mu(0,x_2)+\mathcal{O}(\epsilon^2\zeta_\epsilon^2)$, uniformly in x_2. Our claim is that

$$\lim_{\epsilon\to 0}\int_{D^3_\epsilon}\left(\frac{\epsilon}{2}|\nabla\chi_\epsilon|^2 + \frac{(\chi_\epsilon^2-\mu)^2}{4\epsilon}\right) = \frac{2\sqrt{2}}{3}\int_{-\rho}^{\rho}(\mu(0,x_2))^{3/2}\mathrm{d}x_2. \tag{3.17}$$

Indeed, setting $\tilde{\chi}(x_1,x_2) = \sqrt{\mu(0,x_2)}\tanh(x_1\sqrt{\frac{\mu(0,x_2)}{2}})$, we can see that $\int_{D^3_\epsilon}(\frac{\epsilon}{2}|\nabla\chi_\epsilon|^2 + \frac{(\chi_\epsilon^2-\mu)^2}{4\epsilon})$ is the sum of the following integrals:

$$\int_{D^1_\epsilon}\frac{\mu^2}{4\epsilon} = \int_{D^\epsilon_1}\frac{\mu^2(0,x_2)}{4\epsilon} + \mathcal{O}(\zeta_\epsilon^2\epsilon),$$

$$\begin{aligned}\int_{D^1_\epsilon}\frac{\epsilon}{2}\left|\frac{\partial\chi_\epsilon}{\partial x_1}\right|^2 &= \int_{|x_1|<\zeta_\epsilon,|x_2|<\tau_\epsilon}\frac{1}{2}\frac{l_\epsilon^2(x_2)}{\mu(0,x_2)}\left|\frac{\partial\tilde{\chi}}{\partial x_1}\right|^2\\ &= \int_{|x_1|<\zeta_\epsilon,|x_2|<\tau_\epsilon}\frac{1}{2}\left|\frac{\partial\tilde{\chi}}{\partial x_1}\right|^2 + \mathcal{O}(\epsilon^{4/3}\zeta_\epsilon^2),\end{aligned}$$

$$\int_{D^1_\epsilon}\frac{\epsilon}{2}\left|\frac{\partial\chi_\epsilon}{\partial x_2}\right|^2 = \mathcal{O}(\epsilon^{4/3}\zeta_\epsilon^3),$$

$$\begin{aligned}-\int_{D^1_\epsilon}\frac{\mu}{2\epsilon}\chi_\epsilon^2 &= -\int_{D^1_\epsilon}\frac{\mu(0,x_2)}{2\epsilon}\chi_\epsilon^2 + \mathcal{O}(\epsilon\zeta_\epsilon^2)\\ &= -\int_{|x_1|<\zeta_\epsilon,|x_2|<\tau_\epsilon}\frac{l_\epsilon^2(x_2)}{\mu(0,x_2)}\frac{\mu(0,x_2)\tilde{\chi}^2}{2} + \mathcal{O}(\epsilon\zeta_\epsilon^2)\\ &= -\int_{|x_1|<\zeta_\epsilon,|x_2|<\tau_\epsilon}\frac{\mu(0,x_2)\tilde{\chi}^2}{2} + \mathcal{O}(\epsilon^{4/3}\zeta_\epsilon^3),\end{aligned}$$

$$\int_{D^1_\epsilon}\frac{\chi_\epsilon^4}{4\epsilon} = \int_{|x_1|<\zeta_\epsilon,|x_2|<\tau_\epsilon}\frac{l_\epsilon^4(x_2)}{(\mu(0,x_2))^2}\frac{\tilde{\chi}^4}{4} = \int_{|x_1|<\zeta_\epsilon,|x_2|<\tau_\epsilon}\frac{\tilde{\chi}^4}{4} + \mathcal{O}(\epsilon^{4/3}\zeta_\epsilon^3).$$

Gathering the previous results, it follows that

$$\begin{aligned}\lim_{\epsilon\to0}\int_{D^3_\epsilon}\left(\frac{\epsilon}{2}|\nabla\chi_\epsilon|^2 + \frac{(\chi_\epsilon^2-\mu)^2}{4\epsilon}\right) &= \int_{-\rho}^{\rho}\int_{\mathbb{R}}\left(\frac{1}{2}\left|\frac{\partial\tilde{\chi}}{\partial x_1}\right|^2 + \frac{(\tilde{\chi}^2-\mu(0,x_2))^2}{4}\right)\mathrm{d}x_1\mathrm{d}x_2\\ &= \frac{2\sqrt{2}}{3}\int_{-\rho}^{\rho}(\mu(0,x_2))^{3/2}\mathrm{d}x_2.\end{aligned}$$

Finally, in view of what precedes we deduce that

$$\limsup_{\epsilon\to0}\mathcal{E}_{\epsilon,a}(v_{\epsilon,a}) \le \lim_{\epsilon\to0}\mathcal{E}_{\epsilon,a}(\chi_\epsilon) = \frac{2\sqrt{2}}{3}\int_{-\rho}^{\rho}(\mu_{\mathrm{rad}}(r))^{3/2}\mathrm{d}r - a\int_{D(0;\rho)}|f_1|\sqrt{\mu}.$$

□

At this stage, we are going to compute a lower bound of $\mathcal{E}_{\epsilon,a}(v_{\epsilon,a})$ (cf. (3.25)). This computation reduces to the one-dimensional problem studied in [5]. For every

$x_2 \in (-\rho, \rho)$ fixed, we consider the restriction of the energy to the line $\{(t, x_2) : t \in \mathbb{R}\}$:

$$E^{x_2}(\phi) = \int_{\mathbb{R}} \left(\frac{\epsilon}{2} |\phi'(t)|^2 - \frac{1}{2\epsilon} \mu(t, x_2) \phi^2(t) + \frac{1}{4\epsilon} |\phi(t)|^4 - a f_1(t, x_2) \phi(t) \right) \mathrm{d}t, \quad \phi \in H^1(\mathbb{R}). \tag{3.18}$$

We recall (cf. [5]) that there exists $\psi^{x_2}_{\epsilon,a} \in H^1(\mathbb{R})$ such that $E^{x_2}(\psi^{x_2}_{\epsilon,a}) = \min_{H^1(\mathbb{R})} E^{x_2}$, and moreover setting

$$\mathcal{E}^{x_2}(\phi) := E^{x_2}(\phi) + \int_{|t| < \sqrt{\rho^2 - x_2^2}} \frac{\mu^2(t, x_2)}{4\epsilon} \mathrm{d}t,$$

$$a_*(x_2) := \inf_{t \in (-\sqrt{\rho^2 - x_2^2}, 0]} \frac{\sqrt{2}(\mu(t, x_2))^{3/2}}{3 \int_{-\sqrt{\rho^2 - x_2^2}}^{t} |f_1(t, x_2)| \sqrt{\mu(t, x_2)} \mathrm{d}t},$$

and

$$a^*(x_2) := \sup_{t \in [-\sqrt{\rho^2 - x_2^2}, 0)} \frac{\sqrt{2}((\mu(0, x_2))^{3/2} - (\mu(t, x_2))^{3/2})}{3 \int_{t}^{0} |f_1(t, x_2)| \sqrt{\mu(t, x_2)} \mathrm{d}t},$$

we have

$$\lim_{\epsilon \to 0} \mathcal{E}^{x_2}_{\epsilon,a}(\psi^{x_2}_{\epsilon,a}) = 0, \quad \forall\, x_2 \in (-\rho, \rho),\ \forall\, a \in (0, a_*(x_2)), \tag{3.19}$$

and

$$\lim_{\epsilon \to 0} \mathcal{E}^{x_2}_{\epsilon,a}(\psi^{x_2}_{\epsilon,a}) = \frac{2\sqrt{2}}{3} (\mu(0, x_2))^{3/2} - a \int_{|t| < \sqrt{\rho^2 - x_2^2}} |f_1(t, x_2)| \sqrt{\mu(t, x_2)} \mathrm{d}t, \quad \forall\, x_2 \in (-\rho, \rho), \forall\, a \in (a^*(x_2), \infty). \tag{3.20}$$

Also note that $0 < a_* = \inf_{x_2 \in (-\rho, \rho)} a_*(x_2) \le a_*(x_2) \le a^*(x_2) \le a^* = \sup_{x_2 \in (-\rho, \rho)} a^*(x_2)$, for every $x_2 \in (-\rho, \rho)$. In view of these results we claim:

Lemma 3.4. *We have*

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^2} \epsilon \left| \frac{\partial v_{\epsilon,a}}{\partial x_2} \right|^2 = 0 \quad \textit{when } a \in (0, a_*) \cup (a^*, \infty), \tag{3.21}$$

$$\lim_{\epsilon \to 0} \int_{-\rho}^{\rho} |\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot, x_2))| \mathrm{d}x_2 = 0 \quad \textit{when } a \in (0, a_*), \tag{3.22}$$

and

$$\lim_{\epsilon \to 0} \int_{-\rho}^{\rho} \left| \mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot, x_2)) - \frac{2\sqrt{2}}{3} (\mu(0, x_2))^{3/2} + a \int_{|t| < \sqrt{\rho^2 - x_2^2}} |f_1(t, x_2)| \sqrt{\mu(t, x_2)} \mathrm{d}t \right| \mathrm{d}x_2 = 0 \quad \textit{when } a \in (a^*, \infty). \tag{3.23}$$

Proof. It is clear that $\mathcal{E}_{\epsilon,a}(v_{\epsilon,a}) = \frac{\epsilon}{2}\int_{\mathbb{R}^2}|v_{x_2}|^2 + \int_{-\rho}^{\rho}\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot,x_2))\mathrm{d}x_2 + \int_{|x_2|>\rho}\frac{v^2(v^2-2\mu)}{4\epsilon} - a\int_{|x_2|>\rho} f_1 v$. We are going to examine each of these integrals. In view of Theorem 1.1(iii), we have by dominated convergence

$$\lim_{\epsilon\to 0}\int_{|x_2|>\rho} f_1 v = 0. \tag{3.24}$$

On the other hand, since $\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot,x_2)) \geq \mathcal{E}^{x_2}(\psi^{x_2}_{\epsilon,a})$, and $\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot,x_2))$ is uniformly bounded from below on $(-\rho,\rho)$, it follows from (3.19), (3.20), and Fatou's Lemma that

$$\begin{aligned}
&\liminf_{\epsilon\to 0}\int_{-\rho}^{\rho}\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot,x_2))\mathrm{d}x_2\\
&\quad\geq \int_{-\rho}^{\rho}\liminf_{\epsilon\to 0}\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot,x_2))\mathrm{d}x_2 \geq \int_{-\rho}^{\rho}\liminf_{\epsilon\to 0}\mathcal{E}^{x_2}(\psi^{x_2}_{\epsilon,a})\\
&\quad\geq \begin{cases} 0 & \text{when } a\in(0,a_*),\\ \dfrac{2\sqrt{2}}{3}\displaystyle\int_{-\rho}^{\rho}(\mu_{\mathrm{rad}}(r))^{3/2}\mathrm{d}r - a\int_{D(0;\rho)}|f_1|\sqrt{\mu} & \text{when } a\in(a^*,\infty).\end{cases}
\end{aligned} \tag{3.25}$$

Next, we utilize (3.16), (3.24), and (3.14), to obtain

$$\begin{aligned}
&\limsup_{\epsilon\to 0}\int_{-\rho}^{\rho}\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot,x_2))\mathrm{d}x_2 \leq \limsup_{\epsilon\to 0}\mathcal{E}_{\epsilon,a}(v_{\epsilon,a})\\
&\quad\leq \begin{cases} 0 & \text{when } a\in(0,a_*),\\ \dfrac{2\sqrt{2}}{3}\displaystyle\int_{-\rho}^{\rho}(\mu_{\mathrm{rad}}(r))^{3/2}\mathrm{d}r - a\int_{D(0;\rho)}|f_1|\sqrt{\mu} & \text{when } a\in(a^*,\infty).\end{cases}
\end{aligned} \tag{3.26}$$

Combining (3.25) with (3.26), we deduce that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{-\rho}^{\rho}\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot,x_2))\mathrm{d}x_2\\
&\quad= \begin{cases} 0 & \text{when } a\in(0,a_*),\\ \dfrac{2\sqrt{2}}{3}\displaystyle\int_{-\rho}^{\rho}(\mu_{\mathrm{rad}}(r))^{3/2}\mathrm{d}r - a\int_{D(0;\rho)}|f_1|\sqrt{\mu} & \text{when } a\in(a^*,\infty),\end{cases}
\end{aligned} \tag{3.27}$$

from which (3.21) follows. For a.e. $x_2\in(-\rho,\rho)$, we also obtain (respectively when $a\in(0,a_*)$ and $a\in(a^*,\infty)$), that

$$\begin{aligned}
&\liminf_{\epsilon\to 0}\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot,x_2))\\
&\quad= \begin{cases} 0,\\ \dfrac{2\sqrt{2}}{3}(\mu(0,x_2))^{3/2} - a\displaystyle\int_{|t|<\sqrt{\rho^2-x_2^2}}|f_1(t,x_2)|\sqrt{\mu(t,x_2)}\mathrm{d}t,\end{cases}
\end{aligned} \tag{3.28}$$

thus

$$\begin{cases} \lim_{\epsilon\to 0} \min[\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot,x_2)),0] = 0, \\ \lim_{\epsilon\to 0} \min\left[\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot,x_2)) - \frac{2\sqrt{2}}{3}(\mu(0,x_2))^{3/2}\right. \\ \qquad\left. + a\int_{|t|<\sqrt{\rho^2-x_2^2}} |f_1(t,x_2)|\sqrt{\mu(t,x_2)}\mathrm{d}t, 0\right] = 0, \end{cases} \tag{3.29}$$

and by dominated convergence

$$\begin{cases} \lim_{\epsilon\to 0} \int_{-\rho}^{\rho} \min[\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot,x_2)),0]\mathrm{d}x_2 = 0, \\ \lim_{\epsilon\to 0} \int_{-\rho}^{\rho} \min\left[\mathcal{E}^{x_2}_{\epsilon,a}(v_{\epsilon,a}(\cdot,x_2)) - \frac{2\sqrt{2}}{3}(\mu(0,x_2))^{3/2}\right. \\ \qquad\left. + a\int_{|t|<\sqrt{\rho^2-x_2^2}} |f_1(t,x_2)|\sqrt{\mu(t,x_2)}\mathrm{d}t, 0\right]\mathrm{d}x_2 = 0. \end{cases} \tag{3.30}$$

Combining (3.27) with (3.30), we conclude that (3.22) and (3.23) hold. □

The proof of Theorem 1.2(ii) will follow from

Lemma 3.5. *For fixed $a \in (0, a_*)$, we have*

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^2} f_1 v_{\epsilon,a} = 0, \tag{3.31}$$

and

$$\lim_{\epsilon\to 0} \left(\int_{\mathbb{R}^2} \frac{\epsilon}{2}|\nabla v_{\epsilon,a}|^2 + \int_{|x|<\rho} \frac{(v_{\epsilon,a}^2-\mu)^2}{4\epsilon} \right) = 0. \tag{3.32}$$

Proof. Given a sequence $\epsilon_n \to 0$, we are going to show that we can extract a subsequence $\epsilon'_n \to 0$ such that $\lim_{n\to\infty} \int_{\mathbb{R}^2} f_1 v_{\epsilon'_n,a} = 0$. This will prove (3.31). According to (3.21) and (3.22), there exists a negligible set $N \subset (-\rho,\rho)$ such that for a subsequence called ϵ'_n, and for every $x_2 \in (-\rho,\rho)\backslash N$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} \epsilon'_n \left| \frac{\partial v_n}{\partial x_2}(t,x_2) \right|^2 \mathrm{d}t = 0, \tag{3.33}$$

and

$$\lim_{n\to\infty} \mathcal{E}^{x_2}_{\epsilon'_n,a}(v_n(\cdot,x_2)) = 0, \tag{3.34}$$

where we have set $v_n = v_{\epsilon'_n,a}$. Our claim is that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_1(t,x_2)v_n(t,x_2)\mathrm{d}t = 0, \quad \forall\, x_2 \in (-\rho,\rho)\backslash N. \tag{3.35}$$

From (3.33) and (3.34), it follows that given $x_2 \in (-\rho,\rho)\backslash N$ and $\gamma \in (0, \sqrt{\rho_2^2 - x_2^2})$, there exists $\bar{n}(x_2,\gamma)$ such that

$$n \geq \bar{n}(x_2,\gamma), \quad |t| < \gamma \Rightarrow v_n(t,x_2) \neq 0. \tag{3.36}$$

Indeed, otherwise we can find a subsequence n_k and a sequence $(-\gamma,\gamma) \ni t_k \to t_0$ such that $v_{n_k}(t_k,x_2)=0$. Then, proceeding as in [5, Proof of Theorem 1.1, Step 6] we obtain that $\liminf_{k\to\infty} \mathcal{E}^{x_2}_{\epsilon'_{n_k},a}(v_{n_k}(\cdot,x_2))>0$, which contradicts (3.34). Next, for fixed $t\in(-\gamma,\gamma)$, we set $\tilde v_n(s):=v_n(t+\epsilon'_n s_1, x_2+\epsilon'_n s_2)$, and proceeding as in the proof of Theorem 1.1(i) above, we can see that $\tilde v_n$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to a minimal solution $\tilde V$ of the equation $\Delta\tilde V+(\mu(t,x_2)-\tilde V^2)\tilde V=0$. If $\tilde V(s)=\sqrt{\mu(t,x_2)}\tanh(\sqrt{\mu(t,x_2)/2}(s-s_0)\cdot\nu)$, for some unit vector $\nu=(\nu_1,\nu_2)\in\mathbb{R}^2$, and some $s_0\in\mathbb{R}^2$, then (3.36) excludes the case where $\nu_1\neq 0$, while (3.33) excludes the case where $\nu_2\neq 0$. Thus, $\tilde V(s)\equiv\pm\sqrt{\mu(t,x_2)}$, and in particular $\lim_{n\to\infty}|v_n(t,x_2)|=\sqrt{\mu(t,x_2)}$. Finally, given $\delta>0$, we choose γ such that $2(\sqrt{\rho^2-x_2^2}-\gamma)\|f_1\|_{L^\infty}\sup_n\|v_n\|_{L^\infty}<\delta/2$, and since $\lim_{n\to\infty}\left|\int_{-\gamma}^{\gamma} f_1(t,x_2)v_n(t,x_2)\mathrm{d}t\right| = \lim_{n\to\infty}\left|\int_{-\gamma}^{\gamma} f_1(t,x_2)|v_n(t,x_2)|\mathrm{d}t\right|=0$, we deduce that

$$\left|\int_{|t|<\sqrt{\rho^2-x_2^2}} f_1(t,x_2)v_n(t,x_2)\mathrm{d}t\right|<\delta$$

provided that n is big enough. This proves that $\lim_{n\to\infty}\int_{|t|<\sqrt{\rho 2-x_2^2}} f_1(t,x_2) v_n(t,x_2)\mathrm{d}t=0$, and recalling that $\lim_{n\to\infty}\int_{|t|>\sqrt{\rho 2-x_2^2}} f_1(t,x_2)v_n(t,x_2)\mathrm{d}t=0$ in view of Theorem 1.1(iii), we have established (3.35). Then, we conclude that $\lim_{n\to\infty}\int_{|x_2|<\rho} f_1 v_n=0$ by dominated convergence, and since $\lim_{n\to\infty}\int_{|x_2|>\rho} f_1 v_n=0$ by Theorem 1.1(iii), we have proved (3.31). The limit in (3.32) follows from (3.31) and (3.16). □

Conclusion of the proof of Theorem 1.2(ii). We first show that when $a\in(0,a_*)$, we have $Z\subset\{|x|=\rho\}\cup\{x_1=0,|x_2|\geq\rho\}$. Assume by contradiction that there exist a sequence $\epsilon_n\to 0$, and a sequence $\bar x_n\to x_0\in D(0;\rho_0)$, with $\rho_0<\rho$, such that $v_n:=v_{\epsilon_n,a}$ vanishes at $\bar x_n$. By Lemma 3.1(i), we know that $\bar x_n$ belongs to a smooth branch of zeros that we called Z_{ϵ_n}. Let $D_1(n)=\{x_1:(x_1,x_2)\in Z_{\epsilon_n}\}$, $D_2(n)=\{x_2:(x_1,x_2)\in Z_{\epsilon_n}\}$, and for $i=1,2$, let $\delta_i(n)=\mathcal{L}^1(D_i(n))$, where $\mathcal{L}$ denotes the Lebesgue measure. Since by Lemma 3.1(ii), we have $\frac{(v_n^2(x)-\mu(x))^2}{4\epsilon_n}\geq\frac{9\mu^2_{\mathrm{rad}}(\rho_0)}{4^3\epsilon_n}$, for $x\in\bigcup_{z\in Z_{\epsilon_n}} D(z;l\epsilon_n)$, it follows that $\frac{9\mu^2_{\mathrm{rad}}(\rho_0)l}{4^3}\delta_i(n)\leq\int_{|x|<\rho}\frac{(v_n^2-\mu)^2}{4\epsilon_n}$, and thus $\lim_{n\to\infty}\delta_i(n)=0$, in view of (3.32). This implies in particular that the curves Z_{ϵ_n} do not exist the disc $D(x_0;\frac{\rho-|x_0|}{2})$ when n is large enough. Thus, Z_{ϵ_n} is a compact and connected one-dimensional manifold (without boundary) i.e. a smooth Jordan curve $\Gamma_n\subset D(0;\rho_0)$. Let ω_n be the open set bounded by Γ_n, let $\nu_n(z)$ be the outer unit normal vector at $z\in\Gamma_n$, and let us define the open set $\Omega_n=\{x\in\mathbb{R}^2: d(x,\overline{\omega}_n)<\lambda\epsilon_n\}$, where d stands for the Euclidean distance, and λ is the constant defined in Lemma 3.1. As previously we set $\tilde D_1(n)=\{x_1:(x_1,x_2)\in\Gamma_n\}$, $\tilde D_2(n)=\{x_2:(x_1,x_2)\in\Gamma_n\}$, and for $i=1,2$, $\tilde\delta_i(n)=\mathcal{L}^1(\tilde D_i(n))$. By Lemma 3.1(iii), we have either $v_n\geq\frac{3}{4}\sqrt{\mu_{\mathrm{rad}}(\rho_0)}$ or $v_n\leq-\frac{3}{4}\sqrt{\mu_{\mathrm{rad}}(\rho_0)}$ on $\partial\Omega_n$. Assuming without loss of generality that $v_n\geq\frac{3}{4}\sqrt{\mu_{\mathrm{rad}}(\rho_0)}$

on $\partial\Omega_n$, we introduce the comparison function

$$\chi_n(x) = \begin{cases} v_n(x) & \text{for } x \in \mathbb{R}^2 \setminus \Omega_n, \\ \max(|v_n(x)|, \sqrt{\mu_{\mathrm{rad}}(\rho_0)}/2) & \text{for } x \in \Omega_n, \end{cases} \tag{3.37}$$

and notice that $|\mu - \chi_n^2| \leq |\mu - v_n^2|$. Setting $S_n := \{x : d(x, \Gamma_n) < l\epsilon_n\} \subset \Omega_n$, it is clear that $\mathcal{L}^2(S_n) \geq \tilde{\delta}_i(n) l\epsilon_n$ for $i = 1, 2$. In addition, according to Lemma 3.1(ii) and (iv), the inequalities $|v_n| \leq \sqrt{\mu_{\mathrm{rad}}(\rho_0)}/2$, and $\epsilon_n |\nabla v_n| \geq \lambda'$ hold on S_n. Finally, we also notice that Lemma 3.1(iv) implies that $\tilde{\delta}_i(n) \geq \lambda\epsilon_n$. Gathering these results we reach the following contradiction

$$\begin{aligned} \mathcal{E}_{\epsilon_n,a}(\chi_n) - \mathcal{E}_{\epsilon_n,a}(v_n) &\leq -\frac{\epsilon_n}{2} \int_{|v_n| \leq \sqrt{\mu_{\mathrm{rad}}(\rho_0)}/2} |\nabla v_n|^2 + a \int_{\Omega_n} f(v_n - \chi_n) \\ &\leq -\frac{|\lambda'|^2}{2\epsilon_n} \mathcal{L}^2(S_n) + K\mathcal{L}^2(\Omega_n), \quad \text{where } K > 0 \text{ is a constant} \\ &\leq -\frac{|\lambda'|^2 l}{2} \tilde{\delta}_1(n) + K(\tilde{\delta}_1(n) + 2\lambda\epsilon_n)(\tilde{\delta}_2(n) + 2\lambda\epsilon_n) \\ &\leq \left(9K\tilde{\delta}_2(n) - \frac{|\lambda'|^2 l}{2}\right) \tilde{\delta}_1(n) < 0, \quad \text{for } n \text{ large enough.} \end{aligned} \tag{3.38}$$

This proves that there are no limit points of the zeros of v in $D(0;\rho)$. In view of Theorem 1.1(iii) we deduce that $Z \subset \{|x| = \rho\} \cup \{x_1 = 0, |x_2| \geq \rho\}$. Another consequence is that given $\rho_0 \in (0, \rho)$, there exists $\epsilon_0 > 0$ such that when $\epsilon \in (0, \epsilon_0)$, the minimizer $v_{\epsilon,a}$ does not vanish on $D(0;\rho_0)$. Up to change of $v(x_1, x_2)$ by $-v(-x_1, x_2)$, we may assume that $v_{\epsilon,a} > 0$ on $D(0;\rho_0)$. Then, in view of Theorem 1.1(iii) we have $\{x_1 < 0, |x| = \rho\} \cup \{x_1 = 0, |x_2| \geq \rho\} \subset Z$. Finally, the limit in (1.13) follows from Theorem 1.1(i), in the case where $|x| < \rho$. On the other hand, for fixed x such that $|x| \geq \rho$, the rescaled minimizers $\tilde{v}(s) = v(x + s\epsilon)$ converge to a bounded solution $\tilde{V}$ of the equation $\Delta\tilde{V}(s) + (\mu(x) - \tilde{V}^2(s))\tilde{V}(s) = 0$. As in the proof of Theorem 1.1(iii), the associated potential $W(u) = \frac{u^4}{4} - \frac{\mu(x)}{2}u^2$ is strictly convex, thus $\tilde{V}$ satisfies $W'(\tilde{V}) = 0$, i.e. $\tilde{V} = 0$. □

Now we establish the analog of Lemma 3.5 in the case where $a > a^*$, to complete the proof of Theorem 1.2(iii).

Lemma 3.6. *For fixed $a \in (a^*, \infty)$, and for every $\gamma \in (0, \rho)$, we have*

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^2} f_1 v_{\epsilon,a} = \int_{D(0;\rho)} |f_1|\sqrt{\mu}, \tag{3.39}$$

$$\lim_{\epsilon \to 0} \int_{|x|<\rho, |x_1|<\gamma} \left(\frac{\epsilon}{2} \left| \frac{\partial v_{\epsilon,a}}{\partial x_1} \right|^2 + \frac{(v_{\epsilon,a}^2 - \mu)^2}{4\epsilon} \right) = \int_{-\rho}^{\rho} \frac{2\sqrt{2}}{3} (\mu(0, x_2))^{3/2} \mathrm{d}x_2, \tag{3.40}$$

$$\lim_{\epsilon \to 0} \int_{|x|<\rho, |x_1|>\gamma} \left(\frac{\epsilon}{2} |\nabla v_{\epsilon,a}|^2 + \frac{(v_{\epsilon,a}^2 - \mu)^2}{4\epsilon} \right) = 0. \tag{3.41}$$

Proof. Given a sequence $\epsilon_n \to 0$, we are going to show that we can extract a subsequence $\epsilon'_n \to 0$ such that $\lim_{n\to\infty} \int_{\mathbb{R}^2} f_1 v_{\epsilon'_n,a} = \int_{D(0;\rho)} |f_1|\sqrt{\mu}$, and

$$\lim_{n\to\infty} \int_{|x|<\rho, |x_1|<\gamma} \left(\frac{\epsilon'_n}{2} \left| \frac{\partial v_{\epsilon'_n,a}}{\partial x_1} \right|^2 + \frac{(v^2_{\epsilon'_n,a} - \mu)^2}{4\epsilon'_n} \right) = \int_{-\rho}^{\rho} \frac{2\sqrt{2}}{3} (\mu(0,x_2))^{3/2} \mathrm{d}x_2.$$

This will prove (3.39) and (3.40). According to (3.21) and (3.23), there exists a negligible set $N \subset (-\rho,\rho)$ such that for a subsequence called ϵ'_n, and for every $x_2 \in (-\rho,\rho)\backslash N$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} \epsilon'_n \left| \frac{\partial v_n}{\partial x_2}(t,x_2) \right|^2 \mathrm{d}t = 0, \tag{3.42}$$

and

$$\lim_{n\to\infty} \mathcal{E}^{x_2}_{\epsilon'_n,a}(v_n(\cdot,x_2)) = \frac{2\sqrt{2}}{3}(\mu(0,x_2))^{3/2} - a\int_{|t|<\sqrt{\rho^2-x_2^2}} |f_1(t,x_2)|\sqrt{\mu(t,x_2)}\mathrm{d}t, \tag{3.43}$$

where we have set $v_n = v_{\epsilon'_n,a}$. Our claim is that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_1(t,x_2) v_n(t,x_2)\mathrm{d}t = \int_{|t|<\sqrt{\rho^2-x_2^2}} |f_1(t,x_2)|\sqrt{\mu(t,x_2)}\mathrm{d}t, \quad \forall\, x_2 \in (-\rho,\rho)\backslash N. \tag{3.44}$$

From (3.42) and (3.43), it follows that given $x_2 \in (-\rho,\rho)\backslash N$ and $\gamma \in (0,\rho)$, there exists $\bar{n}(x_2,\gamma)$ such that

$$n \geq \bar{n}(x_2,\gamma), \quad \gamma < |t| < \rho + 1 \Rightarrow v_n(t,x_2) \neq 0. \tag{3.45}$$

Indeed, otherwise we can find a subsequence n_k and a sequence $(-\rho-1,-\gamma)\cup(\gamma,\rho+1) \ni t_k \to t_0$ such that $v_{n_k}(t_k,x_2) = 0$. Then, proceeding as in [5, Proof of Theorem 1.1, Step 6] we obtain that $\liminf_{k\to\infty} \mathcal{E}^{x_2}_{\epsilon'_{n_k},a}(v_{n_k}(\cdot,x_2)) > \frac{2\sqrt{2}}{3}(\mu(0,x_2))^{3/2} - a\int_{|t|<\sqrt{\rho^2-x_2^2}} |f_1(t,x_2)|\sqrt{\mu(t,x_2)}\mathrm{d}t$, which contradicts (3.43). Thus, (3.45) holds, and actually in view of Theorem 1.1(iii) we have

$$\begin{aligned} &n \geq \bar{n}(x_2,\gamma),\ \gamma < t < \rho+1 \Rightarrow v_n(t,x_2) > 0, \text{ and } n \geq \bar{n}(x_2,\gamma), \\ &-\rho - 1 < t < -\gamma \Rightarrow v_n(t,x_2) < 0. \end{aligned} \tag{3.46}$$

Next, for fixed $t \in (-\sqrt{\rho^2-x_2^2},-\gamma) \cup (\gamma,\sqrt{\rho^2-x_2^2})$, we set $\tilde{v}_n(s) := v_n(t + \epsilon'_n s_1, x_2 + \epsilon'_n s_2)$, and proceeding as in the proof of Lemma 3.5, we can see that $\lim_{n\to\infty} v_n(t,x_2) = \sqrt{\mu(t,x_2)}$ for $t \in (\gamma,\sqrt{\rho^2-x_2^2})$, while $\lim_{n\to\infty} v_n(t,x_2) = -\sqrt{\mu(t,x_2)}$ for $t \in (-\sqrt{\rho^2-x_2^2},-\gamma)$. Then, by repeating the arguments in the proof of Lemma 3.5, our claim (3.44) follows. Finally, we conclude that $\lim_{n\to\infty} \int_{|x_2|<\rho} f_1 v_n = \int_{D(0;\rho)} |f_1|\sqrt{\mu}$ by dominated convergence, and since $\lim_{n\to\infty} \int_{|x_2|>\rho} f_1 v_n = 0$ by Theorem 1.1(iii), we have established (3.39). Another consequence of (3.46) is that for every $x_2 \in (-\rho, \rho)\backslash N$, there exists a sequence $\bar{t}_n \to 0$ such that $v_n(\bar{t}_n, x_2) = 0$. Setting $\tilde{v}_n(s) := v_n(\bar{t}_n + \epsilon'_n s_1, x_2 + \epsilon'_n s_2)$, we obtain as in Lemma 3.5, that $\tilde{v}_n$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to $\tilde{V}(s) = \sqrt{\mu(0, x_2)}\tanh(\sqrt{\mu(0, x_2)/2}(s \cdot \nu))$, for some unit vector $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$. Again, (3.42) implies that $\nu = (1, 0)$, and we refer to the detailed computation in [5, Proof of Theorem 1.1, Step 6] to see that

$$\begin{aligned}\liminf_{n\to\infty} \int_{|t|<\min(\gamma, \sqrt{\rho^2 - x_2^2})} \left(\frac{\epsilon'_n}{2} \left| \frac{\partial v_n}{\partial x_1}(t, x_2) \right|^2 + \frac{(v_n^2(t, x_2) - \mu(t, x_2))^2}{4\epsilon'_n} \right) \mathrm{d}t \\ \geq \frac{2\sqrt{2}}{3} (\mu(0, x_2))^{3/2}. \end{aligned} \tag{3.47}$$

Then, it follows from Fatou's Lemma that

$$\liminf_{n\to\infty} \int_{|x|<\rho, |x_1|<\gamma} \left(\frac{\epsilon'_n}{2} \left| \frac{\partial v_{\epsilon'_n, a}}{\partial x_1} \right|^2 + \frac{(v^2_{\epsilon'_n, a} - \mu)^2}{4\epsilon'_n} \right) \geq \int_{-\rho}^{\rho} \frac{2\sqrt{2}}{3} (\mu(0, x_2))^{3/2} \mathrm{d}x_2. \tag{3.48}$$

Finally, combining (3.48) with (3.39) and (3.16), we deduce (3.40) and (3.41). $\square$

Conclusion of the proof of Theorem 1.2(iii). Proceeding as in the conclusion of the proof of Theorem 1.2(ii), we show that there are no limit points of the zeros of v in the set $D(0; \rho) \cap \{(x_1, x_2) : |x_1| > \gamma\}$, where $\gamma > 0$ is small. As a consequence, given $\rho_0 \in (0, \rho)$, there exists $\epsilon_0 > 0$ such that when $\epsilon \in (0, \epsilon_0)$, the minimizer $v_{\epsilon,a}$ is positive on $D(0; \rho_0) \cap \{(x_1, x_2) : x_1 > \gamma\}$. Let $K \subset (\gamma, \rho+1) \times (-\rho_0, \rho_0)$ be a compact set. Our claim is that there exists $\epsilon_K > 0$ such that when $\epsilon \in (0, \epsilon_K)$, the minimizer $v_{\epsilon,a}$ is positive on K. To prove this claim we assume by contradiction that there exist a sequence $\epsilon_n \to 0$, and a sequence $K \ni x_n \to x_0$ such that $v_n(x_n) \leq 0$, where we have set $v_n := v_{\epsilon_n, a}$. Having a closer look at the proof of Lemma 3.6 (cf. in particular (3.46)), we can find $\rho_1 \in (0, \rho_0)$ such that $K \subset (\gamma, \rho + 1) \times (-\rho_1, \rho_1)$, and $\bar{n}(\rho_1, \gamma)$ such that for $n \geq \bar{n}(\rho_1, \gamma)$, and $t \in (\gamma, \rho + 1)$, we have $v_n(t, \pm\rho_1) > 0$. Next, in view of Theorem 1.1(iii), we also obtain that $v_n(\rho + 1, s) > 0$ for every $s \in [-\rho_1, \rho_1]$, provided that n is large enough. Gathering these results, it follows that there exists n_K such that for every $n \geq n_K$, v_n is positive on the boundary of the rectangle $R := (\gamma, \rho + 1) \times (-\rho_1, \rho_1)$. In addition, for $n \geq n_K$, v_n cannot take negative values in R, since otherwise we would have $E(|v_n|, R) < E(v_n, R)$ in contradiction with the minimality of v_n. Thus, v_n has a local minimum at x_n for $n \geq n_K$, and (1.5) implies that $0 \leq \epsilon^2 \Delta v_n(x_n) = -\epsilon a f_1(x_n) \in (-\infty, 0)$, which is a contradiction. This establishes our claim, and now in view of Theorem 1.1(iii) it is clear that $Z = \{x_1 = 0\}$. Finally, the limit in (1.14) is established as in the conclusion of the proof of Theorem 1.2(ii). To prove the limit in (1.15), we proceed as in Lemma 3.6. There exists a subsequence $\epsilon'_n \to 0$, and a negligible set $N \subset (-\rho, \rho)$, such that (3.42) and (3.43) hold for every $x_2 \in (-\rho, \rho)\backslash N$. Now, let $\bar{x}_n = (\bar{t}_{\epsilon'_n, a}, x_2)$ be a zero of v_n with fixed ordinate $x_2 \in (-\rho, \rho)\backslash N$, and set $\tilde{v}_n(s) := v(\bar{x}_n + \epsilon'_n s)$. Then

$\tilde{v}_n$ converges in the $C^2_{\mathrm{loc}}(\mathbb{R})$ sense to $\tilde{V}(s) = \sqrt{\mu(0,x_2)}\tanh(\sqrt{\mu(0,x_2)/2}(s\cdot\nu))$ for some unit vector $\nu \in \mathbb{R}^2$, and (3.42) implies that $\nu = (\pm 1, 0)$, while (3.43) implies that for n large enough v_n has a unique zero with fixed ordinate x_2. Thus, $\nu = (1,0)$ and (1.15) is established. $\square$

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