

Structure of W_4 -immersion free graphs[★]

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Abstract. We study the structure of graphs that do not contain the wheel on 5 vertices W_4 as an immersion, and show that these graphs can be constructed via 1, 2, and 3-edge-sums from subcubic graphs and graphs of bounded treewidth.

Keywords: Immersion Relation, Wheel, Treewidth, Edge-sums, Structural Theorems.

1 Introduction

A recurrent theme in structural graph theory is the study of specific properties that arise in graphs when excluding a fixed pattern. The notion of appearing as a pattern gives rise to various graph containment relations. Maybe the most famous example is the minor relation that has been widely studied, in particular since the fundamental results of Kuratowski and Wagner who proved that planar graphs are exactly those graphs that contain neither K_5 nor $K_{3,3}$ as a (topological) minor. A graph G contains a graph H as a topological minor if H can be obtained from G by a sequence of vertex deletions, edge deletions and replacing internally vertex-disjoint paths by single edges. Wagner also described the structure of the graphs that exclude K_5 as a minor: he proved that K_5 -minor-free graphs can be constructed by “gluing” together (using so-called clique-sums) planar graphs and a specific graph on 8 vertices, called Wagner’s graph.

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Wagner’s theorem was later extended in the seminal Graph Minor series of papers by Robertson and Seymour (see e.g. [23]), which culminated with the proof of Wagner’s conjecture, i.e., that graphs are well-quasi-ordered under minors [24], and ended with the proof of Nash-Williams’ immersion conjecture, i.e., that the graphs are also well-quasi-ordered under immersions [25]. Other major results in graph minor theory include the (Strong) Structure Theorem [23], the Weak Structure Theorem [22], the Excluded Grid Theorem [17,21,26], as well as numerous others, e.g., [5, 15,28]. Moreover, the structural results of graph minor theory have deep algorithmic implications, one of the most significant examples being the existence of cubic time algorithms for the k -DISJOINT PATHS and H -MINOR CONTAINMENT problems [22]. For more applications see, e.g., [1, 3, 6, 16, 18].

However, while the structure of graphs that exclude a fixed graph H as a minor has been extensively studied, the structure of graphs excluding a fixed graph H as a topological minor or as an immersion has not received as much attention. While a general structure theorem for topological minor free graphs was very recently provided by Grohe and Marx [14], finding an exact characterization of the graphs that exclude K_5 as a topological minor remains a notorious open problem. Recently, Wollan gave a structure theorem for graphs excluding complete graphs as immersions [29]. A graph G contains a graph H as a immersion if H can be obtained from G by a sequence of vertex deletions, edge deletions and replacing edge-disjoint paths by single edges. Observe that if a graph G contains a graph H as a topological minor, then G also contains H as an immersion, as vertex-disjoint paths are also edge-disjoint. In 2011, DeVos et al. [7] proved that if the minimum degree of a graph G is at least $200t$ then G contains the complete graph on t vertices as an immersion. In [10] Ferrara et al. provided a lower bound on the minimum degree of any graph G in order to ensure that a given graph H is contained in G as an immersion.

A common drawback of such general results is that they do not provide sharp structural characterizations for concrete instantiations of the excluded graph H . In the particular case of immersion, such structural results are only known when excluding both K_5 and $K_{3,3}$ as immersions [11]. In this paper, we prove a structural characterization of the graphs that exclude W_4 as an immersion and show that they can be constructed from graphs that are either subcubic or have treewidth bounded by a constant. We denote by W_4 the wheel with 4 spokes, i.e., the graph obtained from a cycle on 4 vertices by adding a universal vertex. The structure of graphs that exclude W_4 as a topological minor has been studied by Farr [9]. He proved that these graphs can be constructed via clique-sums of order at most 3 from graphs that are either subcubic or have diameter at most 3. However, this characterization only applies to simple graphs. In our study we exclude W_4 as an immersion while allowing multiple edges.

As with the minor relation, many algorithmic results have also started appearing in terms of immersions. In [13], Grohe et al. gave an cubic time algorithm that decides whether a fixed graph H immerses in any input graph G . This algorithm, combined with the well-quasi-ordering of immersions [25], implies that the membership of a graph in any graph class that is closed under taking immersions can be decided in cubic time. However, the construction of such an algorithm requires the ad-hoc knowledge of the finite set of excluded immersions that characterizes this graph class (which is called

obstruction set). While no general way to compute an obstruction set is known, in [12], Giannopoulou et al. proved that the obstruction set of an immersion-closed graph class can be computed when an upper bound on the treewidth of the obstructions and a description of the graph class in Monadic Second Order Logic are given. Another example of explicit construction of immersion obstruction sets is given by Belmonte et al. [2], where the set of immersion obstructions is given for graphs of carving-width 3.

Our paper is organized as follows: in Section 2, we give necessary definitions and previous results. In Section 3, we show that containment of W_4 as an immersion is preserved under 1, 2 and 3-edge-sums. Then, in Section 4, we provide our main result, i.e., a decomposition theorem for graphs excluding W_4 as an immersion. Finally, we conclude with remarks and open problems.

2 Preliminaries

For undefined terminology and notation, we refer to the textbook of Diestel [8]. For every integer n , we let $[n] = \{1, 2, \dots, n\}$. All graphs we consider are finite, undirected, and without self-loops but may have multiple edges. Given a graph G we denote by $V(G)$ and $E(G)$ its *vertex* and *edge set* respectively. Given a set $F \subseteq E(G)$ (resp. $S \subseteq V(G)$), we denote by $G \setminus F$ (resp. $G \setminus S$) the graph obtained from G if we remove the edges in F (resp. the vertices in S along with their incident edges). We denote by $C(G)$ the set of the *connected components* of G . Given two vertices $v, u \in V(G)$, we also use the notation $G - v = G \setminus \{v\}$ and the notation uv for the edge $\{u, v\}$. The *neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices in G that are adjacent to v . We denote by $E_G(v)$ the set of the edges of G that are incident with v . The *degree* of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is the number of edges that are incident with it, that is, $\deg_G(v) = |E_G(v)|$. Notice that, as we are working with multigraphs, $|N_G(v)| \leq \deg_G(v)$. Given two vertices v and u with $u \in N(v)$ we say that u is an i -neighbor of v if $E(G)$ contains exactly i copies of the edge $\{u, v\}$. Let P be a path and $v, u \in V(P)$. We denote by $P[v, u]$ the subpath of P with endpoints v and u . The *maximum degree* of a graph G , denoted by $\Delta(G)$ is the maximum of the degrees of the vertices of G , that is, $\Delta(G) = \max_{v \in V(G)} \deg_G(v)$.

We denote by W_{k-1} the *wheel* on k vertices, that is, the graph obtained from the cycle of length $k - 1$ after adding a new vertex and making it adjacent to all of its vertices. We call the new vertex *center* of the wheel.

Definition 1. *An immersion of H in G is a function α with domain $V(H) \cup E(H)$, such that:*

- $\alpha(v) \in V(G)$ for all $v \in V(H)$, and $\alpha(u) \neq \alpha(v)$ for all distinct $u, v \in V(H)$;
- for each edge e of H , $\alpha(e)$ is a path of G with ends $\alpha(u), \alpha(v)$;
- for all distinct $e, f \in E(H)$, $E(\alpha(e)) \cap \alpha(f) = \emptyset$.

We call the image of every such function α in G *model* of the graph H in G and the vertices of the set $\alpha(V(H))$ *branch* vertices of α .

An *edge cut* in a graph G is a non-empty set F of edges that belong to the same connected component of G and such that $G \setminus F$ has more connected components than

G . If $G \setminus F$ has one more connected component than G and no proper subset of F is an edge cut of G , then we say that F is a *minimal* edge cut. Given a connected set S such that $G \setminus S$ is also connected, we denote by $(S, G \setminus S)$ the cut between S and $G \setminus S$. Let F be an edge cut of a graph G and let G be the connected component of G containing the edges of F . We say that F is an *internal* edge cut if it is minimal and both connected components of $G \setminus F$ contain at least 2 vertices. An edge cut is also called *i-edge cut* if it has order at most i .

Definition 2. Let G, G_1 , and G_2 be graphs. Let $t \geq 1$ be a positive integer. The graph G is a *t-edge-sum* of G_1 and G_2 if the following holds. There exist vertices $v_i \in V(G_i)$ such that $|E_{G_i}(v_i)| = t$ for $i \in [2]$ and a bijection $\pi : E_{G_1}(v_1) \rightarrow E_{G_2}(v_2)$ such that G is obtained from $(G_1 - v_1) \cup (G_2 - v_2)$ by adding an edge xy for every pair of edges e_1 and e_2 such that $e_1 = xv_1$, $e_2 = yv_2$, and $v_2 = \pi(v_1)$. We say that the edge-sum is *internal* if both G_1 and G_2 contain at least 2 vertices and denote the internal t -edge-sum of G_1 and G_2 by $G_1 \hat{\oplus}_t G_2$.

Note that if G is the t -edge-sum of graphs G_1 and G_2 for some $t \geq 0$, then the set of edges $\{\{u, v\} \in E(G) \mid u \in V(G_1), v \in V(G_2)\}$ forms a minimal edge cut of G of order t .

Let r be a positive integer. The (r, r) -grid is the graph with vertex set $\{(i, j) \mid i, j \in [r]\}$ and edge set $\{\{(i, j), (i', j')\} \mid |i - i'| + |j - j'| = 1\}$. The (*elementary*) *wall* of height r is the graph W_r with vertex set $V(W_r) = \{(i, j) \mid i \in [r + 1], j \in [2r + 2]\}$ in which we make two vertices (i, j) and (i', j') adjacent if and only if either $i = i'$ and $j' \in \{j - 1, j + 1\}$ or $j' = j$ and $i' = i + (-1)^{i+j}$, and then remove all vertices of degree 1; see Figure 1 for some examples. The vertices of this vertex set are called *original* vertices of the wall. A *subdivided wall* of height r is the graph obtained from W_r after replacing some of its vertices by internally vertex-disjoint paths.

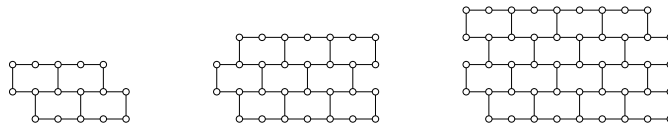


Fig. 1. Elementary walls of height 2, 3, and 4.

Let r be a positive integer and notice that the wall of height r is contained in the $(2r + 2) \times (2r + 2)$ -grid as a subgraph. This implies that any graph containing the $(2r + 2) \times (2r + 2)$ -grid as a minor also contains the wall of height r as a minor. Furthermore, from a folklore result, for any simple graph H such that $\Delta(H) \leq 3$ it holds that H is a minor of a graph G if and only if H is a topological minor of G .

Theorem 1. [19] Let G and H be two graphs, where H is connected and simple, not a tree, and has h vertices. Let also g be a positive integer. If G has treewidth greater than $3(8h(h - 2)(2g + h)(2g + 1))^{|E(H)| - V(H)} + \frac{3h}{2}$ then G contains either the $g \times g$ -grid or H as a minor.

Theorem 1, in the case where $g = 2r + 2$ and H is the wall of height r , can be restated as the well known fact that large treewidth ensures the existence of a large wall as a topological minor:

Theorem 2. [19] *Let G be a graph and $r \geq 2$ be an integer. If the treewidth of G is greater than $2^{18r^2 \log r}$ then G contains the wall of height r as a topological minor.*

3 Invariance of W_4 containment under small edge-sums

In this section, we show that immersion of W_4 is completely preserved under edge-sums of order at most 3, i.e., that W_4 immerses in a graph G if and only if it immerses in at least one of the graphs obtained by decomposing G along edge-sums. Theorem 3 will be necessary in Section 4 to ensure that our decomposition does not change whether the graphs consider contain W_4 as an immersion or not. We first prove the following general lemma.

Lemma 1. *If G , G_1 , and G_2 are graphs such that $G = G_1 \hat{\oplus}_t G_2$, $t \in [3]$, then both G_1 and G_2 are immersed in G .*

Proof. Notice that it is enough to prove that G_1 is an immersion of G . Let v_1 and v_2 denote the unique vertex of $V(G_1) \setminus V(G)$ and $V(G_2) \setminus V(G)$ respectively. In the case where $G = G_1 \hat{\oplus}_1 G_2$, let u_i be the unique neighbor of v_i in G_i , $i \in [2]$. Then the function $\{(v, v) \mid v \in V(G_1 - v_1)\} \cup \{v_1, u_2\}$ is an isomorphism from G_1 to the graph $G \setminus V(G_2 - u_2)$ (by the definition of the edge-sum $u_2 u_1 \in E(G)$) which is a subgraph of G . Therefore, $G_1 \subseteq G$ and thus G_1 also immerses in G .

We now assume that $G = G_1 \hat{\oplus}_t G_2$, $t = 2, 3$. Let e_j , $j \in [|E_{G_1}(v_1)|]$, be the edges of $E_{G_1}(v_1)$ and let u_j be the (not necessarily distinct) endpoints of the edges e_j , $j \in [|E_{G_1}(v_1)|]$, in $G_1 - v_1$. Notice that in both cases, in order to obtain G_1 as an immersion of G , it is enough to find a vertex u in $V(G) \setminus V(G_1)$ and for each edge e_j of $E_{G_1}(v_1)$ find a path P_j from u to e_j in $E(G) \setminus E(G_1)$ such that these paths are edge-disjoint. In what follows we find such vertex and paths. We distinguish the following cases.

Case 1. $N_{G_2}(v_2) = \{y\}$. Then, by the definition of the edge-sum, G contains the edges ye_j^1 , $j \in [|E_{G_1}(v_1)|]$. Notice that neither the vertex y belongs to $V(G_1)$ nor the edges yu_j^1 , $j \in [|E_{G_1}(v_1)|]$, belong to $E(G_1)$ and therefore the claim holds for $u = y$.

Case 2. $N_{G_2}(v_2) = \{x, y\}$. First notice that in the case where $G = G_1 \hat{\oplus}_3 G_2$ one of the x, y , say x , is a 2-neighbor of v_2 . As the edge-sum is internal, the set $E = E(G) \setminus (E(G_1) \cup E(G_2))$ of edges created after the edge-sum is a minimal separator of G . Without loss of generality let yu_1^1 , xu_2^1 , and (in the case where $G = G_1 \hat{\oplus}_3 G_2$) xu_3^1 be its edges. By the minimality of the separator E , $G_2 - v_2$ is connected. Therefore there exists a (x, y) -path P in $G_2 - v_2$. Observe that the path $P \cup \{yu_1^1\}$, the path consisting only of the edge xu_2^1 and (in the case where $G = G_1 \hat{\oplus}_3 G_2$) the path consisting only of the edge xu_3^1 are edge-disjoint paths who do not have any edge from $E(G_1)$ and share x as a common endpoint. Then the claim holds for $u = x$.

Case 3. $N_{G_2}(v_2) = \{x, y, z\}$. In this case, it holds that $G = G_1 \hat{\oplus}_3 G_2$. As above, consider the set $E = E(G) \setminus (E(G_1) \cup E(G_2))$ of the edges created by the edge-sum and without loss of generality, let $E = \{xu_1, yu_2, zu_3\}$. Since E is a minimal separator, the graph $G_2 - v_2$ is connected. Therefore, there are a (x, y) -path P and a (y, z) -path Q in $G_2 - v_2$. Let z' be the vertex in $V(P) \cap V(Q)$ such that $V(Q[z, z']) \cap V(P) = \{z'\}$ and consider the paths $Q[z, z']$, $P[x, z']$, and $P[z', y]$ (in the case where $z' = y$ the path $P[z', y]$ is the graph consisting of only one vertex). Observe that these graphs are edge-disjoint. Therefore the paths $P[x, z'] \cup \{xu_1\}$, $P[y, z'] \cup \{yu_2\}$, and $Q[z, z'] \cup \{zu_3\}$ are edge-disjoint, do not contain any edge from $E(G_1)$, and share the vertex z' as an endpoint. Thus, the claim holds for $u = z'$. It then follows that G_1 is an immersion of G and this completes the proof of the lemma. \square

Theorem 3. *Let G , G_1 , and G_2 be graphs such that $G = G_1 \hat{\oplus}_t G_2$, with $t \in [3]$. Then, G contains W_4 as an immersion if and only if G_1 or G_2 does as well.*

Proof. If G_1 or G_2 contains W_4 as an immersion, then G does as well due to Lemma 1. It remains to prove the converse direction.

Let α be an immersion of W_4 in G . We first prove that either $|\alpha(V(W_4)) \cap (V(G_1) - v_1)| \geq 4$, or $|\alpha(V(W_4)) \cap (V(G_2) - v_2)| \geq 4$. Indeed, this is due to the fact that any cut $(S, G \setminus S)$ of W_4 with $|S| = 3$ has order at least 4, whereas the cut $F = E(G) \setminus (E(G_1) \cup E(G_2))$ in G between $V(G_1) - v_1$ and $V(G_2) - v_2$ has order at most 3. Moreover, the same argument implies that the image of the center of W_4 , that is, the unique vertex of degree 4 of W_4 , say x_0 , belongs to the connected component of $G - F$ that contains at least 4 of the branch vertices of the immersion α . Let us assume without loss of generality that $x_0 \in V(G_1) - v_1$.

Assume first that $\alpha(V(W_4)) \cap (V(G_1) - v_1) = 5$. If for every edge e of W_4 it holds that $\alpha(e) \cap V(G_2 - v_2) = \emptyset$, then clearly α is an immersion of W_4 in $G_1 - v_1$, and therefore in G_1 . Moreover, it is easy to observe that there cannot be two distinct edges e, e' of W_4 whose image path in G contains vertices of $G_2 - v_2$, since each such path must contain at least 2 edges of F , and $|F| \leq 3$. Hence we may assume that there exists a unique edge e with $\alpha(e) \cap V(G_2 - v_2) \neq \emptyset$. Note that $\alpha(e)$ must intersect the cut F in an even number of edges, since otherwise the path would end in $G_2 - v_2$, contradicting our assumption that all branch vertices of α lie in $G_1 - v_1$. Let P be the maximum subpath of $\alpha(e)$ such that $E(P') \cap E(G_1 - v_1) = \emptyset$. Notice that the first and the last edge of such a path are edges of F . Let u_1 and u_2 be the endpoints of P . This implies that we may obtain an immersion α' of W_4 in G_1 by replacing in α the path P by the path $u_1 v_1 u_2$.

Now, we assume that $\alpha(V(W_4)) \cap (V(G_1) - v_1) = 4$, and denote by x the unique branch vertex of α lying in $V(G_2 - v_2)$. We claim that it is possible to create an immersion function α' of W_4 in G_1 by replacing the vertex x in α with v_1 . To show this, we apply the following operations to G : let P_1, P_2, P_3 be the paths of α whose associated edges in W_4 are incident with $\alpha^{-1}(x_4)$, and let P'_1, P'_2, P'_3 be the subpaths of P_1, P_2 , and P_3 that do not contain edges of $G_1 - v_1$. The paths P'_1, P'_2, P'_3 are easily observed to be edge-disjoint, and therefore we may lift the edges in each of these paths. We complete the construction by deleting the vertices in $V(G_2) - \{v_2, x_4\}$. The graph obtained from this construction is readily observed to be isomorphic to G_1 by mapping every vertex of $G_1 - v_1$ to itself, and v_1 to x . Therefore W_4 immerses in G_1 . This concludes the proof of the theorem. \square

4 Structure of graphs excluding W_4 as an immersion

In this section, we prove the main result of our paper, namely we provide a structure theorem for graphs that exclude W_4 as an immersion. We first provide a technical lemma that will be crucial for the proof of Theorem 5.

Lemma 2. *There exists a function f such that for every integer $r \geq 60000$ and every graph G that does not contain W_4 as an immersion, has no internal 3-edge cut, and has a vertex u with $d(u) \geq 4$, if $tw(G) \geq f(r)$, then there exist sets $Z = \{z_1, \dots, z_r\}$, S_1, \dots, S_r , and X , that satisfy the following properties:*

- (i) $z_i \in S_i, \forall i \in \{1, \dots, r\}$;
- (ii) $z_i \in S_j, \forall i \neq j \in \{1, \dots, r\}$;
- (iii) $u \in \bigcap_{i \in \{1, \dots, r\}} S_i$;
- (iv) $\partial(S_i) \leq 6$;
- (v) $G[S_i]$ is connected, $\forall i \in \{1, \dots, r\}$;
- (vi) $X \cap S_i = \emptyset, \forall i \in \{1, \dots, r\}$;
- (vii) For every $Z' \subseteq Z$ such that $|Z'| \geq 7$, there is a 7-flow from Z' to X ;

Proof. Assume that G has treewidth at least $2^{18(6r)^2 \log(6r)}$. Then, from Theorem 2, $G - u$ contains an elementary wall of height $6r$ as a topological minor. We define the cycles C_1, \dots, C_{6r} as the ones formed by vertices $w_{5+20i,3+2j}$ to $w_{11+20i,3+2j}$ and $w_{11+20i,4+2j}$ to $w_{5+20i,4+2j}$, for every $i, j \in \{0, \dots, \lceil \sqrt{6r} \rceil - 1\}$. Observe that C_1, \dots, C_{6r} is a set of vertex disjoint cycles of length 14 in $G - u$. For every $i \in [6r]$, we denote by G_{C_i} the graph obtained from G by removing the edges of C_i and adding a vertex v_i adjacent exactly to the vertices of C_i . Since W_4 does not immerse in G , there exists an edge cut F_i of order at most 3 that separates u and v_i , and since both u and v_i have degree at least 4, this edge cut is internal. We now define the set T_i , for every $i \in [6r]$, as the set of vertices that lie in the same connected component of $G_{C_i} - F_i$ as u .

Claim 1 *For every $i \in \{1, \dots, 6r\}$, $1 \leq |T_i \cap V(C_i)| \leq 3$.*

Proof of Claim 1. The fact that $|T_i \cap V(C_i)| \geq 1$ follows from the observation that if $T_i \cap V(C_i) = \emptyset$, then the cut F_i is not only a cut in G_{C_i} , but also in G , which contradicts the assumption that G is internally 4 edge-connected. On the other hand, observe that for every vertex $w \in V(C_i) \cap T_i$, the edge vw must belong to the cut F_i . Hence no more than 3 vertices of C_i may lie in T_i , which concludes the proof of the claim. \diamond

We now define the set $Z = \{z_1, \dots, z_{6r}\}$: for every $i \in [6r]$, we choose arbitrarily one vertex of $T_i \cap V(C_i)$ to be the vertex z_i . The existence of the vertices z_i follows from Claim 1. Observe that, by construction of Z , it holds that $z_i \in T_i, \forall i \in [6r]$, i.e., the sets T_i satisfy property (i).

Observe that, by construction, $G[T_i]$ is connected. Moreover, the only edges of G that are not edges of G_{C_i} are the edges of the cycle C_i . Thus, the only edges in the cut $(T_i, G \setminus T_i)$ of G that are not edges of the cut F_i in G_{C_i} , are the edges of C_i incident with the vertices of $T_i \cap V(C_i)$. Furthermore, for every vertex w of $T_i \cap V(C_i)$, the edge wv_i belongs to the cut F_i in G_{C_i} , but not to the cut $(T_i, G \setminus T_i)$ in G . Hence, the number of

edges of the cut $(T_i, G \setminus T_i)$ in G is at most $|F_i| + 2|T_i \cap V(C_i)| - |T_i \cap V(C_i)|$. Since F_i and $T_i \cap V(C_i)$ both have order at most 3, it follows that the cut $(T_i, G \setminus T_i)$ in G has order at most 6. We have therefore proved that properties (iii)-(v) hold for the sets $T_i, i \in [6r]$.

We may now define the set X . We first start with the subwall formed by vertices $w_{i,j}$, with $6r + 1 - 36(\lceil \sqrt{6r} \rceil + 1) \leq i \leq 6r + 1 + 36(\lceil \sqrt{6r} \rceil + 1)$ and $6r + 1 - (\lceil \sqrt{6r} \rceil + 1) \leq j \leq 6r + 1 - 73(\lceil \sqrt{6r} \rceil + 1)$. This set, denoted X_0 , contains at least $72(\sqrt{6r} + 1)^2 \geq 72r + 73$ vertices, due to $r \geq 1$. We now need the following:

Claim 2 For every $i \in [6r]$, $|X_0 \cap T_i| \leq 72$.

Sketch of proof of Claim 2. We prove the claim by showing that for every $i \in [6r]$ and every subset X'_0 of X_0 such that $|X'_0| \geq 73$, there are 7 disjoint paths from vertices of $C_i \setminus T_i$ to vertices of X'_0 in G . Together with property (iv), this will imply validity of Claim 2. Consider a subset X'_0 of X_0 of cardinality at least 73. Observe that there must be 13 vertices that lie on the same horizontal path, or 7 vertices that lie on different horizontal paths. From there, taking into account the dimensions of the wall and the position of the vertices of C_i and X_0 , it is easy to observe that there always exist vertices y_1, \dots, y_7 in $C_i \setminus T_i$ and x_1, \dots, x_7 in X'_0 such that there are 7 disjoint paths between y_1, \dots, y_7 and x_1, \dots, x_7 . \diamond

Therefore, the set $X_0 \cap \bigcup T_i$ contains at most $72r$ vertices, which implies that there exists a subset X of X_0 containing at least 73 vertices such that $X \cap T_i = \emptyset$ for every $i \in [6r]$. This proves property (vi) for the sets $T_i, i \in [6r]$. The validity of property (vii) follows from arguments similar to those given in the proof of Claim 2.

Finally, we show how to select sets S_1, \dots, S_r among T_1, \dots, T_{6r} so that property (ii) holds, namely that for every $1 \leq i \neq j \leq r, z_i \notin S_j$. In order to find such sets, we proceed as follows: let H be a directed graph such that $V(H) = \{T_1, \dots, T_{6r}\}$, and (T_i, T_j) is an arc of H if and only if $z_i \in S_j$. We now claim that vertices of H have indegree at most 6. This is shown by combining properties (iv), (vi), and (vii). Assume for contradiction that there is a vertex in H having indegree at least 7, then there exist distinct indices i_1, \dots, i_7 and j such that $z_{i_1}, \dots, z_{i_7} \in S_j$. However, we know that there exist 7 disjoint paths from $\{z_{i_1}, \dots, z_{i_7}\}$ to X by property (vii). Together with property (vi), we obtain a contradiction with property (iv). Therefore, we conclude that the directed graph H has maximum indegree at most 6. Thus, $|E(H)| \leq 36r$, which implies that the average degree of H is at most 6. Hence, H is 6-degenerate and thus contains an independent set of size at least $\frac{|V(H)|}{6} = r$. The vertices of such an independent set correspond to sets T_{i_1}, \dots, T_{i_r} such that, for every $1 \leq p \neq q \leq r, z_{i_p} \notin T_{i_q}$. Therefore, we choose $S_p := T_{i_p}$ for every $p \in [r]$ and observe that the set S_1, \dots, S_r as defined indeed satisfy property (ii).

Finally, since every set T_i satisfies properties (i) and (iii)-(vi), and for every $j \in [r]$ there exists $i \in [6r]$ such that $S_j = T_i$, we obtain that the sets S_i satisfy these properties as well. This concludes the proof of the lemma. \square

Lemma 2 essentially states that large treewidth yields a large number of vertex disjoint cycles that are highly connected to each other, and an additional disjoint set that is highly connected to these cycles. However, this, together with the assumption that W_4 does not immerse in G , implies that there cannot be a large flow between a vertex

of degree at least 4 and one of the cycles. We will combine this fact with the notion of important separators to obtain Lemma 3.

Definition 3. Let $X, Y \subseteq V(G)$ be vertices, $S \subseteq E(G)$ be an (X, Y) -separator, and let R be the set of vertices reachable from X in $G \setminus S$. We say that S is an important (X, Y) -separator if it is inclusion-wise minimal and there is no (X, Y) -separator S' with $|S'| \leq |S|$ such that $R' \subset R$, where R' is the set of vertices reachable from X in $G \setminus S'$.

Theorem 4. [4, 20] Let $X, Y \subseteq V(G)$ be two sets of vertices in graph G , let $k \geq 0$ be an integer, and let S_k be the set of all (X, Y) -important separators of size at most k . Then $|S_k| \leq 4^k$ and S_k can be constructed in time $|S_k| \cdot n^{O(1)}$.

Theorem 4 states that the number of important separators of a certain size is bounded. The next lemma combines this fact with Lemma 2.

Lemma 3. Let G be a graph such that G does not contain W_4 as an immersion, has no internal 3-edge cut and has a vertex u with $d(u) \geq 4$. Then the treewidth of G is upper bounded by a constant.

Proof. If G has treewidth at least $2^{18(6r)^2 \log(6r)}$ for $r \geq 60000$, then there exist sets $Z = \{z_1, \dots, z_r\}$, S_1, \dots, S_r and X that satisfy the properties of Lemma 2. Recall that F is an important separator if there is no separator F' such that $|F'| \leq |F|$ and the connected component of $G - F$ that contains u is properly contained in the connected component of $G - F'$ that contains u . Additionally, observe that for every set S_i , there is an important separator F of order at most 6 such that S_i lies in the same connected component as $\{u\}$ in $G - F$. Moreover, for any cut F of order at most 6 such that S_i is contained in the same connected component as u in $G - F$, there cannot be 7 disjoint paths from u to X through F . Combined with property (vii) of Lemma 2 and the fact that every set S_i contains a vertex z_i , this implies that for every important separator F , there are at most 6 sets S_{i_1}, \dots, S_{i_p} , $p \leq 6$, that are contained in the same connected component as u in $G - F$. However, Theorem 4 ensures that there are at most 4^6 important $(X, \{u\})$ -separators of size at most 6 in G . Therefore, if $r \geq 60000 > 6 \cdot 4^6$, there is a set S_i such that the cut $(S_i, G - S_i)$ has order at least 7. Thus, we conclude that either G has an internal edge cut of order at most 3, or it has no vertex of degree at least 4, or it contains W_4 as an immersion. Hence the lemma holds. \square

We are now ready to prove the main theorem of our paper.

Theorem 5. Let G be a graph that does not contain W_4 as an immersion. Then the prime graphs of a decomposition of G via i -edge-sums, $i \in [3]$, are either subcubic graphs, or have treewidth upper bounded by a constant.

Proof. Let us consider a decomposition of G via i -edge-sums, $i \in [3]$, and let H be a prime graph of such a decomposition. Note first that, since G does not contain W_4 as an immersion, then H does not contain it either, due to Theorem 3. Now, assume that H is not subcubic. Then there is a vertex u of degree at least 4 in H . Moreover, it is clear from Theorem 3 that H is internally 4 edge-connected. Hence, we may apply Lemma 3 and conclude that H has treewidth at most $2^{2^{13} \cdot 3^6 \cdot 5^8 \cdot \log(2^6 \cdot 3^2 \cdot 5^4)}$. Thus, the theorem holds. \square

We conclude this section by noting that Theorem 5 is in a sense tight: indeed, both the fact that we decompose along edge-sums of order at most 3 and the requirement that a unique vertex of degree at least 4 is sufficient to enforce small treewidth are necessary. The fact that decomposing along internal 3-edge-sums is necessary can be seen from the fact that there are internally 3 edge-connected graphs that have vertices of degree at least 4 and yet do not contain W_4 as an immersion, e.g., a cycle where every edge is doubled.

5 Concluding remarks

Following the proof of Theorem 5, the first task is to improve the bound on the treewidth of internally 4 edge-connected graphs that exclude W_4 as an immersion and have a vertex of degree at least 4. Our proof of Theorem 5 relies on the fact that large treewidth ensures the existence of a large number of vertex disjoint cycles that are highly connected to each other. In order to obtain these cycles, we use the fact that graphs of large treewidth contain a large wall as a topological minor. However, the value of treewidth required to find a sufficiently large wall is currently enormous. Avoiding to rely on the existence of a large wall would be an efficient way to drastically reduce the constants in Lemma 2 and Theorem 5.

Another question that we leave open is to prove a similar result for larger wheels, i.e., W_k for $k \geq 5$. Providing a decomposition theorem for larger wheel seems to be a challenging task, as edge-sums no longer seem to be the proper way to proceed, since, as argued in Section 4, k edge connectivity is necessary, but W_k -immersion is not preserved under edge-sums of order $k - 1$, as seen in Figure 2 and 3.

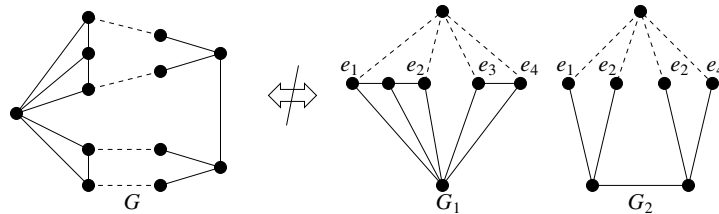


Fig. 2. G_1 contains W_5 as an immersion but $G = G_1 \hat{\oplus}_4 G_2$ does not. The unique vertex in G_i incident with dotted edges is the vertex v_i , and the edge-sum maps to each other edges of G_1 and G_2 with the same label.

Decomposition theorems exist when small wheels are excluded as topological minors [9, 27], however these results do not apply when excluding wheels as immersions, as in this case we must consider multigraphs. A similar important question is to characterize graphs excluding K_5 as an immersion.

Finally, note that the general algorithm to test immersion containment runs in cubic time for every fixed target graph H . We believe that Theorem 5 can be used to devise

efficient algorithms to recognize graphs that exclude W_4 as an immersion. A direct application of the construction of Theorem 5 implies that this can be done in time $\tilde{O}(n^2)$. However, we believe that this can be further improved.

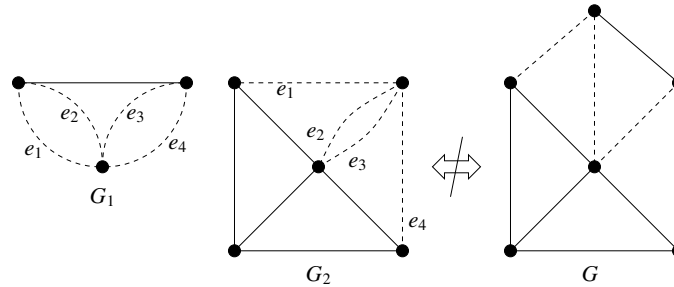


Fig. 3. Neither G_1 nor G_2 contain W_4 as immersion but $G = G_1 \hat{\oplus}_4 G_2$ does. The unique vertex in G_i incident with dotted edges is the vertex v_i , and the edge-sum maps to each other edges of G_i and G_2 with the same label.

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