



## On disconnected cuts and separators<sup>☆</sup>

Takehiro Ito<sup>a</sup>, Marcin Kamiński<sup>b</sup>, Daniël Paulusma<sup>c,\*</sup>, Dimitrios M. Thilikos<sup>d</sup>

<sup>a</sup> Graduate School of Information Sciences, Tohoku University, Aobayama 6-6-05, Sendai, 980-8579, Japan

<sup>b</sup> Computer Science Department, Université Libre de Bruxelles, Boulevard du Triomphe CP212, B-1050 Brussels, Belgium

<sup>c</sup> Department of Computer Science, University of Durham, Science Laboratories, South Road, Durham DH1 3LE, England, United Kingdom

<sup>d</sup> Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis, GR15784 Athens, Greece

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### ABSTRACT

For a connected graph  $G = (V, E)$ , a subset  $U \subseteq V$  is called a disconnected cut if  $U$  disconnects the graph, and the subgraph induced by  $U$  is disconnected as well. A natural condition is to impose that for any  $u \in U$ , the subgraph induced by  $(V \setminus U) \cup \{u\}$  is connected. In that case,  $U$  is called a minimal disconnected cut. We show that the problem of testing whether a graph has a minimal disconnected cut is NP-complete. We also show that the problem of testing whether a graph has a disconnected cut separating two specified vertices,  $s$  and  $t$ , is NP-complete.

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### 1. Introduction

Graph connectivity is a fundamental graph-theoretic property that is well studied in the context of network robustness. In the literature, several measures for graph connectivity are known, such as requiring hamiltonicity, edge-disjoint spanning trees, and edge- or vertex-cuts of sufficiently large size.

Let  $G = (V, E)$  be a connected simple graph. For a subset  $U \subseteq V$ , we denote by  $G[U]$  the subgraph of  $G$  induced by  $U$ . We say that  $U$  is a *cut* of  $G$  if  $U$  disconnects  $G$ , that is,  $G[V \setminus U]$  contains at least two (connected) components. A cut  $U$  is *connected* if  $G[U]$  contains exactly one component, and *disconnected* if  $G[U]$  contains at least two components. We observe that  $G[U]$  is a disconnected cut if and only if  $G[V \setminus U]$  is a disconnected cut.

In our paper [6], we studied the following three problems. The DISCONNECTED CUT problem is to test whether a connected graph has a disconnected cut. For a fixed integer  $k$ , the  $k$ -CUT problem is to test whether a connected graph  $G = (V, E)$  has a cut  $U$  such that  $G[U]$  contains exactly  $k$  components. For fixed integers  $k, \ell$ , the  $(k, \ell)$ -CUT problem is to test whether a connected graph  $G = (V, E)$  has a cut  $U$  such that  $G[U]$  and  $G[V \setminus U]$  contain exactly  $k$  and  $\ell$  components, respectively. We showed that the  $k$ -CUT problem is polynomial-time solvable if  $k = 1$ , and NP-complete if  $k \geq 2$ . We also showed that the  $(k, \ell)$ -CUT problem is polynomial-time solvable if  $k = 1$  or  $\ell = 1$ , and NP-complete otherwise.

The complexity of the DISCONNECTED CUT problem is still open for general graphs, but we showed that the problem can be solved in polynomial time for planar graphs, claw-free graphs and chordal graphs [6]. In addition, Fleischner et al. [5] showed that DISCONNECTED CUT is polynomial-time solvable for triangle-free graphs, graphs with bounded maximum degree, graphs

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\* Corresponding author. Tel.: +44 (0)191 33 41723; fax: +44 (0)191 33 41701.

E-mail addresses: [takehiro@ecei.tohoku.ac.jp](mailto:takehiro@ecei.tohoku.ac.jp) (T. Ito), [marcin.kaminski@ulb.ac.be](mailto:marcin.kaminski@ulb.ac.be) (M. Kamiński), [daniel.paulusma@durham.ac.uk](mailto:daniel.paulusma@durham.ac.uk) (D. Paulusma), [sedthilk@math.uoa.gr](mailto:sedthilk@math.uoa.gr) (D.M. Thilikos).

with a dominating edge (including co-graphs) and graphs that are not locally connected. In particular, they show that every graph of diameter at least three has a disconnected cut.

The DISCONNECTED CUT problem is equivalent to several other problems posed in the literature. A graph  $G$  has a disconnected cut if and only if  $G$  allows a vertex-surjective homomorphism to the reflexive 4-vertex cycle. Furthermore, if  $G$  has diameter two, then  $G$  has a disconnected cut if and only if  $G$  allows a compaction to the reflexive 4-vertex cycle if and only if  $G$  can be contracted to some biclique. We refer to our paper [6] for more details. Here, we also mention that a graph  $G = (V, E)$  has a disconnected cut if and only if its complement  $\bar{G} = (V, \{uv \mid uv \notin E\})$  has a spanning subgraph that consists of two bicliques [5].

The DISCONNECTED CUT problem is also studied in the context of  $H$ -partitions as introduced by Dantas et al. [1]. A model graph  $H$  with  $V_H = \{h_1, \dots, h_k\}$  has two types of edges: solid and dotted edges, and an  $H$ -partition of a graph  $G$  is a partition of  $V_G$  into  $k$  (nonempty) sets  $V_1, \dots, V_k$ , such that for all vertices  $u \in V_i, v \in V_j$  and for all  $1 \leq i < j \leq k$ , the following two conditions hold. First, if  $h_i h_j$  is a solid edge of  $H$ , then  $uv \in E_G$ . Second, if  $h_i h_j$  is a dotted edge of  $H$ , then  $uv \notin E_G$ . There are no such restrictions when  $h_i$  and  $h_j$  are not adjacent. Let  $2K_2$  be the model graph with vertices  $h_1, \dots, h_4$  and two solid edges  $h_1 h_3, h_2 h_4$ , and  $2S_2$  be the model graph with vertices  $h_1, \dots, h_4$  and two dotted edges  $h_1 h_3, h_2 h_4$ . Then a graph  $G$  has a disconnected cut if and only if  $G$  has a  $2S_2$ -partition if and only if its complement  $\bar{G}$  has a  $2K_2$ -partition. The (equivalent) cases  $H = 2K_2$  and  $H = 2S_2$  are the only two cases of model graphs on at most four vertices, whose computational complexity is still open. Especially,  $2K_2$ -partitions have been well studied; see e.g. two very recent papers of Dantas et al. [2] and Teixeira et al. and de Figueiredo [7]. The first paper [2] studies the  $2K_2$ -PARTITION problem for several graph classes and the second paper [7] defines a new class of problems called  $2K_2$ -hard.

In this manuscript, we study three natural variants of the DISCONNECTED CUT problem in order to increase our understanding of this problem. Our study is also motivated by the following example. Let  $P_n$  denote the path on  $n$  vertices. We observe that  $P_4 = p_1 p_2 p_3 p_4$  has a disconnected cut  $\{p_1, p_3\}$  and a disconnected cut  $\{p_2, p_4\}$ . We observe that both these cuts contain a vertex, namely  $p_1$  and  $p_4$ , respectively, such that moving this vertex from the cut back into the graph keeps the graph disconnected. As such, the property of the cut being disconnected can be viewed to be somewhat artificial in this case. Therefore, we can define the following problem, where we call a disconnected cut  $U$  of a connected graph  $G = (V, E)$  minimal if  $G[(V \setminus U) \cup \{u\}]$  is connected for every  $u \in U$ .

**MINIMAL DISCONNECTED CUT**

*Instance:* a connected graph  $G$

*Question:* does  $G$  have a minimal disconnected cut  $U$ ?

We can relax the minimality by defining a disconnected cut  $U$  of a connected graph  $G = (V, E)$  to be *semi-minimal* if  $G[(V \setminus U) \cup \{u\}]$  contains fewer components than  $G[V \setminus U]$  for every  $u \in U$ . This leads to the problem:

**SEMI-MINIMAL DISCONNECTED CUT**

*Instance:* a connected graph  $G$

*Question:* does  $G$  have a semi-minimal disconnected cut  $U$ ?

We note that any minimal disconnected cut is semi-minimal. However, the reverse is not true; to illustrate the differences between these two problems and the DISCONNECTED CUT problem we observe the following:

- (i) The path  $P_k$  has a disconnected cut if and only if  $k \geq 4$ .
- (ii) The path  $P_k$  has a semi-minimal disconnected cut if and only if  $k \geq 5$ .
- (iii) The path  $P_k$  does not have a minimal disconnected cut for any  $k \geq 1$ .

Because a minimal disconnected cut of a graph  $G$  does not contain a cut-vertex of  $G$ , we can generalize (iii) to the following statement: every connected graph that contains a cut-vertex in all its cuts has no minimal disconnected cut. We will show that the MINIMAL CUT and SEMI-MINIMAL CUT problem are NP-complete.

An  $s$ - $t$  separator of a connected graph  $G$  with two specified vertices  $s$  and  $t$  is a cut  $U$  such that  $s$  and  $t$  belong to two different components of  $G[V \setminus U]$ . We say that an  $s$ - $t$  separator  $U$  is *disconnected* if  $U$  is a disconnected cut.

**DISCONNECTED SEPARATOR**

*Instance:* a graph  $G = (V, E)$  and two vertices  $s, t \in V$

*Question:* does  $G$  have a disconnected  $s$ - $t$  separator  $U$ ?

We will prove that the DISCONNECTED SEPARATOR problem is NP-complete.

## 2. Preliminaries

The graphs that we consider are undirected and without multiple edges. We assume that they may contain self-loops. For undefined (standard) graph terminology we refer to [3].

Let  $G = (V, E)$  be a graph. Each maximal connected subgraph of  $G$  is called a *component* of  $G$ . For a vertex  $u \in V$ , we denote its neighborhood, i.e., the set of its adjacent vertices, by  $N(u) = \{v \mid uv \in E\}$ . Two disjoint nonempty subsets  $U, U' \subset V$  are *adjacent* if there exist vertices  $u \in U$  and  $u' \in U'$  with  $uu' \in E$ . The *distance*  $d_G(u, v)$  between two vertices  $u$  and  $v$  in a graph  $G$  is the number of edges in a shortest path between them. The *diameter*  $\text{diam}(G)$  is defined as  $\max\{d_G(u, v) \mid u, v \in V\}$ . We say that  $S \subset V$  is *separated* from  $T \subset V$  by  $W \subset V \setminus (S \cup T)$  if every path that starts in a vertex of  $S$  and that ends in a vertex of  $T$  uses at least one vertex from  $W$ .

Let  $U$  be a cut of a graph  $G$ . If  $G[(V \setminus U) \cup \{u\}]$  is connected, we say that  $u$  is a *minimal* vertex of  $U$ . If  $G[(V \setminus U) \cup \{u\}]$  contains fewer components than  $G[V \setminus U]$ , we say that  $u$  is a *semi-minimal* vertex of  $U$ .

A graph is *reflexive* if it has a self-loop in every vertex. We denote the reflexive  $n$ -vertex cycle by  $C_n$ . A graph with no self-loops is called *irreflexive*.

Let  $f : V_G \rightarrow V_H$  be a (graph) *homomorphism* from a graph  $G$  to a graph  $H$ , i.e.,  $f(u)f(v) \in E_H$  whenever  $uv \in E_G$ . We say that  $f$  is *vertex-surjective* if  $f(V_G) = V_H$ . Here we used the shorthand notation  $f(S) = \{f(u) \mid u \in S\}$  for a subset  $S \subseteq V$ . We say that  $f$  is a *compaction* if  $f$  is *edge-surjective*, i.e., for every edge  $xy \in E_H$  with  $x \neq y$ , there exist two adjacent vertices  $u, v$  with  $f(u) = x$  and  $f(v) = y$ . We stress that the surjectivity condition only holds for edges  $xy \in E_H$ ; there is no such condition on the self-loops  $xx \in E_H$ . If  $f$  is a compaction from  $G$  to  $H$ , we also say that  $G$  *compacts* to  $H$ .

Let  $H$  be an induced subgraph of a graph  $G$ . A homomorphism  $f$  from a graph  $G$  to  $H$  is a *retraction* from  $G$  to  $H$  if  $f(h) = h$  for all  $h \in V_H$ . In that case, we say that  $G$  *retracts* to  $H$ .

The  $H$ -COMPACTION problem asks if a graph  $G$  compacts to a fixed graph  $H$ , i.e.,  $H$  is not part of the input. The  $H$ -RETRACTION problem asks if a graph  $G$  retracts to a fixed graph  $H$ . The following two results proven by Feder and Hell [4] and Vikas [8], respectively, are of importance to us.

**Theorem 1** ([4]). *The  $C_4$ -RETRACTION problem is NP-complete.*

**Theorem 2** ([8]). *The  $C_4$ -COMPACTION problem is NP-complete.*

### 3. Gadgets

In the remainder of this paper, the graph  $H$  denotes the reflexive 4-vertex cycle  $h_0h_1h_2h_3h_0$  with self-loops  $h_ih_i$  for  $i = 0, \dots, 3$ , and the graph  $G = (V, E)$  denotes a graph that contains  $H$  as an induced subgraph.

For each vertex  $v \in V_G \setminus V_H$ , we add three new vertices  $u_v, w_v, y_v$  with edges  $h_0u_v, h_0y_v, h_1u_v, h_2w_v, h_2y_v, h_3w_v, u_vv, u_vw_v, u_vy_v, vw_v, w_vy_v$ . We also add all edges between any two vertices  $u_v, u_{v'}$  and between any two vertices  $w_v, w_{v'}$  with  $v \neq v'$ . For each edge  $vv'$  in  $E_G \setminus E_H$ , we choose one arbitrary direction, say from  $v$  to  $v'$ , and then add a new vertex  $x_{vv'}$  with edges  $vx_{vv'}, v'x_{vv'}, u_vx_{vv'}, w_{v'}x_{vv'}$ . We call the new graph  $G'$  obtained from  $G$  an  $H$ -*compactor* of  $G$ . See Fig. 1 for an example. This figure does not depict any self-loops, although formally  $G$  must have at least four self-loops, because  $G$  contains  $H$  as an induced subgraph. However, this is irrelevant for our problems, and we may just as well assume that  $G$  is irreflexive.

Vikas [8] proves Theorem 2 by a reduction from  $H$ -RETRACTION, which is NP-complete by Theorem 1. In his proof he shows the following result, which we will use as well.

**Lemma 1** ([8]). *Let  $G'$  be an  $H$ -compactor of a graph  $G$  that has  $H$  as an induced subgraph. Then the following statements are equivalent:*

- (i)  $G$  retracts to  $H$ ;
- (ii)  $G'$  retracts to  $H$ ;
- (iii)  $G'$  compacts to  $H$ .

Below we explore the properties of a retraction  $f$  from an  $H$ -compactor  $G'$  to  $H$ . We call a subgraph of  $G'$ , every vertex of which is mapped to the same vertex  $h_i$  by  $f$  *monochromatic*.

**Lemma 2.** *Let  $G'$  be an  $H$ -compactor of a graph  $G$  that has  $H$  as an induced subgraph. Any retraction  $f$  from  $G'$  to  $H$  satisfies:*

- (i) for  $i = 0, \dots, 3$ , the subgraph  $G'_i$  induced by  $\{u \in V_{G'} \mid f(u) = h_i\}$  is connected;
- (ii) for  $i = 0, \dots, 3$ , each vertex  $u$  with  $f(u) = h_i$  has a neighbor  $v$  with  $f(v) = h_j$  for some  $j \neq i$ .

**Proof.** Let  $G'$  be an  $H$ -compactor of a graph  $G$  with  $H$  as an induced subgraph. Let  $f$  be a retraction from  $G'$  to  $H$ . We prove that (i) and (ii) hold.

*Proof of (i).* By definition,  $f(h_i) = h_i$  for  $i = 0, \dots, 3$ . This means that  $f$  maps  $u_v$ -vertices to  $h_0$  and  $h_1$ , and  $w_v$ -vertices to  $h_2$  and  $h_3$ . It also means that  $f$  maps  $y_v$ -vertices to  $h_1$  or  $h_3$ .

We first prove the following claim.

*Claim 1.* For every  $v \in V_G \setminus V_H$ , if  $f(v) \in \{h_0, h_1\}$  then  $f(u_v) = f(v)$ , and if  $f(v) \in \{h_2, h_3\}$  then  $f(w_v) = f(v)$ .

We prove Claim 1 as follows. Suppose  $f(v) \in \{h_0, h_1\}$  and  $f(u_v) \neq f(v)$ . Recall that  $f(u_v) \in \{h_0, h_1\}$  and  $f(w_v) \in \{h_2, h_3\}$ . Then  $f$  maps  $u_v, v, w_v$  to three different vertices of  $H$ . This is not possible, because  $u_v, v, w_v$  form a triangle in  $G'$ . By the same argument, we can show that  $f(w_v) = f(v)$  if  $f(v) \in \{h_2, h_3\}$ . This proves Claim 1.

We now show that  $G'_0$  is connected. Let  $V_0$  denote the vertex set of  $G'_0$ . Let  $z \neq h_0$  be a vertex in  $V_0$ , so  $f(z) = h_0$ . We show that  $z$  is in the same component of  $G'_0$  as  $h_0$ . This means that  $G'_0$  is connected as desired.

Suppose  $z$  is a  $u_v$ -vertex. Then  $z$  is adjacent to  $h_0$ . Note that  $z$  is neither a  $w_v$ -vertex nor a  $y_v$ -vertex, because such a vertex is mapped to a vertex in  $\{h_2, h_3\}$  or  $\{h_1, h_3\}$ , respectively. Suppose  $z = v$  for some  $v \in V_G \setminus V_H$ . By Claim 1, we find that  $f(u_v) = f(v) = h_0$ . Then  $v$  is in the same component of  $G[V_0]$  as  $h_0$  due to the monochromatic path  $vu_vh_0$ .

Finally, suppose  $z = x_{vv'}$  for two adjacent vertices  $v, v' \in V_G \setminus V_H$ . If  $f(u_v) = h_0$ , then  $x_{vv'}$  is connected to  $h_0$  in  $G[V_0]$  due to the path  $x_{vv'}u_vh_0$ . If  $f(u_v) \neq h_0$  then  $f(u_v) = h_1$ . Because  $v$  is adjacent to  $x_{vv'}$  with  $f(x_{vv'}) = h_0$  and to  $u_v$  with  $f(u_v) = h_1$ ,

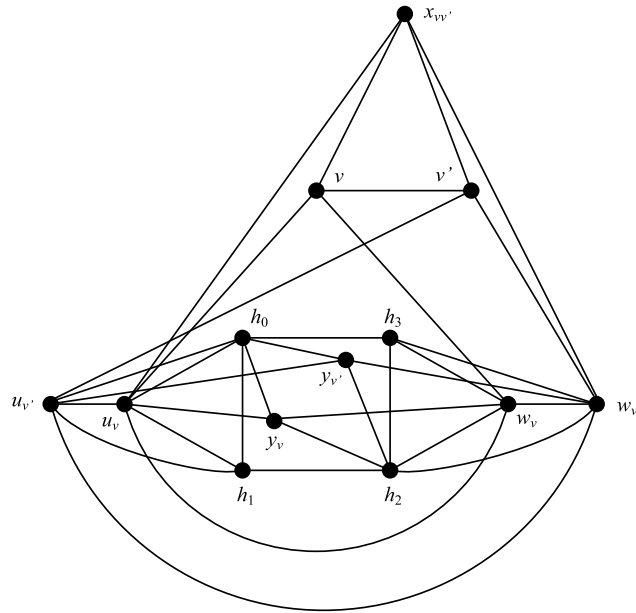


Fig. 1. The part of  $G'$  that corresponds to edge  $vv' \in E_G \setminus E_H$  as displayed in [8].

we obtain  $f(v) \in \{h_0, h_1\}$ . Then by Claim 1,  $f(v) = f(u_v) = h_1$ . Because  $f(x_{vv'}) = h_0$  and  $f(w_{v'}) \in \{h_2, h_3\}$ , we find that  $f(w_{v'}) = h_3$ . Then  $v'$  is adjacent to three vertices, namely  $x_{vv'}$ ,  $v$ ,  $w_{v'}$ , that are mapped to  $h_0$ ,  $h_1$ ,  $h_3$ , respectively. This means that  $f(v') = h_0$ . Consequently,  $f(u_{v'}) = f(v') = h_0$  by Claim 1. Hence,  $x_{vv'}$  is in the same component of  $G[V_0]$  as  $h_0$  due to the monochromatic path  $x_{vv'}v'u_{v'}h_0$ .

From the above we conclude that  $G'_0$  is connected. By symmetry, we find that  $G'_2$  is connected as well. We now show that  $G'_1$  is connected.

Let  $z \neq h_0$  be a vertex in  $V_1$ , so  $f(z) = h_1$ . We show that  $z$  is in the same component of  $G'_1$  as  $h_1$  by the same arguments as we used for  $i = 0$ ; the only difference is the argument for the case in which  $z$  is a  $y_v$ -vertex. In that case  $z$  is connected to  $h_1$  by the edge  $h_1z$ . Hence, we conclude that  $G'_1$  is connected. By symmetry, we find that  $G'_3$  is connected as well. Consequently, we have shown (i).

*Proof of (ii).* Let  $z$  be a vertex in  $G'$ . Suppose  $f(z) = h_0$ . Then  $z$  is neither a  $w_v$ -vertex nor a  $y_v$ -vertex, and  $z$  is not in  $\{h_1, h_2, h_3\}$  either, because  $f$  does not map such vertices to  $h_0$ . If  $z$  is  $h_0$  or a  $u_v$ -vertex, then  $z$  is adjacent to  $h_1$  with  $f(z) = h_1$ . Otherwise,  $z \in V_G \setminus V_H$  or  $z = x_{vv'}$  for some  $vv' \in E_G \setminus E_H$ . In both cases,  $z$  is adjacent to a  $w_v$ -vertex, which  $f$  maps to  $h_2$  or  $h_3$ . The case  $f(z) = h_2$  follows by symmetry.

Suppose  $f(z) = h_1$ . We can use the same arguments as in the previous case; the only difference is when  $z$  is a  $y_v$ -vertex. In that case  $z$  is adjacent to  $h_0$  with  $f(h_0) = h_0$ . The case  $f(z) = h_3$  follows by symmetry. Consequently, we have shown (ii). This completes the proof of Lemma 2.  $\square$

The following lemma will be used later on as well, in order to strengthen our NP-hardness results. We note that it also strengthens Theorem 2, i.e., the  $H$ -COMPACTION problem is NP-complete, even for graphs of diameter 3.

**Lemma 3.** *Let  $G$  be a graph that has  $H$  as induced subgraph. The  $H$ -compactor of  $G$  has diameter three.*

**Proof.** Let  $G'$  be the  $H$ -compactor of  $G$  that has  $H$  as an induced subgraph. We choose that  $G'$  has diameter 3 by a straightforward case analysis.

Consider a vertex  $h_i \in V_H$ . By symmetry, we may assume  $i \in \{0, 1\}$ . As  $H$  is isomorphic to  $C_4$ , we have  $d(h_i, h_j) \leq 2$  for all  $h_j \in H \setminus \{h_i\}$ . Suppose  $v \in V_G \setminus V_H$ . Then  $d(h_i, v) \leq 2$  and  $d(h_i, u_v) = 1$  due to the path  $h_iu_vv$ . We also deduce  $d(h_i, w_v) = 2$  due to the path  $h_iu_vw_v$ , and  $d(h_i, y_v) \leq 2$  due to the path  $h_iy_v$  if  $i = 0$  or  $h_ih_{i-1}y_v$  if  $i = 1$ . Furthermore,  $d(h_i, x_{v'v''}) = 2$  holds for any  $v'v'' \in E_G \setminus E_H$  due to the path  $h_iu_{v'}x_{v'v''}$ .

Consider a vertex  $v \in V_G \setminus V_H$ . By construction,  $d(v, u_v) = d(v, u_w) = 1$ . We deduce  $d(v, y_v) = 2$  due to the path  $vu_vy_v$ , and for all  $vv' \in E_G \setminus E_H$  we have  $d(v, x_{vv'}) = 1$  due to the edge  $vx_{vv'}$ . Suppose  $v' \in V_G \setminus (V_H \cup \{v\})$ . Then  $d(v, v') \leq 3$  and  $d(v, u_{v'}) = 2$  due to the path  $vu_vu_{v'}v'$ . Also,  $d(v, w_{v'}) = 2$  due to the path  $vw_vw_{v'}$ , and  $d(v, y_{v'}) \leq 3$  due to the path  $vu_vu_{v'}y_{v'}$ . Furthermore,  $d(v, x_{v'v''}) \leq 3$  for all  $v'v'' \in E_G \setminus E_H$  due to the path  $vu_vu_{v'}x_{v'v''}$ .

Consider a vertex  $u_v$  for some  $v \in V_G \setminus V_H$ . By construction,  $d(u_v, w_v) = d(u_v, y_v) = 1$  and also  $d(u_v, x_{vv'}) = 1$  for all  $vv' \in E_G \setminus E_H$ . Suppose  $v' \in V_G \setminus (V_H \cup \{v\})$ . Then  $d(u_v, u_{v'}) = 1$  by the edge  $u_vu_{v'}$ , and  $d(u_v, w_{v'}) = 2$  due to the path  $u_vu_{v'}w_{v'}$ , and  $d(u_v, y_{v'}) = 2$  due to the path  $u_vu_{v'}y_{v'}$ . Furthermore,  $d(u_v, x_{v'v''}) = 2$  for all  $v'v'' \in E_G \setminus E_H$  with  $v' \neq v$  due to the path  $u_vu_{v'}x_{v'v''}$ .

Consider a vertex  $w_v$  for some  $v \in V_G \setminus V_H$ . By symmetry, we return to the previous case.

Consider a vertex  $y_v$  for some  $v \in V_G \setminus V_H$ . Then  $d(y_v, x_{vv'}) \leq 2$  for all  $vv' \in E_G \setminus E_H$  due to the path  $y_v u_v x_{vv'}$ . Suppose  $v' \in V_G \setminus (V_H \cup \{v\})$ . Then  $d(y_v, y_{v'}) = 2$  due to the path  $y_v h_0 y_{v'}$ . Furthermore,  $d(y_v, x_{v'v''}) \leq 3$  for all  $v'v'' \in E_G \setminus E_H$  with  $v' \neq v$  due to the path  $y_v u_v u_{v'} x_{v'v''}$ .

Consider a vertex  $x_{vv'}$  for some  $vv' \in E_G \setminus E_H$ . Suppose  $v''v^* \in E_G \setminus (E_H \cup \{vv'\})$ . Then  $d(x_{vv'}, x_{v''v^*}) \leq 3$  due to the path  $x_{vv'} u_v u_{v''} x_{v''v^*}$  if  $v \neq v''$ ; otherwise we can take the path  $x_{vv'} u_v x_{v''v^*}$ . This completes our case analysis, and we have proven Lemma 3.  $\square$

#### 4. NP-completeness proofs

We first prove the following result on  $H$ -compactors.

**Lemma 4.** *Let  $G'$  be the  $H$ -compactor of a graph  $G$  that has  $H$  as an induced subgraph. Then the following three statements are equivalent.*

- (i)  $G'$  compacts to  $H$ .
- (ii)  $G'$  has a minimal disconnected cut.
- (iii)  $G'$  has a semi-minimal disconnected cut.

**Proof.** Let  $G'$  be the  $H$ -compactor of a graph  $G$  that has  $H$  as an induced subgraph.

(i)  $\Rightarrow$  (ii). Suppose  $G'$  compacts to  $H$ . Then by Lemma 1 there exists a retraction  $f$  from  $G'$  to  $H$ . Then  $f$  partitions  $V_{G'}$  into four classes  $V_i = \{u \in V \mid f(u) = h_i\}$  for  $i = 0, \dots, 3$ . By Lemma 2 (i), each  $V_i$  induces a connected subgraph of  $G'$ .

Consider  $V_0$ . We repeatedly perform the following operation as long as possible. Let  $v \in V_0$ . By Lemma 2 (ii),  $v$  has at least one neighbor in  $V_1 \cup V_3$ . If  $v$  is adjacent to a vertex in  $V_1$  but not adjacent to any vertex in  $V_3$ , then put  $v$  in  $V_1$ . Similarly, if  $v$  is adjacent to a vertex in  $V_3$  but not adjacent to any vertex in  $V_1$ , put  $v$  in  $V_3$ . Afterward we end up with a subset  $V'_0 \subseteq V_0$  that only contains vertices that have a neighbor in both  $V_1$  and  $V_3$ . We note that  $h_0 \in V'_0$ , because  $h_0$  is in  $V_0$ , and  $h_0$  is adjacent to  $h_1 \in V_1$  and  $h_3 \in V_3$ . Hence,  $V'_0 \neq \emptyset$ .

By the same arguments, we modify  $V_2$  into a nonempty set  $V'_2$  in which all vertices have a neighbor in  $V_1$  and a neighbor in  $V_3$ . Note that the above operations do not introduce an edge between  $V_1$  and  $V_3$ . They do not introduce an edge between  $V'_0$  and  $V'_2$  either. Furthermore,  $V_1$  and  $V_3$  still induce connected subgraphs of  $G'$ . Because every vertex in  $V'_0 \cup V'_2$  is adjacent to a vertex in  $V_1$  and to a vertex in  $V_3$ , this means that  $V'_0 \cup V'_2$  is a minimal disconnected cut of  $G'$ .

(ii)  $\Rightarrow$  (iii). This follows directly from the two definitions.

(iii)  $\Rightarrow$  (i). Suppose  $G'$  has a semi-minimal disconnected cut  $U$ . Let the components of  $G'[U]$  be  $A_1, \dots, A_k$  for some  $k \geq 2$ . Let the components of  $G'[V \setminus U]$  be  $B_1, \dots, B_\ell$  for some  $\ell \geq 2$ . Because  $U$  is semi-minimal, every vertex  $u \in A_1$  has a neighbor in at least two components  $B_i$  and  $B_j$  for some  $1 \leq i < j \leq \ell$ . By the same reasoning, every vertex  $v \in A_2$  has a neighbor in at least two components  $B_{i'}$  and  $B_{j'}$  for some  $1 \leq i' < j' \leq \ell$ . Because  $i \neq j$  and  $i' \neq j'$ , we may assume without loss of generality that  $i \neq j'$  and  $i' \neq j$ ; otherwise we swap two indices.

We define the function  $f$  that maps each vertex in  $A_1$  to  $h_0$ , each vertex in  $A_2 \cup \dots \cup A_k$  to  $h_2$ , each vertex in  $B_i \cup B_{i'}$  to  $h_1$ , and each vertex in  $B_j \cup B_{j'}$  to  $h_3$ . We let  $f$  map all remaining vertices of  $V_{G'} \setminus U$  to  $h_3$  as well. By our choice of indices  $i, i', j, j'$ , we find that  $f$  is a compaction from  $G'$  to  $H$ . This finishes the proof of Lemma 4.  $\square$

We are now able to show the first main result of this section.

**Theorem 3.** *The MINIMAL DISCONNECTED CUT and the SEMI-MINIMAL DISCONNECTED CUT problem are NP-complete, even for the class of graphs of diameter three.*

**Proof.** Note that both problems are in NP. To prove NP-completeness, we use a reduction from the  $\mathcal{C}_4$ -RETRACTION problem, which is NP-complete by Theorem 1. Let  $G$  be a graph that has  $H$  as an induced subgraph. Let  $G'$  be an  $H$ -compactor of  $G$ . By Lemma 3,  $G'$  has diameter three. By Lemmas 1 and 4 we find that  $G$  retracts to  $H$  if and only if  $G'$  compacts to  $H$  if and only if  $G'$  has a minimal disconnected cut if and only if  $G'$  has a semi-minimal disconnected cut. This proves Theorem 3.  $\square$

Here is our second main result.

**Theorem 4.** *The DISCONNECTED SEPARATOR problem is NP-complete even for the class of graphs of diameter 3.*

**Proof.** Note that this problem is in NP. To prove NP-completeness, we use a reduction from the  $\mathcal{C}_4$ -RETRACTION problem, which is NP-complete by Theorem 1. Let  $G$  be a graph that has  $H$  as an induced subgraph. Let  $G'$  be an  $H$ -compactor of  $G$ . By Lemma 3,  $G'$  has diameter three. We claim that  $G$  retracts to  $H$  if and only if  $G'$  has a disconnected  $h_0$ - $h_2$  separator.

Suppose  $G$  retracts to  $H$ . By Lemma 1, there exists a retraction  $f$  from  $G'$  to  $H$ . Let  $V_i = \{x \in V_{G'} \mid f(x) = h_i \text{ for } i = 0, \dots, 3\}$ . By definition,  $h_0 \in V_0$  and  $h_2 \in V_2$ , and there are no edges between  $V_0$  and  $V_2$ , and no edges between  $V_1$  and  $V_3$ . Because  $h_1 \in V_1$  and  $h_3 \in V_3$  by definition,  $V_1$  is nonempty and  $V_3$  is nonempty. Hence  $V_1 \cup V_3$  is a disconnected  $h_0$ - $h_2$  separator of  $G'$ .

In order to prove the reverse implication, suppose  $G'$  has a disconnected  $h_0$ - $h_2$  separator  $U$ . Let  $A_1, \dots, A_k$  be the vertex sets of the components of  $G[U]$  and let  $B_1, \dots, B_\ell$  be the vertex sets of the components of  $G[V \setminus U]$ . As  $U$  is an  $h_0$ - $h_2$  separator,

we may without loss of generality assume that  $h_0 \in B_1$  and  $h_2 \in B_2$ . Because  $h_1$  and  $h_3$  are each adjacent to both  $h_0$  and  $h_1$ , we find that  $h_1$  and  $h_3$  are in  $V \setminus U$ , say  $h_1 \in A_1$  and  $h_3 \in A_i$  for some  $i \geq 1$ ; note that we must consider the case  $h_3 \in A_1$  as a possibility.

Define  $f : V_G \rightarrow V_H$  as follows. Let  $f$  map each vertex of  $B_1$  to  $h_0$ , each vertex of  $B_2 \cup \dots \cup B_\ell$  to  $h_2$ , each vertex of  $A_1$  to  $h_1$  and each vertex of  $A_2 \cup \dots \cup A_k$  to  $h_3$ . We observe that  $f$  is a homomorphism to  $H$  with  $f(h_i) = h_i$  for  $0 \leq i \leq 2$ . Because  $A_2 \cup \dots \cup A_k$  is nonempty, it contains a vertex  $z$ , which is mapped to  $h_3$ . If we can show that  $z$  is adjacent to a vertex mapped to  $h_0$  and to a vertex mapped to  $h_2$ , then we find that  $f$  is a compaction from  $G$  to  $H$ . Then, by Lemma 1,  $G$  retracts to  $H$ , and we are done. Below we consider each possibility.

We first note that  $z$  cannot be a  $u_v$ -vertex. The reason is that a  $u_v$ -vertex is mapped to a vertex in  $\{h_0, h_1\}$ , because it is adjacent to  $h_0$  with  $f(h_0) = h_0$  and to  $h_1$  with  $f(h_1) = h_1$ .

Suppose  $z = h_3$ . Then  $z$  is adjacent to  $h_0$ , which is mapped to  $h_0$ , and to  $h_2$ , which is mapped to  $h_2$ , as desired. Suppose we cannot choose  $z$  to be  $h_3$ . Then  $h_3 \in A_1$ , and consequently,  $f(h_3) = h_1$ .

Because  $f(h_3) = h_1$ , we find that  $z$  cannot be a  $w_v$ -vertex. The reason is that a  $w_v$ -vertex is also adjacent to  $h_2$  with  $f(h_2) = h_2$ . Hence, it must be mapped to a vertex in  $\{h_1, h_2\}$ .

Suppose  $z$  is a  $y_v$ -vertex. Then  $z$  is adjacent to both  $h_0$  with  $f(h_0) = h_0$  and  $h_2$  with  $f(h_2) = h_2$ , as desired. Suppose this is not the case.

Suppose  $z = v$  for some  $v \in V_G \setminus V_H$ . Recall that  $u_v$  is adjacent to  $h_0, h_1, v$ . Because  $f(h_0) = h_0$  and  $f(h_1) = 1$ , we then find that  $f(u_v) = h_0$ . Recall that  $w_v$  is adjacent to  $h_2, h_3, u_v, v$ , which are mapped to  $h_2, h_1, h_0, h_3$ , respectively. This is not possible. Hence,  $z$  cannot be in  $V_G \setminus V_H$ .

Suppose  $z = x_{vv'}$  for some  $vv' \in E_G \setminus E_H$ . Recall that  $x_{vv'}$  is adjacent to  $u_v$  and  $w_{v'}$ . Because  $u_v$  is also adjacent to  $h_0$  with  $f(h_0) = h_0$  and to  $h_1$  with  $f(h_1) = 1$ , we find that  $f(u_v) = h_0$ . Because  $w_{v'}$  is also adjacent to  $h_2$  with  $f(h_2) = h_2$  and to  $h_3$  with  $f(h_3) = h_1$ , we find that  $f(w_{v'}) = h_2$ . Hence,  $z$  is adjacent to a vertex that is mapped to  $h_0$ , namely  $u_v$ , and to a vertex that is mapped to  $h_2$ , namely  $w_{v'}$ , as desired. This completes our case analysis. Hence, we have proven Theorem 4.  $\square$

## 5. Further work

The main open problem is to determine the computational complexity of the DISCONNECTED CUT problem. Graphs with diameter at least three have a disconnected cut [5]. Graphs with diameter one are complete graphs and do not have a disconnected cut. Hence, we may restrict ourselves to graphs of diameter two. For this reason the following result is of interest. It shows that the four problems DISCONNECTED CUT, MINIMAL DISCONNECTED CUT, SEMI-MINIMAL DISCONNECTED CUT and  $\mathcal{C}_4$ -COMPACTION are polynomially equivalent to each other for graphs of diameter two.

**Proposition 1.** *Let  $G$  be a graph of diameter two. Then the following statements are equivalent.*

- (i)  $G$  has a disconnected cut;
- (ii)  $G$  has a minimal disconnected cut;
- (iii)  $G$  has a semi-minimal disconnected cut.
- (iv)  $G$  compacts to  $\mathcal{C}_4$ .

**Proof.** By definition, any minimal disconnected cut is a semi-minimal disconnected cut, and any semi-minimal disconnected cut is a disconnected cut. The equivalence “(i)  $\Leftrightarrow$  (iv)” is straightforward and has been shown in [5]. Hence, we only need to prove “(i)  $\Rightarrow$  (ii)”.

Suppose  $G = (V, E)$  has a disconnected cut  $U$ . As long as  $U$  stays a disconnected cut we move vertices from  $U$  to  $V \setminus U$ . Denote the resulting disconnected cut by  $U'$ . We claim that  $U'$  is minimal. Suppose not. Then  $U'$  contains a vertex  $u$  that is not minimal. Then  $G[U']$  consists of a component  $A$  and a component  $\{u\}$ ; if not we would have added  $u$  to  $V \setminus U'$ . As  $u$  is not minimal, there exists a component  $B$  of  $G[V \setminus U']$  such that  $u$  is not adjacent to  $V_B$ . Let  $v$  be a vertex in  $B$ . Then a shortest path from  $v$  to  $u$  must use at least one vertex from  $A$  and some other component  $B' \neq B$  of  $G[V \setminus U']$ . Hence  $d_G(u, v) \geq 3$ . This is not possible as  $\text{diam}(G) = 2$ .  $\square$

The following two questions are of interest as well.

1. What is the computational complexity of the DISCONNECTED SEPARATOR problem for graphs of diameter two?
2. What is the computational complexity of the  $\mathcal{C}_4$ -RETRACTION problem for graphs of diameter two?

Regarding question 2, recall that the  $\mathcal{C}_4$ -RETRACTION problem is NP-complete by Theorem 1. Below we show that  $\mathcal{C}_4$ -RETRACTION problem is NP-complete even for graphs of diameter three.

**Proposition 2.** *The  $\mathcal{C}_4$ -RETRACTION problem is NP-complete even for graphs of diameter three.*

**Proof.** We reduce from  $\mathcal{C}_4$ -RETRACTION for general graphs. Let  $G = (V, E)$  be a graph that has  $H$  as an induced subgraph. Let  $V = \{v_1, \dots, v_n\}$ . For each pair of different vertices  $v_i, v_j$  we add a new vertex  $a_{ij}$  only adjacent to  $v_i$  and  $v_j$ . We add a vertex  $b$  and edges  $a_{ij}b$  for all  $1 \leq i < j \leq n$ . We denote the resulting graph by  $G^* = (V^*, E^*)$ .

We show that  $G^*$  has diameter 3. Consider a vertex  $v_i \in V$ . Then  $v_i$  is of distance at most two from each vertex  $v_j \in V$  due to the path  $v_i a_{ij} v_j$ . Furthermore,  $v_i$  is of distance at most three from each vertex  $a_{jk}$  due to the path  $v_i a_{ij} b a_{jk}$ . As  $b$  is on this

path,  $v_i$  has distance two to  $b$ . A vertex  $a_{ij}$  is of distance one from  $b$  due to the edge  $a_{ij}b$  and of distance two from a vertex  $a_{k\ell}$  due to the path  $a_{ij}ba_{k\ell}$ . Hence,  $G^*$  has diameter 3 indeed. Below we prove that  $G$  retracts to  $H$  if and only if  $G^*$  retracts to  $H$ .

Suppose  $G$  retracts to  $H$  via  $f$ . Consider a vertex  $a_{ij}$ . Suppose  $h_0 \in f(\{v_i, v_j\})$ . If  $f(\{v_i, v_j\})$  does not contain  $h_2$ , then we map  $a_{ij}$  to  $h_0$ . Otherwise we map  $a_{ij}$  to  $h_1$ . Suppose  $h_0 \notin f(\{v_i, v_j\})$  and  $h_1 \in f(\{v_i, v_j\})$ . If  $f(\{v_i, v_j\})$  does not contain  $h_3$ , then we map  $a_{ij}$  to  $h_1$ . Otherwise we map  $a_{ij}$  to  $h_2$ . Suppose  $\{h_0, h_1\} \cap f(\{v_i, v_j\}) = \emptyset$ . Then we map  $a_{ij}$  to  $h_2$ . Finally, we map  $b$  to  $h_1$ . This way we have extended  $f$  to a homomorphism  $f^*$  from  $G^*$  to  $H$  with  $f^*(h_i) = f(h_i) = h_i$  for  $i = 0, \dots, 3$ . Hence  $G^*$  retracts to  $H$ .

Suppose  $G^*$  retracts to  $H$ . Because  $G$  is a subgraph of  $G^*$  and  $H$  is a subgraph of  $G$ , we find that  $G$  retracts to  $H$ . This completes the proof of Proposition 2.  $\square$

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