European Journal of Combinatorics 33 (2012) 969–979

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Forbidden graphs for tree-depth ☆

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ARTICLE INFO

Article history:
Available online 10 October 2011

ABSTRACT

For every $k \geq 0$, we define $\mathcal{G}_k$ as the class of graphs with tree-depth at most k, i.e. the class containing every graph G admitting a valid colouring $\rho : V(G) \to \{1, \ldots, k\}$ such that every (x, y)-path between two vertices where $\rho(x) = \rho(y)$ contains a vertex z where $\rho(z) > \rho(x)$. In this paper, we study the set of graphs not belonging in $\mathcal{G}_k$ that are minimal with respect to the minor/subgraph/induced subgraph relation (obstructions of $\mathcal{G}_k$). We determine these sets for $k \leq 3$ for each relation and prove a structural lemma for creating obstructions from simpler ones. As a consequence, we obtain a precise characterization of all acyclic obstructions of $\mathcal{G}_k$ and we prove that there are exactly $\frac{1}{2}2^{2^{k-1}-k}(1+2^{2^{k-1}-k})$. Finally, we prove that each obstruction of $\mathcal{G}_k$ has at most $2^{2^{k-1}}$ vertices.

1. Introduction

The graph parameter of tree-depth (also known as the vertex ranking problem [1], or the ordered colouring problem [5]) has received much attention, mostly because of the theory of graph classes of bounded expansion, developed by Nešetřil and Ossona de Mendez in [8,11,9,10,7]. Furthermore, the tree-depth of a graph is equivalent to the minimum-height of an elimination tree of a graph [2,3,8] (this measure is of importance for the parallel Cholesky factorization of matrices [6]).

The *tree-depth* of a graph G is defined as the minimum k for which there is a valid colouring $\rho : V(G) \to \{1, \ldots, k\}$ such that every (x, y)-path between two vertices where $\rho(x) = \rho(y)$ contains a vertex z where $\rho(z) > \rho(x)$. Given a non-negative integer k, we define $\mathcal{G}_k$ as the class of all graphs with tree-depth at most k.

☆ A preliminary version of this paper was presented in Giannopoulou and Thilikos (2009) [4].

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doi:10.1016/j.ejc.2011.09.014

We say that a graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by applying edge contractions. We use the notation $\mathbf{obs}_{\leq}(\mathcal{G}_k)$ for the set of minor-minimal graphs not in $\mathcal{G}_k$. If instead of the minor relation, we consider the subgraph or the induced subgraph relation, we define the sets $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ and $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ respectively.

In this paper, we examine the sets $\mathbf{obs}_{\leq}(\mathcal{G}_k)$, $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$, and $\mathbf{obs}_{\sqsubseteq}(G_k)$. From the Robertson and Seymour theorem [13], it follows that $\mathbf{obs}_{\leq}(\mathcal{G}_k)$ is finite for each $k \geq 0$. The finiteness of $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ follows from [8]. Also, it is easy to verify that $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ is finite (see Observation 4).

Our first result is an upper bound of $2^{2^{k-1}}$ to the order of the graphs in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ for $k \geq 0$. This bound also holds for $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ and $\mathbf{obs}_{\leq}(\mathcal{G}_k)$ as $\mathbf{obs}_{\leq}(\mathcal{G}_k) \subseteq \mathbf{obs}_{\subseteq}(\mathcal{G}_k) \subseteq \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ (Observation 3). Our next result is a structural lemma that constructs new obstructions from simpler ones. This permits us to identify, for each $k \geq 0$, all acyclic obstructions and prove that are exactly $\frac{1}{2}2^{2^{k-1}-k}(1+2^{2^{k-1}-k})$ for all relations. So far, such a parameterized set of acyclic obstructions is known only for classes of bounded pathwidth [15] and variations of it such as search number [12], proper-pathwidth [15], linear-width [16] (see [14] for similar results on graphs with bounded feedback vertex set number). However, this is the first time where an exact enumeration of parameterized obstructions is derived. Our final result is the identification of the sets $\mathbf{obs}_{\leq}(\mathcal{G}_k)$, $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$, and $\mathbf{obs}_{\sqsubseteq}(G_k)$ for $k \leq 3$. For $k = 3$, these sets have 12, 14, and 29 graphs respectively.

2. Preliminaries

In this paper, we consider simple graphs without loops and parallel edges. We denote by P_n the path that has n vertices and length $n-1$ and by $\mathcal{C}(G)$ the connected components of a graph G. We say that two graphs G_1, G_2 are *hom-equivalent* if G_1 is homomorphic to G_2 and G_2 is homomorphic to G_1. Moreover, an automorphism f of a graph is called *involution* if and only if $f \circ f = \mathbf{id}$.

For a graph H, we say that it is

- an *induced subgraph* of a graph G, denoted by $H \sqsubseteq G$, if it can be obtained from G by applying vertex deletions
- a *subgraph* of a graph G, denoted by $H \subseteq G$, if it can be obtained from G by applying edge and vertex deletions
- a *minor* of a graph G, denoted by $H \leq G$, if it can be obtained from G by applying edge and vertex deletions and edge contractions, where, to contract an edge $e = \{x, y\}$ of a graph G is to remove it and then replace its ends by a single vertex incident to all the edges which were incident to either x or y without allowing parallel edges.

A graph G admits a *k-vertex ranking* if there exists a valid colouring $\rho : V(G) \to \{1, \ldots, k\}$ such that every (x, y)-path between two vertices, where $\rho(x) = \rho(y)$ contains a vertex z where $\rho(z) > \rho(x)$. The *tree-depth* of a graph G, $\mathbf{td}(G)$, is defined as the minimum k such that G admits a k-vertex ranking [8]. Moreover, we give the following (equivalent) definition for the tree-depth of a connected graph G.

$$\mathbf{td}(G) = \begin{cases} 1 & \text{if } |V(G)| = 1 \\ 1 + \min_{v \in V(G)} \mathbf{td}(G \setminus v) & \text{if } |V(G)| > 1. \end{cases}$$

If follows from that, for any non-negative integer n, $\mathbf{td}(P_n) = \lceil \log_2(n+1) \rceil$ (see [8]). For every non-negative integer k, we denote by $\mathcal{G}_k$ the class of graphs with tree-depth at most k, i.e. $\mathcal{G}_k = \{G \mid \mathbf{td}(G) \leq k\}$. It is known from [1,8] that, if H is a minor of G, then $\mathbf{td}(H) \leq \mathbf{td}(G)$. A direct consequence is that for any non-negative integer k, $\mathcal{G}_k$ is minor-closed. For every $\mathsf{R} \in \{\sqsubseteq, \subseteq, \leq\}$, we denote by $\mathbf{obs}_{\mathsf{R}}(\mathcal{G}_k)$ the set of graphs with tree-depth strictly bigger than k that are minimal with respect to the relation R.

Lemma 1 ([8]). *Let $k \geq 1$ be an integer. Then, the class $\mathcal{G}_k$ includes a finite subset $\hat{\mathcal{G}}_k$ such that, for every graph $G \in \mathcal{G}_k$, there exists $\hat{G} \in \hat{\mathcal{G}}_k$ which is hom-equivalent to G and isomorphic to an induced subgraph of G.*

By Lemma 1, a tower function bound can be derived for the order of the forbidden subgraphs. However, as we prove in the next section, a direct argument shows a much better bound.

3. Upper bound on the order of obstructions for $\mathcal{G}_k$

Observation 1. *For any graph G, $\mathbf{td}(G) = \max\{\mathbf{td}(C) \mid C \in \mathcal{C}(G)\}$.*

Observation 2. *For every $k \geq 0$, all graphs in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$, $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ and $\mathbf{obs}_{\leq}(\mathcal{G}_k)$ are connected.*

Proof. Follows directly from Observation 1. □

Observation 3. *For every non-negative integer k, $\mathbf{obs}_{\leq}(\mathcal{G}_k) \subseteq \mathbf{obs}_{\subseteq}(\mathcal{G}_k) \subseteq \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$.*

Observation 4. *Let G be a graph such that $G \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$, for some integer k. Then there exists $G' \in \mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ such that $V(G) = V(G')$ and $E(G') \subseteq E(G)$.*

Proof. Let G be a counterexample of minimal size. Then there exists an edge e such that $G' = G \setminus e$ also belongs to $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ and $V(G) = V(G')$ and $E(G') \subseteq E(G)$. □

Theorem 1. *For any integer $k > 0$, if G is a graph with $\mathbf{td}(G) > k$, then G contains a connected subgraph H with $\mathbf{td}(H) > k$ and $|V(H)| \leq 2^{2^{k-1}}$.*

Proof. We may assume that G is connected, otherwise from Observation 1, we focus on the component of G that determines its tree-depth. Also, without loss of generality, $\mathbf{td}(G) = k + 1$. We prove the statement by induction:

If $\mathbf{td}(G) = 2$, then G contains at least one edge, and we may set $H = K_2$. If $\mathbf{td}(G) = 3$, then G is not a star forest, i.e., it contains P_4 or K_3 as a subgraph.

Now, suppose that $\mathbf{td}(G) = k + 1$ for $k \geq 3$, and assume that the statement holds for all smaller values of tree-depth. If G contains P_{2^k} as a subgraph, then we may set $H = P_{2^k}$. Otherwise, each two vertices in G are connected by a path of length at most $2^k - 2$.

Since $\mathbf{td}(G) > k - 1$, by induction hypothesis, G contains a subgraph H_0 with $\mathbf{td}(H_0) \geq k$ and $m \leq 2^{2^{k-2}}$ vertices $v_1, \ldots, v_m$. For each $i = 1, \ldots, m$, the graph $G \setminus v_i$ has tree-depth greater than $k - 1$, hence $G \setminus v_i$ contains a subgraph H_i with at most $2^{2^{k-2}}$ vertices and tree-depth at least k.

If there exists i such that $V(H_0) \cap V(H_i) = \emptyset$, then we let H consist of H_0, H_i and the shortest path that connects them. For every vertex v of H, the graph $H \setminus v$ contains H_0 or H_i as a subgraph, hence the tree-depth of $H \setminus v$ is at least k and $\mathbf{td}(H) > k$. Also, $|V(H)| \leq 2^{2^{k-2}+1} + 2^k - 3 \leq 2^{2^{k-1}}$ (for $k \geq 3$).

On the other hand, if all the graphs H_i intersect H_0, then we set $H = H_0 \cup H_1 \cup \cdots \cup H_m$. Since all the graphs H_i are connected, the graph H is connected as well, and it has at most $m + m(2^{2^{k-2}} - 1) \leq 2^{2^{k-1}}$ vertices. Similar to the previous case, the graphs $H \setminus v_i$ contain H_i as a subgraph (for $i = 1, \ldots, m$), and the graph $H \setminus v$ for v different from $v_1, \ldots, v_m$ contains H_0 as a subgraph, hence $\mathbf{td}(H) > k$. □

From Theorem 1 and Observations 3 and 4, we obtain the following corollary,

Corollary 1. *All graphs in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ (and therefore, also in $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ and $\mathbf{obs}_{\leq}(\mathcal{G}_k)$) have at most $2^{2^{k-1}}$ vertices.*

4. A structural lemma for the obstructions of tree-depth

In this section, we prove a lemma for tree-depth that permits us to build obstructions from simpler ones. We first consider the following observations.

Observation 5. *Let G be a connected graph such that $\mathbf{td}(G) = k$ and $\rho : V(G) \to [k]$ a k-vertex ranking of G. Then $|\rho^{-1}(k)| = 1$.*

Proof. If v_1 and v_2 are two (non-adjacent) vertices in $\rho^{-1}(k)$, then there exists a path with end-vertices v_1, v_2. Observe that, all internal vertices of this path have colour smaller than k, a contradiction. □

Observation 6. *If $G \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ (or $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ or $\mathbf{obs}_{\leq}(\mathcal{G}_k)$), then for every $v \in V(G)$ there exists a $(k+1)$-vertex ranking ρ such that $\rho(v) = k + 1$.*

Proof. As $G \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ (or $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ or $\mathbf{obs}_{\leq}(\mathcal{G}_k)$), $G \setminus v$ admits a k-vertex ranking ρ. Then $\rho \cup (v, k + 1)$ is the required $(k + 1)$-vertex ranking of G. □

Let G_1 and G_2 be two disjoint graphs and let $v_i \in V(G_i)$, for $i = 1, 2$. We define $\mathbf{j}(G_1, G_2, v_1, v_2) = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2) \cup \{\{v_1, v_2\}\})$.

Observation 7. *Let G_1 and G_2 be disjoint graphs where $\mathbf{td}(G_1) \le k$ and $\mathbf{td}(G_2) \le k$. Let $v_i \in V(G_i),\ i = 1, 2$. Then the graph $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ has tree-depth at most $k + 1$.*

Proof. Let ρ_i be a k-vertex ranking of $G_i,\ i = 1, 2$. Then $\rho = \rho_1 \cup \rho_2 \setminus \{(v_1, \rho_1(v_1))\} \cup \{(v_1, k + 1)\}$ is a $(k + 1)$-vertex ranking of G. □

Observation 8. *Let G_1 and G_2 be disjoint, connected graphs such that $\mathbf{td}(G_1) \ge k$ and $\mathbf{td}(G_2) \ge k$. Let $v_i \in V(G_i),\ i = 1, 2$. Then the graph $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ has tree-depth at least $k + 1$.*

Proof. Assume, on the contrary, that there exists a k-vertex ranking $\rho : V(G) \to [k]$. Note that $\rho^{-1}(k) \neq \emptyset$, otherwise $\mathbf{td}(G) < k$ contradicting the fact that $\mathbf{td}(G_1) \ge k$. Combining this fact with Observation 5, G has a unique vertex v where $\rho(v) = k$. W.l.o.g. we assume that $v \in V(G_1)$. Then the restriction of ρ to G_2 gives a $(k - 1)$-vertex ranking of it, a contradiction. □

Lemma 2. *Let k be a positive integer and let $\mathsf{R} \in \{\sqsubseteq, \subseteq, \le\}$. Let G_1 and G_2 be disjoint graphs such that $G_1, G_2 \in \mathbf{obs}_{\mathsf{R}}(\mathcal{G}_{k-1})$ and let $v_1 \in V(G_1), v_2 \in V(G_2)$. Then $\mathbf{j}(G_1, G_2, v_1, v_2) \in \mathbf{obs}_{\mathsf{R}}(\mathcal{G}_k)$.*

Proof. Let G_1 and G_2 be such that $G_1, G_2 \in \mathbf{obs}_{\mathsf{R}}(\mathcal{G}_{k-1})$ and let $v_i \in V(G_i),\ i = 1, 2$. We set $G = \mathbf{j}(G_1, G_2, v_1, v_2)$. We first prove that $\mathbf{td}(G) = k + 1$. Indeed, Observation 7 yields $\mathbf{td}(G) \le k + 1$ and Observation 8 yields $\mathbf{td}(G) \ge k + 1$.

Now, we have to prove that if G' is the result of the removal or the contraction of some edge e in G, then $\mathbf{td}(G') \le k$ (this also covers the case of a vertex removal as, from Observation 1, G is connected and thus the removal of a vertex implies the removal of at least one edge).

First, we examine the case where $e = \{v_1, v_2\}$. If $G' = G \setminus e$, then from Observation 1, $\mathbf{td}(G) = \max\{\mathbf{td}(G_1), \mathbf{td}(G_2)\} \le k$. If $G' = G/e$, then from Observation 6, there exists a k-vertex ranking ρ_i of G_i such that $\rho_i(v_i) = k,\ i = 1, 2$. Then if v_{new} is the result of the contraction of e, we have that $\rho : V(G') \to [k]$ where

$$\rho(x) = \begin{cases} \rho_1(x) & \text{if } x \in V(G_1) \setminus \{v_1\} \\ \rho_2(x) & \text{if } x \in V(G_2) \setminus \{v_2\} \\ k & \text{if } x = v_{\text{new}} \end{cases}$$

is a k-vertex ranking of G', therefore $\mathbf{td}(G') \le k$.

Finally, we examine the case where e is an edge of G_1 or G_2. Without loss of generality, we assume that $e_1 \in E(G_1)$. Because $G_1 \in \mathbf{obs}_{\subseteq}(\mathcal{G}_{k-1})$, there exists a $(k - 1)$-vertex ranking ρ_1' of $G_1 \setminus e$ (and G_1/e). By Observation 6, since $G_2 \in \mathbf{obs}_{\subseteq}(\mathcal{G}_{k-1})$, there exists a k-vertex ranking ρ_2 of G_2 such that $\rho_2(v_2) = k$. It is easy to see that $\rho_1' \cup \rho_2$ is a k-vertex ranking of G', thus $\mathbf{td}(G') \le k$ and this completes the proof of the lemma. □

5. Acyclic obstructions for tree-depth

For every integer $k \ge 0$, we recursively define the class $\mathcal{T}_k$ as follows. Let $\mathcal{T}_0 = \{K_1\}$ and for every $k \ge 1$, we set

$$\mathcal{T}_k = \{\mathbf{j}(G_1, G_2, v_1, v_2) \mid G_1, G_2 \in \mathcal{T}_{k-1},\ v_i \in V(G_i),\ i = 1, 2\}.$$

The above definition permits us to state Lemma 2 as follows.

Observation 9. *For every integer $k \ge 0$ and every $\mathsf{R} \in \{\sqsubseteq, \subseteq, \le\}$, $\mathcal{T}_k \subseteq \mathbf{obs}_{\mathsf{R}}(\mathcal{G}_k)$.*

Lemma 3. *For any positive integer k, if $G \in \mathcal{T}_k$, then for any vertex $v \in V(G)$ there exists a leaf $u \neq v$ of G such that the tree created from $G \setminus u$ by adding a leaf adjacent to v also belongs to $\mathcal{T}_k$.*

Proof. Assume that this holds for any tree in $\mathcal{T}_{k-1}$, $k \ge 2$. Let $G_1, G_2 \in \mathcal{T}_{k-1}$ and $v_i \in V(G_i),\ i = 1, 2$ such that $G = \mathbf{j}(G_1, G_2, v_1, v_2)$. Consider an arbitrary vertex $v \in V(G)$, and let us show that there exists a leaf u of G that we can move to v while preserving membership in $\mathcal{T}_k$. Without loss of generality, we may assume that $v \in V(G_1)$. By the induction hypothesis, there exists a vertex $u' \in V(G_1)$ such that the tree created from $G_1 \setminus u'$ by adding a leaf adjacent to v is also in $\mathcal{T}_{k-1}$. If $u' \neq v_1$, we may set $u = u'$. Otherwise, let u'' be the leaf of G_2 that can be moved to v_2. In this case, we can set $u = u''$: Moving the leaf u'' to v has the same result as moving it to v_2, moving the leaf u' to v, and replacing the edge e by an edge between u'' and the vertex of G_1 that used to be adjacent to u'. □

In Lemma 2, we described a procedure that for any non-negative integer k constructs graphs $G \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k+1})$ from disjoint graphs $G_1, G_2 \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ (adding an edge that connects a vertex v_1 of G_1 and a vertex v_2 of G_2). With the following lemma, we fully characterize and construct all the acyclic graphs in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k+1})$ for every non-negative integer k.

Lemma 4. *Let G be a tree in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ for $k \geq 1$. Then there exists an edge $e \in E(G)$ such that if $\{G_1, G_2\} = \mathcal{C}(G \setminus \{e\})$ then $G_1, G_2 \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k-1})$.*

Proof. We examine the non-trivial case, where $k \geq 2$ assuming that the statement holds for all acyclic obstructions of smaller tree-depth. From Observation 7, we obtain that for each edge $e = \{v_1, v_2\} \in E(G)$, at least one of the connected components G_1, G_2 of $G \setminus e$ has tree-depth at least k. We claim that G contains at least one edge $e = \{v_1, v_2\}$ such that both connected components of $G \setminus e$ have tree-depth k. Suppose that this is not correct. Then we can direct each edge $e = \{v_1, v_2\}$ of $E(G)$ such that its tail belongs to the connected component of $G \setminus e$ that has tree-depth $<k$. We denote this directed tree by $\tilde{T}$, as $k \geq 2$, $\tilde{T}$ contains internal vertices. Moreover, all edges of $\tilde{T}$ that are incident to a leaf are directed away from it. It follows that $\tilde{T}$ contains an internal vertex v of out-degree 0. This means that each, say G_i, connected component of $G \setminus v$ has a $(k-1)$-vertex ranking ρ_i. Then $\rho = \{(v, k)\} \cup \bigcup_{i=1,\dots,m} \rho_i$ is a k-vertex ranking of G, a contradiction and this completes the proof of the claim.

Now, let G_i be the connected component of $G \setminus e$ that contains v_i, $i = 1, 2$. If one, say G_1, is not in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k-1})$ then it contains an induced subgraph G_1' such that $G_1' \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k-1})$. Additionally, there is a unique path P in G that connects G_1' with G_2. Observe that, since $G \in \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k-1})$, G is exactly the union of G_1', G_2 and P. We need to show that P has no inner vertices. Suppose that this is not the case, and let w be the inner vertex of P adjacent to a vertex $v \in V(G_1)$. By the induction hypothesis, G_1' and G_2 satisfy the conditions of Lemma 3, thus G_1 contains a leaf u such that the graph obtained from G_1 by moving the leaf u to v belongs to $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_{k-1})$. This implies that we may remove the vertex u from G and consider w to be its replacement. The created graph is a proper induced subgraph of G and has tree-depth $k+1$, a contradiction. This completes the proof of the lemma. □

Now, observe that the following is a direct consequence of Lemmata 2 and 4.

Theorem 2. *Let k be a non-negative integer. Then $\mathcal{T}_k$ is the set of all acyclic graphs in $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$.*

Corollary 2. *For every non-negative integer k, $\mathcal{T}_k$ is the set of all acyclic graphs in $\mathbf{obs}_{\subseteq}(\mathcal{G}_k)$ (or in $\mathbf{obs}_{\leq}(\mathcal{G}_k)$).*

Proof. Follows directly from Observations 3 and 9. □

6. Lower bound on the number of obstructions for $\mathcal{G}_k$

In this section, we prove that $|\mathcal{T}_k| = \frac{1}{2}2^{2^{k-1}-k}(1 + 2^{2^{k-1}-k})$, $k \geq 1$. This gives a lower bound on $|\mathbf{obs}_{\leq}(\mathcal{G}_k)|$, $k \geq 2$. As we shall see later we can identify the elements of the sets $\mathbf{obs}_{\subseteq}(\mathcal{G}_i)$, $\mathbf{obs}_{\leq}(\mathcal{G}_i)$, $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_i)$ for $i = 0, 1, 2, 3$.

For a tree $G \in \mathcal{T}_k$ such that $G = \mathbf{j}(G_1, G_2, v_1, v_2)$, we call v_1v_2 the *middle edge* of G.

Observation 10. *If k is a non-negative integer, then every graph in $\mathcal{T}_k$ has exactly 2^k vertices. This implies that the middle edge of a graph $G \in \mathcal{T}_k$ is unique.*

Also, consider the following.

Observation 11. *Let T^1, T^2 be two trees and $e^i = \{v_1^i, v_2^i\} \in E(T^i)$, $i = 1, 2$. If ϕ is an isomorphism from T^1 to T^2 such that $\phi(v_i^1) = v_i^2$, $i = 1, 2$ and T_i^j is the connected component of $T^j \setminus e^j$ that contains v_i^j, $i = 1, 2$, $j = 1, 2$, then $\phi_i = \{(x, y) \in \phi \mid x \in V(T_i^1)\}$ is an isomorphism from T_i^1 to T_i^2, $i = 1, 2$.*

We use notation $\mathbf{Aut}(G)$ for the automorphism group of a graph G. Observation 11 easily implies the following.

Observation 12. *Let T be a tree and $e = \{v_1, v_2\} \in E(T)$. If $\phi \in \mathbf{Aut}(T)$ satisfies $\phi(v_i) = v_{3-i},\ i = 1, 2$ and T_i is the connected component of $T \setminus e$ that contains $v_i,\ i = 1, 2$, then $\phi' = \{(x, y) \in \phi \mid x \in V(T_1)\}$ is an isomorphism from T_1 to T_2.*

Observation 13. *Let G_1, G_2 be disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k,\ k \geq 1$ and $v_i \in V(G_i),\ i = 1, 2$. If $\phi \in \mathbf{Aut}(G)$, where $G = \mathbf{j}(G_1, G_2, v_1, v_2)$, then $\phi(e) = e$.*

Proof. Follows directly from Observation 10. □

Lemma 5. *Let $G \in \mathcal{T}_k$ for $k \geq 1$, $e = \{v_1, v_2\} \in E(G)$ the middle edge and $\phi \in \mathbf{Aut}(G)$. If there exists $v \in V(G)$ such that $\phi(v) = v$, then $\phi(v_i) = v_i,\ i = 1, 2$.*

Proof. We examine the non-trivial case where $k \geq 2$. Suppose, in contrary, that $\phi(v_i) = v_{3-i},\ i = 1, 2$. We denote by G_1, G_2 the connected components of $G \setminus e$ where, w.l.o.g, $v,\ v_1 \in V(G_1)$. By Observation 12, $\phi' = \{(v_1, v_2) \in \phi \mid v_1 \in V(G_1)\}$ is an isomorphism of G_1 to G_2, a contradiction since $\phi'(v) = \phi(v) = v$. □

Now, we proceed to the proof of the following.

Lemma 6. *Let k be a non-negative integer. For any $G \in \mathcal{T}_k$ and $\phi \in \mathbf{Aut}(G)$, if there exists $v \in V(G)$ such that $\phi(v) = v$ then $\phi = \mathbf{id}$.*

Proof. We use induction on k. For $k = 0$ the claim is trivial. Now, assume that the claim holds for $k = n \geq 0$. Let $k = n + 1$. We denote by $e = \{v_1, v_2\} \in E(G)$ the middle edge and by G_1, G_2 the connected components of $G \setminus e$, where $v_i \in V(G_i),\ i = 1, 2$. Since $\phi \in \mathbf{Aut}(G)$, by Lemma 5, it follows that $\phi(v_i) = v_i,\ i = 1, 2$. Hence ϕ is an isomorphism from $G \setminus e$ to $G \setminus e$. From Observation 11, $\phi_i = \{(v, u) \in \phi \mid v \in V(G_i)\} \in \mathbf{Aut}(G_i),\ i = 1, 2$. Observe that $\phi_i(v_i) = \phi(v_i) = v_i,\ i = 1, 2$. Since $G_i \in \mathcal{T}_n,\ i = 1, 2$, by the induction hypothesis, $\phi_i,\ i = 1, 2$ is the trivial automorphism of G_i. Therefore, $\phi = \mathbf{id}$. □

Let G be a graph and $v \in V(G)$. We denote by $\mathbf{tr}_G(v)$ the orbit of the automorphism group of G that contains v, i.e. $\mathbf{tr}_G(v) = \{u \in V(G) \mid \exists \phi \in \mathbf{Aut}(G)$ such that $\phi(u) = v\}$.

Lemma 7. *Let G_1, G_2 be disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k$, $v_2, v_2' \in V(G_2)$ such that $v_2 \in \mathbf{tr}_{G_2}(v_2')$ and $v_1 \in V(G_1)$. Then $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ and $G' = \mathbf{j}(G_1, G_2, v_1, v_2')$ are isomorphic.*

Proof. Let $\mathbf{id} \in \mathbf{Aut}(G_1)$ and $\phi \in \mathbf{Aut}(G_2)$, such that $\phi(v_2) = v_2'$. Then $\mathbf{id} \cup \phi$ is an isomorphism from G to G'. □

Lemma 8. *Let G_1, G_2 be disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k$, $v_2, v_2' \in V(G_2)$ such that $v_2 \notin \mathbf{tr}_{G_2}(v_2')$ and $v_1 \in V(G_1)$. Then $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ and $G' = \mathbf{j}(G_1, G_2, v_1, v_2')$ are not isomorphic.*

Proof. Assume, on the contrary, that ϕ is an isomorphism from G to G'. Observation 13 implies that either $\phi(v_1) = v_1$ and $\phi(v_2) = v_2'$ or $\phi(v_1) = v_2'$ and $\phi(v_2) = v_1$. We first exclude the case where $\phi(v_1) = v_1$ and $\phi(v_2) = v_2'$. Indeed, by Observation 11, $\phi' = \{(x, y) \in \phi \mid x \in V(G_2)\} \in \mathbf{Aut}(G_2)$ and moreover $\phi'(v_2) = \phi(v_2) = v_2'$, a contradiction since $v_2 \notin \mathbf{tr}_{G_2}(v_2')$. Therefore, $\phi(v_1) = v_2'$ and $\phi(v_2) = v_1$. By Observation 11, $\phi_i = \{(x, y) \in \phi \mid x \in V(G_i)\}$ is an isomorphism from G_i to $G_{3-i},\ i = 1, 2$. Then $\psi = \phi_1 \circ \phi_2 \in \mathbf{Aut}(G_2)$ and $\psi(v_2) = \phi_1(\phi_2(v_2)) = \phi_1(\phi(v_2)) = \phi_1(v_1) = v_2'$. It follows that $v_2 \in \mathbf{tr}_{G_2}(v_2')$, a contradiction. □

Given a graph G. We say that G is *asymmetric* if it has a trivial automorphism group. Moreover, we say that a graph G is 2-*asymmetric* if its only non-trivial automorphism is an involution without fixed points.

Lemma 9. *Let k be a non-negative integer and let G_1, G_2 be two disjoint non-isomorphic graphs such that $G_1, G_2 \in \mathcal{T}_k$. Then the graph $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ is asymmetric.*

Proof. Suppose that $\phi \in \mathbf{Aut}(G)$ and $\phi \neq \mathbf{id}$. From Lemma 6, $\phi(v) \neq v$ for all $v \in V(G)$ and from Observation 13, $\phi(v_i) = v_{3-i}$, $i = 1, 2$. From Observation 12, G_1 is isomorphic to G_2, a contradiction. □

Lemma 10. *Let k be a non-negative integer and let G_1, G_2 be two disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k$. If ϕ is an isomorphism from G_1 to G_2 and $v_i \in V(G_i)$, $i = 1, 2$ such that $\phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)$, then $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ is asymmetric.*

Proof. Suppose that $\psi \in \mathbf{Aut}(G)$ and $\psi \neq \mathbf{id}$. From Lemma 6, $\psi(v) \neq v$ for all $v \in V(G)$ and from Observation 13, $\psi(v_1) = v_2$ and $\psi(v_2) = v_1$. From Observation 12, $\chi = \{(x, y) \in \psi \mid x \in V(G_1)\}$ is an isomorphism from G_1 to G_2. Moreover, $\phi \circ \chi^{-1}$ is an automorphism of G_2 mapping v_2 to $\phi(v_1)$, contradicting the assumption that $\phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)$. □

Lemma 11. *Let k be a non-negative integer and let G_1, G_2 be two disjoint graphs such that $G_1, G_2 \in \mathcal{T}_k$. If $\phi : V(G_1) \to V(G_2)$ is an isomorphism from G_1 to G_2 and $v_i \in V(G_i)$, $i = 1, 2$ are two vertices such that $\phi(v_1) \in \mathbf{tr}_{G_2}(v_2)$, then $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ is 2-asymmetric.*

Proof. Since $\phi(v_1) \in \mathbf{tr}_{G_2}(v_2)$, there exists an isomorphism $\psi : V(G_1) \to V(G_2)$ such that $\psi(v_1) = v_2$. Observe that $\chi = \psi \cup \psi^{-1}$ is an automorphism of G, and that χ is an involution without fixed points. Consider an automorphism $\chi' \neq \mathbf{id}$ of G. By Lemma 6 and Observation 13, $\chi'(v_1) = v_2$ and by Observation 12, $\chi'_1 = \{(x, y) \in \chi' | x \in V(G_1)\}$ is an isomorphism of G_1 and G_2. Then $\chi'_1 \circ \psi^{-1}$ is an automorphism of G_2 that fixes v_2, and by Lemma 6, $\chi'_1 = \psi$. We conclude that $\chi' = \chi$, and thus $\mathbf{Aut}(G) = \{\mathbf{id}, \chi\}$ and G is 2-asymmetric. □

From Theorem 2 and Lemmata 9–11 it follows directly.

Observation 14. *If G is a graph such that $G \in \mathcal{T}_k$, then G is either asymmetric or 2-asymmetric.*

For every integer $k \geq 0$, we define for following partition of $\mathcal{T}_k$:

$$\mathcal{A}_k = \{G \in \mathcal{T}_k \mid \mathbf{Aut}(G) = \{\mathbf{id}\}\} \quad \text{and} \quad \mathcal{B}_k = \{G \in \mathcal{T}_k \mid \mathbf{Aut}(G) \neq \{\mathbf{id}\}\}.$$

We denote that $\alpha_k = |\mathcal{A}_k|$, $\beta_k = |\mathcal{B}_k|$ and $\tau_k = |\mathcal{T}_k| = \alpha_k + \beta_k$. We also set $\gamma_k = 2^{k-2}$. A direct consequence of Observations 10 and 14 is the following.

Observation 15. *Let $k \geq 2$ be an integer. Then the automorphism group of each graph $G \in \mathcal{A}_k$ (resp. $G \in \mathcal{B}_k$) has exactly γ_{k+2} (resp. γ_{k+1}) orbits.*

Observation 16. $\beta_0 = \alpha_1 = \alpha_2 = 0$ *and* $\alpha_0 = \beta_1 = \beta_2 = 1$.

Theorem 3. *For every integer, $k \geq 1$, $\tau_k = 2^{2^k-(2k+1)} + 2^{2^{k-1}-(k+1)}$.*

Proof. First, observe that for $k = 1, 2$ the claim holds. Let G be a graph. Recall that $G \in \mathcal{T}_k$ iff $G = \mathbf{j}(G_1, G_2, v_1, v_2)$ for some $G_i \in \mathcal{T}_{k-1}$, and $v_i \in V(G_i)$, $i = 1, 2$. Therefore, in order to count τ_k, it is sufficient to count the ways to choose $G_1, G_2 \in \mathcal{T}_{k-1}$ and $v_i \in V(G_i)$, $i = 1, 2$ and not end up with isomorphic graphs. Let G_1, G_2 be graphs such that $G_i \in \mathcal{T}_{k-1}$ and $v_i \in V(G_i)$, $i = 1, 2$. We define

$$\mathcal{A}_k^1 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \not\simeq G_2, G_i \in \mathcal{A}_{k-1} \text{ and } v_i \in V(G_i),\ i = 1, 2\} \quad (1)$$

$$\mathcal{A}_k^2 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \not\simeq G_2, G_i \in \mathcal{B}_{k-1} \text{ and } v_i \in V(G_i),\ i = 1, 2\} \quad (2)$$

$$\mathcal{A}_k^3 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \not\simeq G_2, G_1 \in \mathcal{A}_{k-1}, G_2 \in \mathcal{B}_{k-1}, \text{ and } v_i \in V(G_i),\ i = 1, 2\} \quad (3)$$

$$\mathcal{A}_k^4 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \simeq_\phi G_2, G_i \in \mathcal{A}_{k-1}, \text{ and } v_i \in V(G_i),\ i = 1, 2, \text{ such that } \phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)\} \quad (4)$$

$$\mathcal{A}_k^5 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \simeq_\phi G_2, G_i \in \mathcal{B}_{k-1}, \text{ and } v_i \in V(G_i),\ i = 1, 2, \text{ such that } \phi(v_1) \notin \mathbf{tr}_{G_2}(v_2)\} \quad (5)$$

$$\mathcal{B}_k^1 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \simeq_\phi G_2, G_i \in \mathcal{A}_{k-1}, \text{ and } v_i \in V(G_i), \ i = 1, 2, \text{ such that } \phi(v_1) \in \mathbf{tr}_{G_2}(v_2)\} \tag{6}$$

$$\mathcal{B}_k^2 = \{G \mid G = \mathbf{j}(G_1, G_2, v_1, v_2), G_1 \simeq_\phi G_2, G_i \in \mathcal{B}_{k-1}, \text{ and } v_i \in V(G_i), \ i = 1, 2 \text{ such that } \phi(v_1) \in \mathbf{tr}_{G_2}(v_2)\}. \tag{7}$$

By their definitions, the above sets are a partition of $\mathcal{T}_k$. By Lemma 9 (for Relations (1)–(3)) and Lemma 10 (for Relations (4) and (5)), the union of the first five is a subset of $\mathcal{A}_k$. Moreover, by Lemma 11 (applied to Relations (6) and (7)), the union of the last two is a subset of $\mathcal{B}_k$. We conclude that $\mathcal{A}_k = \bigcup_{i=1,\dots,5} \mathcal{A}_k^i$ and $\mathcal{B}_k = \mathcal{B}_k^1 \cup \mathcal{B}_k^2$.

From Observation 15, Lemmata 7 and 8, and Relations (1)–(7) we derive that

$$|\mathcal{A}_k^1| = \binom{\alpha_{k-1}}{2} \cdot \gamma_{k+1}^2,$$
$$|\mathcal{A}_k^2| = \binom{\beta_{k-1}}{2} \cdot \gamma_k^2,$$
$$|\mathcal{A}_k^3| = \alpha_{k-1} \cdot \gamma_{k+1} \cdot \beta_{k-1} \cdot \gamma_k,$$
$$|\mathcal{A}_k^4| = \alpha_{k-1} \cdot \binom{\gamma_{k+1}}{2}$$
$$|\mathcal{A}_k^5| = \beta_{k-1} \cdot \binom{\gamma_k}{2}$$
$$|\mathcal{B}_k^1| = \alpha_{k-1} \cdot \gamma_{k+1}$$
$$|\mathcal{B}_k^2| = \beta_{k-1} \cdot \gamma_k.$$

Therefore,

$$\alpha_k = \binom{\alpha_{k-1}}{2} \gamma_{k+1}^2 + \binom{\beta_{k-1}}{2} \gamma_k^2 + \alpha_{k-1} \binom{\gamma_{k+1}}{2} + \beta_{k-1} \binom{\gamma_k}{2} + \alpha_{k-1}\beta_{k-1}\gamma_k\gamma_{k+1} \tag{8}$$

$$\beta_k = \alpha_{k-1}\gamma_{k+1} + \beta_{k-1}\gamma_k. \tag{9}$$

By simplifying (8),

$$\alpha_k = \frac{1}{2} \left[\left(\gamma_{k+1}^2 \alpha_{k-1}^2 + \gamma_k^2 \beta_{k-1}^2 + 2\alpha_{k-1}\beta_{k-1}\gamma_k\gamma_{k+1}\right) - (\alpha_{k-1}\gamma_{k+1} + \beta_{k-1}\gamma_k) \right] = \frac{1}{2} \left(\beta_k^2 - \beta_k\right).$$

It follows (using Relation (9)) that,

$$\tau_k = \frac{1}{2} \left(\beta_k^2 + \beta_k\right) \quad \text{and } \beta_k = \gamma_k \beta_{k-1}^2.$$

Let $\delta_k = 2^{k-1} - k$ and observe that $\beta_k = 2^{\delta_k} = 2^{2^{k-1}-k}$, for every integer $k \geq 2$. Then $\tau_k = 2^{2^k-(2k+1)} + 2^{2^{k-1}-(k+1)}$, $k \geq 3$ and the theorem follows. □

7. Obstructions for $\mathcal{G}_k$, $k \leq 3$

It is easy to prove that

- $\mathbf{obs}_{\leq}(\mathcal{G}_0) = \mathbf{obs}_{\subseteq}(\mathcal{G}_0) = \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_0) = \{K_1\}$,
- $\mathbf{obs}_{\leq}(\mathcal{G}_1) = \mathbf{obs}_{\subseteq}(\mathcal{G}_1) = \mathbf{obs}_{\sqsubseteq}(\mathcal{G}_1) = \{K_2\}$,
- $\mathbf{obs}_{\leq}(\mathcal{G}_2) = \mathbf{obs}_{\subseteq}(\mathcal{G}_1) = \{K_3, P_4\}$ and $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_2) = \{K_3, P_4, C_4\}$.

Let $\mathcal{D}$ be the set of the graphs that appear inside the outer polygon in Fig. 1. In this section, we prove that $\mathbf{obs}_{\subseteq}(\mathcal{G}_3) = \mathcal{D}$.

Theorem 4. *For any graph G, $\mathbf{td}(G) > 3$ if and only if G contains one of the graphs in $\mathcal{D}$ as a subgraph.*

Proof. Since each graph in $\mathcal{D}$ is connected and has tree-depth four, it suffices to show that any connected graph with tree-depth four contains one of them as a subgraph. Suppose for contradiction

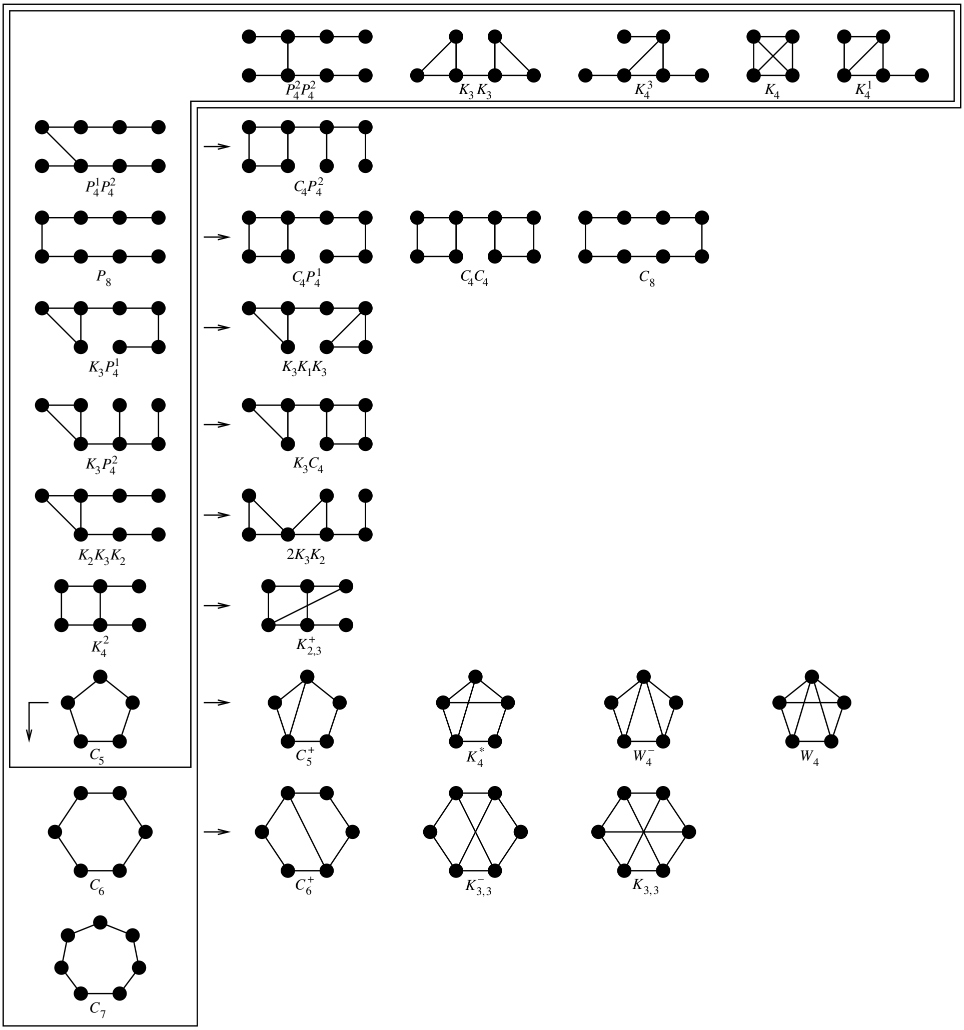

Fig. 1. The forbidden graphs for $\mathcal{G}_3$.

that this is not the case, and let G be a connected graph with tree-depth four that contains none of the graphs in $\mathcal{D}$ as a subgraph. We may assume that G is minimal, i.e., $\mathbf{td}(G \setminus e) = 3$ and $\mathbf{td}(G \setminus v) = 3$ for any edge $e \in E(G)$ and any vertex $v \in V(G)$. The graph G cannot contain any cycles of length greater than four, otherwise, it would contain C_5, C_6, C_7, or P_8 as a subgraph.

Let G' be a 2-connected subgraph of G, and suppose that $|V(G')| \geq 5$. Observe that G' contains a 4-cycle $C = v_1v_2v_3v_4$. Consider a vertex $v_5 \in V(G') \setminus V(C)$. Since G' is 2-connected, there exists a path P with distinct end-vertices in C such that $v_5 \in V(P)$ and $|V(P) \cap V(C)| = 2$. Since G does not contain cycles of length at least 5, P has length two and joins two opposite vertices of C, say v_1 and v_3. If the subgraph induced by $V(C) \cup \{v_5\}$ contains any of the edges $\{v_2, v_4\}$, $\{v_2, v_5\}$ or $\{v_4, v_5\}$, then G contains C_5 as a subgraph, hence we may assume that this is not the case. Also, none of v_2, v_4 and v_5 may be incident with any other vertex of G, otherwise G would contain K_4^2. Consider the graph H obtained from G by removing the edge $\{v_1, v_5\}$. By the minimality of G, $\mathbf{td}(H) = 3$. The graph H is connected, hence H contains a vertex v such that $H \setminus v$ is a star forest. If $v = v_1$ or $v = v_3$, then $G \setminus v$ is a star forest, which is a contradiction with $\mathbf{td}(G) = 4$. However, $H \setminus v$ for any other vertex v contains

P_4 as a subgraph. This is a contradiction, hence we may assume that any 2-connected subgraph of G has at most four vertices.

Now, let us consider the case where G contains a 4-cycle $C = v_1v_2v_3v_4$. If both edges $\{v_1, v_3\}$ and $\{v_2, v_4\}$ are in G, then G contains K_4 as a subgraph, thus we may assume that this is not the case. First, suppose that $\{v_1, v_3\}$ is an edge (thus $\{v_2, v_4\}$ is not an edge). If v_2 or v_4 is adjacent to a vertex outside of C, then G contains K_4^1 as a subgraph. Otherwise, consider the graph H obtained from G by removing the edge $\{v_1, v_3\}$. By the minimality of G, there exists a vertex v such that $H \setminus v$ is a star forest. The vertex v must belong to C. Since $G \setminus v$ is not a star forest, $v \neq v_1$ and $v \neq v_3$, hence we may assume that $v = v_2$. Since $H \setminus v_2$ is a star forest, v_4 is the only neighbour of v_1 and v_3 in $H \setminus v_2$. But then $H = C$, and tree-depth of G would be only three, which is a contradiction; therefore, any 4-cycle in G is induced.

Let $C = v_1v_2v_3v_4$ be an induced 4-cycle in G. Since G does not contain K_4^2 as a subgraph, the vertices of $V(G) \setminus V(C)$ can only be adjacent to two non-adjacent vertices of C, say v_1 and v_3. Since $\mathbf{td}(G) = 4$, we have $G \neq C$ and we may assume that there exists a vertex $v_5 \in V(G) \setminus V(C)$ adjacent to v_1. Let us consider the graph H obtained from G by removing the edge v_1v_4. By the minimality of G, there exists a vertex v such that $H \setminus v$ is a star forest. Since $v_5v_1v_2v_3v_4$ is a path, v must be v_1, v_2 or v_3. If $v = v_1$ or $v = v_3$, then $G \setminus v$ is a star forest, hence $v = v_2$. However, this means that $G \setminus v_1$ is a star forest, which is a contradiction, thus G does not contain any 4-cycle.

Now, consider the case where G contains a triangle $C = v_1v_2v_3$. The graph G cannot contain another triangle disjoint from C, since otherwise it would contain $K_3P_4^1$ or K_3K_3 as a subgraph. Together with the fact that each nontrivial 2-connected subgraph of G is a triangle, this implies that all the triangles in G intersect in one vertex. We may assume that there is at least one vertex v_4 not belonging to C adjacent to v_1, and that all triangles in G contain the vertex v_1.

The vertex v_1 is a cut-vertex in G. The graph $G \setminus v_1$ is not a star forest, hence one of its components contains a triangle or P_4. All triangles in G contain the vertex v_1, hence one of the components of $G \setminus v_1$ contains a path P of length three.

If P is disjoint with C, then G contains a subgraph $K_3P_4^1$ or $K_3P_4^2$. It follows that C is the only triangle in G and that the path P intersects $C \setminus v_1$. If the degree of both v_2 and v_3 is greater than two, then G contains the subgraph K_4^3, thus we may assume that the degree of v_2 is two and that $P = v_2v_3v_5v_6$ for some vertices v_5 and v_6. Similarly, $G \setminus v_3$ contains P_4 as a subgraph, hence we may assume that there is a vertex v_7 adjacent to v_4. However, the graph G then would contain $K_2K_3K_2$ as a subgraph. Therefore, G does not contain a triangle, and it must be a tree.

It is however easy to verify using Theorem 2 that the only tree-depth critical trees with tree-depth four are P_8, $P_4^1P_4^2$ and $P_4^2P_4^2$. It follows that any graph with $\mathbf{td}(G) > 3$ contains one of the graphs in $\mathcal{D}$ as a subgraph. □

Corollary 3. *The set* $\mathbf{obs}_{\leq}(\mathcal{G}_3)$ *contains exactly all the graphs depicted in the inner polygon in Fig. 1.*

Proof. Follows directly from Observation 3 and the fact that $C_5 \leq C_6$ and $C_5 \leq C_7$. □

Corollary 4. *The set* $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_3)$ *contains exactly all the graphs in Fig. 1.*

Proof. Follows by inspection, using Observation 4. □

Notice that the obstructions for $\mathcal{G}_k$ have at most 2^k vertices for $k \leq 3$. Hence Theorem 1 is not sharp even in this case (it only claims that the obstructions have at most 16 vertices).

We conclude with the following conjecture.

Conjecture 1. *For every* $k \geq 1$, *the order of the graphs in* $\mathbf{obs}_{\sqsubseteq}(\mathcal{G}_k)$ *is bounded by* 2^k.

Acknowledgements

The first author was supported by the ERC starting grant CCOSA (grant agreement no. 259385), and by the Institute for Theoretical Computer Science (ITI), project 1M0021620808 of the Ministry of Education of Czech Republic.

The third author was supported by the project "Kapodistrias" (AΠ 02839/28.07.2008) of the National and Kapodistrian University of Athens (project code: 70/4/8757).

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