Structure and Enumeration of K_4-free links and link diagrams

Juanjo Rué [1,2]

Department of Mathematics, Universitat Politècnica de Catalunya, Barcelona, Spain

Dimitrios M. Thilikos [1,2]

AlGCo project-team, LIRMM, Université de Montpellier, CNRS, Montpellier, France
Department of Mathematics, National and Kapodistrian University of Athens, Athens, Greece

Vasiliki Velona [1,2]

Department of Economics, Universitat Pompeu Fabra, Barcelona, Spain
Department of Mathematics, Universitat Politècnica de Catalunya, Barcelona, Spain

Abstract

We study the class $\mathcal{L}$ of link types that admit a K_4-minor-free diagram, i.e., they can be projected on the plane so that the resulting graph does not contain any subdivision of K_4. We prove that $\mathcal{L}$ is the closure of torus links under the operation of connected sum. Using this structural result, we enumerate $\mathcal{L}$ and subclasses of it, with respect to the minimal number of crossings in a projection of $L \in \mathcal{L}$. Further, we enumerate (both exactly and asymptotically) all K_4-minor-free link diagrams, all minimal K_4-free link diagrams, and all K_4-free diagrams of the unknot.

Keywords: series-parallel graphs, links, knots, generating functions, asymptotic enumeration, map enumeration

1 Introduction

The exhaustive generation of knots and links is a well-established problem in low dimensional geometry (see [7, Ch.5], for instance). However, there are very few enumerative results in the literature and they are relatively recent, (see [10] and [6]). Moreover, there seems to be no known results connecting graph theoretic classes with link classes.

Let us start with some formal definitions. A *knot* K is a smooth embedding of the 1-dimensional sphere $\mathbb{S}^1$ in $\mathbb{R}^3$. A *link* is a finite disjoint union of knots $L = K_1 \cup ... \cup K_n$. Two links L_1 and L_2 are equivalent if there is a continuous and injective function $h : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$, such that $h(L_1, 0) = L_1$ and $h(L_1, 1) = L_2$. A link equivalent to a set of non intertwined circles is called a *trivial link*. If it is a knot, we also call it the *unknot*.

Consider a link L and a sphere $\mathbb{S}^2$ embedded in such a way that it meets the link transversely in exactly two points P_1 and P_2. Then we can discern two different links L_1, L_2, after connecting P_1 and P_2. The first corresponds to the part of L in the interior of the sphere and the second to the part in the exterior. We then call L a *connected sum* with *factors* L_1, L_2, denoted $L_1 \# L_2$. A link that does not have non-trivial factors is called *prime*, otherwise *composite*. A link is *split* if there is a sphere embedded in the link complement that separates the link. Each of the components is called a *disjoint component* of the link and, conversely, the link is their *disjoint sum*.

Let $L \in \mathbb{R}^3$ be a link and let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection map. If for all $x \in L$, $|\pi^{-1}(x)| = 2$ or 1, and all the double points are finite and transverse, then the projection is said to be *regular*. For all knots defined here, there exists a regular projection [2, Ch. 3]. This allows to work with *link-diagrams*, i.e. a triple (V, E, σ), where (V, E) is a 4-regular plane graph with $V \neq \emptyset$ and $\sigma : V(G) \to \binom{E(G)}{2}$, where, for every $v \in V(G)$, $\sigma(v)$ is a set of two opposite edges of the embedding of G and encodes which pair is overcrossing. We call *crossing number* of a link L the minimum number of crossings that can have a diagram of it, where we suppose that trivial components have diagrams of size equal to zero. Such a diagram is called *minimal*. We also say that a diagram (V, E, σ) is K_4-minor free if the graph (V, E) does not contain any subgraph

[1] The first and the last author have partially been supported by the MINECO projects MTM2014-54745-P, MTM2017-82166-P and the María de Maetzu research grant MDM-2014-0445 (BGSMath). The last author has also been supported under an FPI grant from the MINECO research project MTM2015-67304-PI. The second author has been supported by the project "ESIGMA" (ANR-17-CE40-0028).
[2] Emails: juan.jose.rue@upc.edu, sedthilk@thilikos.info, vasiliki.velona@upf.edu

that is a subdivision of a K_4.

A *torus link* is a link that can be embedded on the torus. They are denoted by $T(p,q)$, $p,q \in \mathbb{Z}$, where p and q are the number of times that the link crosses the meridian and the longitude cycle, respectively. Let $\mathcal{T}_2$ be the closure of torus links of type $T(2,q)$, $q \in \mathbb{Z} - \{0\}$, under the connected sum operation. Let $\mathcal{L}$ be the class of links that have a K_4-minor free link-diagram and $\mathbf{mcl}(\mathcal{T}_2)$ be the closure of $\mathcal{T}_2$ under disjoint sum.

2 Structure of K_4-links and enumeration

Our first result gives the type of links that admit a K_4 minor-free diagram.

Theorem 2.1 *The links that admit a K_4 minor-free diagram are exactly the ones in $\mathbf{mcl}(\mathcal{T}_2)$, i.e. $\mathcal{L} = \mathbf{mcl}(\mathcal{T}_2)$.*

Based on the above, we derive a precise description of links that admit a K_4-minor free diagram. Let $\bar{\mathcal{L}}$ be the class of non-split links in $\mathcal{L}$ and $\hat{\mathcal{L}}$ the class of links in $\mathcal{L}$ with no trivial disjoint components, with size defined as their crossing number. We will obtain the asymptotic growth of $\mathcal{L}$ with respect to the number of edges in a minimal diagram (not the crossing number, so as to account also for trivial disjoint components). Let $L(z)$, $\bar{L}(z)$, and $\hat{L}(z)$ be the respective generating functions. Finally, let $\mathcal{T}$ be the class of non plane, unrooted trees, where the vertices are labelled with multisets of the set $\{1\} \cup \{\pm 3, \pm 5, ...\}$ and the edges are labelled with an element of the set $\{2, \pm 4, \pm 6, ...\}$. These labels will encode crossing numbers, hence a label i corresponds to size $|i|$. The size of $T \in \mathcal{T}$ is the sum of its labels. See Figure 1 for an example of such a tree.

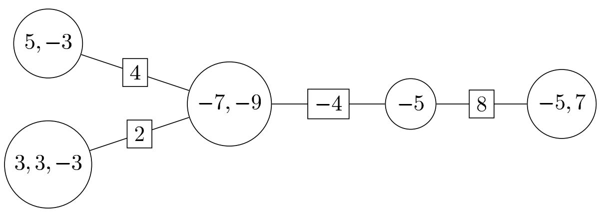

Figure 1. An object of $\mathcal{T}$.

Proposition 2.2 *$\bar{\mathcal{L}} \cong \mathcal{T}$ and* MSet$_{\geq 1}(\mathcal{T} \backslash T_1) \cong \hat{\mathcal{L}}$, *where* MSet *is the multiset operator with at least one element and $T_1 \in \mathcal{T}$ is the tree with one vertex and label 1.*

Using Proposition 2.2 and the framework of the *Symbolic Method* (see [4]), we can create combinatorial expressions for $\mathcal{L}$ that translate to generating functions. Let $\bar{\mathcal{K}}$ and $\mathcal{K}$ be the classes of prime knots and composite knots plus the trivial knot in $\mathcal{L}$, respectively. For prime knots in $\mathcal{L}$, i.e. prime torus knots $T(2, 2i+1)$, $i \in \mathbb{Z}\backslash\{0, -1\}$, it holds that $\bar{K}(z) = 2\sum_{i\geq 1} z^{2i+1} = \frac{2z^3}{1-z^2}$, counting according to their crossing number. For $\mathcal{K}$, it is enough to consider non-empty multisets of prime torus knots, therefore $\mathcal{K} = \mathrm{MSet}_{\geq 1}(\bar{\mathcal{K}})$, which translates to the expression

$$K(z) = \exp\left(\sum_{k\geq 1}\frac{1}{k}\bar{K}(z^k)\right) - 1 = \exp\left(\sum_{k\geq 1}\frac{1}{k}\frac{2z^{3k}}{1-z^{2k}}\right) - 1.$$

Let $\mathcal{G}$ be the class of unrooted and unlabelled non-plane trees. Notice that one cannot replace immediately z for $\mathcal{K}$ in the ogf $G(z)$, since the vertices are not distinguishable. Hence, to continue, one needs to use cycle indices. Let $\mathcal{G}^\bullet$ be the class of rooted and unlabelled non-plane trees. For the cycle index of $\mathcal{G}^\bullet$, it is known (see [1]) that

$$\mathcal{Z}_{\mathcal{G}^\bullet}(s_1, s_2, ...) = s_1 \exp\Big(\sum_{k\geq 1}\frac{1}{k}\mathcal{Z}_{\mathcal{G}^\bullet}(s_k, s_{2k}, ...)\Big).$$

Let $\mathcal{E}$ be the combinatorial class of the edge labels of $\mathcal{T}$, hence $E(z) = z^2 + \frac{2z^4}{1-z^2}$, and let $f(z) = E(z)K(z)$. We can now obtain the ordinary generating function of $\mathcal{F} = \mathcal{T}^\bullet \circ (\mathcal{E} \times \mathcal{L}_c)$. By Polya's Theorem, the latter satisfies the equation $F(z) = \mathcal{Z}_{\mathcal{G}^\bullet}(f(z), f(z^2), ...)$. Then, it holds that

$$\bar{L}(z) = \frac{F(z)}{E(z)} + \frac{E(z)}{2}\Big(-\frac{F(z)^2}{E(z)^2} + \frac{F(z^2)}{E(z^2)}\Big),$$

by the Dissymmetry Theorem (see [1]). The first terms of $\bar{L}(z)$ are as follows:

$$\bar{L} = 1 + z^2 + 2\,z^3 + 3\,z^4 + 4\,z^5 + 9\,z^6 + 12\,z^7 + 26\,z^8 + 40\,z^9 + ...$$

We can then get asymptotic estimates by means of complex analytic tools:

Theorem 2.3 *$\bar{L}$ has asymptotic growth of the form:*

$$[z^n]\bar{L} \sim \frac{cn^{-3/2}}{\Gamma(-1/2)}\rho^{-n}, \quad \rho \approx 0.44074, \quad c \approx 1.45557.$$

Additionally, $\hat{L}$ has a similar exponential growth but with $c \approx 3.61691$.

The generating function $L(z)$ is equal to $\hat{L}(z^2)\frac{1}{1-z}$, since a link diagram of n vertices has $2n$ edges and one needs to account the number of trivial components. We thus obtain the following corollary.

Corollary 2.4 *$\mathcal{L}$ has asymptotic growth of the form:*

$$[z^n]L(z) \sim \frac{cn^{-3/2}}{\Gamma(-1/2)}\rho^{-n},$$

where $\rho \approx 0.44074$ and $c \approx 8.97779$ or $c \approx 3.95687$, when n is even or odd, respectively.

Finally, we also get asymptotic estimates for the coefficients of $K(z)$. The following result is a consequence of Meinardus' Theorem (see [4, VIII.23]).

Theorem 2.5 *$\mathcal{L}$ has asymptotic growth of the form:*

$$[z^n]K(z) \sim Cn^k \exp(Kn^{1/2}),$$

where $K = 2\sqrt{\Gamma(2)\zeta(2)} \approx 2.56509$, $C = \frac{e^{2\ln(2)\zeta(0)}(\Gamma(2)\zeta(2))^{5/4}}{2\sqrt{\pi}} \approx 0.26275$, $k \approx -7/4$, and $\zeta(z)$ is the Riemann zeta function.

3 Enumeration of families of link diagrams

Enumerating K_4-free link-diagrams is equivalent to enumerating 4-regular unrooted planar maps which are K_4-minor free, that we call $\bar{\mathcal{M}}_1$. We first give a construction for the rooted ones, $\mathcal{M}_1$, with respect to edges, adapting the construction for rooted 4-regular maps in [8]. With this we obtain a functional system of equations that can be analyzed by using the analytic machinery developed by Drmota in [3]. Finally, by adapting an argument by Richmond and Wormald in [9], we are able to show that we can unroot the maps under study, showing that the maps in our class with non-trivial automorphisms are exponentially few. This study gives the following result:

Theorem 3.1 *The class of K_4-free link diagrams $\bar{\mathcal{M}}_1$ satisfies that*

$$[z^n]\bar{M}_1(z) \sim \frac{1}{2n}\frac{cn^{-3/2}}{\Gamma(-1/2)}\rho^{-n}2^n, \ \rho \approx 0.31184, \, c \approx 1.52265$$

Refining the combinatoric analysis in the previous case (refining it with more auxiliary classes) and using again an adaptation of the argument of Richmond and Wormald in [9], we are able to asymptotically enumerate also the class of minimal link diagrams $\bar{\mathcal{M}}_2$, as well as the class of K_4-free link diagrams of the unknot, $\bar{\mathcal{M}}_3$:

Theorem 3.2 *The class of K_4-free minimal link diagrams $\bar{\mathcal{M}}_2$, and the class* *of K_4-free link diagrams of the unknot, $\bar{\mathcal{M}}_3$ satisfy:*

$$[z^n]\bar{M}_2(z) \sim \frac{1}{2n}\frac{c_2 n^{-3/2}}{\Gamma(-1/2)}\rho_2^{-n},\ \rho_2 \approx 0.41456,\ c_2 \approx 0.81415,$$

$$[z^n]\bar{M}_3(z) \sim \frac{1}{2n}\frac{c_3 n^{-3/2}}{\Gamma(-1/2)}\rho_3^{-n},\ \rho_3 \approx 0.23188,\ c_3 \approx 2.19020.$$

Acknowledgement. All authors like to thank Lefteris Kirousis that contributed to the inception of this research.

References

[1] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial species and tree-like structures*, volume 67. Cambridge University Press, 1998.

[2] P. R. Cromwell. *Knots and links.* Cambridge University Press, 2004.

[3] M. Drmota. *Systems of functional equations. Random Structures and Algorithms*, 10, 103-124, 1997.

[4] P. Flajolet and R. Sedgewick. *Analytic combinatorics.* Cambridge University press, 2009.

[5] A. Kawauchi. *A survey of knot theory.* Birkhäuser, 2012.

[6] S. Kunz-Jacques and G. Schaeffer. The asymptotic number of prime alternating links. In *FPSAC'2001*, pages 10–p. Arizona State University, 2001.

[7] W. Menasco and M. Thistlethwaite. *Handbook of knot theory.* Elsevier, 2005.

[8] M. Noy, C. Requilé, and J. Rué. Enumeration of labeled 4-regular planar graphs. *Submitted*, available on-line at `arXiv:` 1709.04678.

[9] L. B. Richmond and N. C. Wormald. Almost all maps are asymmetric. *J. of Combinatorial Theory, Series B*, 63(1):1–7, 1995.

[10] C. Sundberg and M. Thistlethwaite. The rate of growth of the number of prime alternating links and tangles. *Pacific J. of mathematics*, 182(2):329–358, 1998.