

# A linear kernel for planar red-blue dominating set\*

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## Abstract

In the RED-BLUE DOMINATING SET problem, we are given a bipartite graph  $G = (V_B \cup V_R, E)$  and an integer  $k$ , and asked whether  $G$  has a subset  $D \subseteq V_B$  of at most  $k$  “blue” vertices such that each “red” vertex from  $V_R$  is adjacent to a vertex in  $D$ . We provide the first explicit linear kernel for this problem on planar graphs, of size at most  $46k$ .

**Keywords:** parameterized complexity, planar graphs, linear kernels, red-blue domination.

## 1 Introduction

**Motivation.** The field of parameterized complexity (see [6, 7, 16]) deals with algorithms for decision problems whose instances consist of a pair  $(x, k)$ , where  $k$  is known as the *parameter*. A fundamental concept in this area is that of *kernelization*. A kernelization algorithm, or *kernel*, for a parameterized problem takes an instance  $(x, k)$  of the problem and, in time polynomial in  $|x| + k$ , outputs an equivalent instance  $(x', k')$  such that  $|x'|, k' \leq g(k)$  for some function  $g$ . The function  $g$  is called the *size* of the kernel and may be viewed as a measure of the “compressibility” of a problem using polynomial-time preprocessing rules. A natural problem in this context is to find polynomial or linear kernels for problems that admit such kernelization algorithms.

A celebrated result in this area is the linear kernel for DOMINATING SET on planar graphs by Alber *et al.* [2], which gave rise to an explosion of (meta-)results on linear kernels on planar graphs [12] and other sparse graph classes [3, 8, 13]. Although of great theoretical importance, these meta-theorems have two important drawbacks from a practical point of view. On the one hand, these results rely on a problem property called *Finite Integer Index*, which guarantees the *existence* of a linear kernel, but nowadays it is still not clear how and when such a kernel can be effectively *constructed*. On the other hand, at the price of generality one cannot hope that general results of this type may directly provide explicit reduction rules and small constants for particular graph problems. Summarizing, as mentioned explicitly by Bodlaender *et al.* [3], these meta-theorems provide simple criteria to decide whether a problem admits a linear kernel on a graph class, but finding linear kernels with reasonably small constant factors for concrete problems remains a worthy investigation topic.

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**Our result.** In this article we follow this research avenue and focus on the RED-BLUE DOMINATING SET problem (RBDS for short) on planar graphs. In the RED-BLUE DOMINATING SET problem, we are given a bipartite<sup>1</sup> graph  $G = (V_B \cup V_R, E)$  and an integer  $k$ , and asked whether  $G$  has a subset  $D \subseteq V_B$  of at most  $k$  “blue” vertices such that each “red” vertex from  $V_R$  is adjacent to a vertex in  $D$ . This problem appeared in the context of the European railroad network [17]. From a (classical) complexity point of view, finding a red-blue dominating set (or rbds for short) of minimum size is NP-hard on planar graphs [1]. From a parameterized complexity perspective, RBDS parameterized by the size of the solution is  $W[2]$ -complete on general graphs and FPT on planar graphs [6]. It is worth mentioning that RBDS plays an important role in the theory of non-existence of polynomial kernels for parameterized problems [5].

The fact that RBDS involves a *coloring* of the vertices of the input graph makes it unclear how to make the problem fit into the general frameworks of [3, 8, 12, 13]. In this article we provide the first explicit (and quite simple) polynomial-time data reduction rules for RED-BLUE DOMINATING SET on planar graphs, which lead to a linear kernel for the problem.

**Theorem 1** RED-BLUE DOMINATING SET *parameterized by the solution size has a linear kernel on planar graphs. More precisely, there exists a polynomial-time algorithm that for each planar instance  $(G, k)$ , returns an equivalence instance  $(G', k')$  such that  $k' \leq k$  and either  $(G', k')$  is a negative instance or  $|V(G')| \leq 46 \cdot k'$ .*

This result complements several explicit linear kernels on planar graphs for other domination problems such as DOMINATING SET [2], EDGE DOMINATING SET [12, 18], EFFICIENT DOMINATING SET [12], CONNECTED DOMINATING SET [11, 15], or TOTAL DOMINATING SET [10]. It is worth mentioning that our constant is considerably smaller than most of the constants provided by these results. Since one can easily reduce the FACE COVER problem on a planar graph to RBDS (without changing the parameter)<sup>2</sup>, the result of Theorem 1 also provides a linear *bikernel* for FACE COVER (i.e., a polynomial-time algorithm that given an input of FACE COVER, outputs an equivalent instance of RBDS with a graph whose size is linear in  $k$ ). To the best of our knowledge, the best existing kernel for FACE COVER is quadratic [14]. Our techniques are much inspired by those of Alber *et al.* [2] for DOMINATING SET, although our reduction rules and analysis are slightly simpler. We start by describing in Section 2 our reduction rules for RED-BLUE DOMINATING SET when the input graph is embedded in the plane, and in Section 3 we prove that the size of a reduced plane YES-instance is linear in the size of the desired red-blue dominating set, thus proving Theorem 1. Finally, we conclude with some directions for further research in Section 4.

## 2 Reduction rules

In this section we propose reduction rules for RED-BLUE DOMINATING SET, which are largely inspired by the rules that yielded the first linear kernel for DOMINATING SET on planar graphs [2]. The idea is to either replace the neighborhood of some blue vertices by appropriate gadgets, or to remove some blue vertices and their neighborhood when we can assume that these blue vertices belong to the dominating set. We would like to point out that our rules have also some points in common with the ones for the current best kernel for DOMINATING SET [4]. In

<sup>1</sup>In fact, this assumption is not necessary, as if the input graph  $G$  is not bipartite, we can safely remove all edges between vertices of the same color.

<sup>2</sup>Just consider the *radial graph* corresponding to the input graph  $G$  and its dual  $G^*$ , and color the vertices of  $G$  (resp.  $G^*$ ) as red (resp. blue).

Subsection 2.1 we present two easy elementary rules that turn out to be helpful in simplifying the instance, and then in Subsections 2.2 and 2.3 we present the rules for a single vertex and a pair of vertices, respectively.

**Definition 1** *A graph  $G$  is reduced if none of the reduction Rules 1, 2, 3, or 4 can be applied to  $G$ .*

## 2.1 Elementary rules

The following two simple rules enable us to simplify an instance of RBDS. We would like to point out that similar rules have been provided by Weihe [17] in a more applied setting. We first need the definition of neighborhood.

**Definition 2** *Let  $G = (V_B \cup V_R, E)$  be a graph. The neighborhood of a vertex  $v \in V_B \cup V_R$  is the set  $N(v) = \{u : \{v, u\} \in E\}$ . The neighborhood of a pair of vertices  $v, w \in V_B$  is the set  $N(v, w) = N(v) \cup N(w)$ .*

**Rule 1** *Remove any blue vertex  $b$  such that  $N(b) \subseteq N(b')$  for some other blue vertex  $b'$ .*

**Rule 2** *Remove any red vertex  $r$  such that  $N(r) \supseteq N(r')$  for some other red vertex  $r'$ .*

**Lemma 1** *Let  $G = (V_B \cup V_R, E)$  be a graph. If  $G'$  is the graph obtained from  $G$  by the application of Rule 1 or 2, then there is a rbds in  $G'$  of size at most  $k$  if and only if there is one in  $G$ .*

*Proof.* For Rule 1, if  $N(b) \subseteq N(b')$  for two blue vertices  $b$  and  $b'$ , then any solution containing  $b$  can be transformed to a solution containing  $b'$  in which the set of dominated red vertices may have only increased. For Rule 2, if  $N(r') \subseteq N(r)$  for two red vertices  $r$  and  $r'$ , then any blue vertex dominating  $r'$  dominates also  $r$ .  $\square$

## 2.2 Rule for a single vertex

We present a rule for removing a blue vertex when it is necessarily a dominating vertex. For this we need the definition of private neighborhood.

**Definition 3** *Let  $G = (V_B \cup V_R, E)$  be a graph. The private neighborhood of a blue vertex  $b$  is the set  $P(b) = \{r \in N(b) : N(N(r)) \subseteq N(b)\}$ .*

Let us remark that for (classical) DOMINATING SET, each neighborhood is split into three subsets [2]. The third one corresponds to our private neighborhood, but since non-private neighbors can be used to dominate the private ones, an intermediary set is necessary for (classical) DOMINATING SET. In our problem it does not occur because non-private vertices are red and thus cannot belong to a rbds. This is one of the reasons why our rules are simpler.

**Rule 3** *Let  $v \in V_B$  be a blue vertex. If  $|P(v)| \geq 1$ :*

- *remove  $v$  and  $N(v)$  from  $G$ ,*
- *decrease the parameter  $k$  by 1.*

Our Rule 3 corresponds to Rule 1 for (classical) DOMINATING SET [2]. In both rules we know that  $v$  is in the dominating set, but for RBDS we can remove it and its neighborhood (and decrease the parameter accordingly); this is not possible for DOMINATING SET, since vertices of the neighborhood possibly belong to the dominating set, and hence they cannot be removed **thus** a gadget is added to enforce  $v$  to be dominating. We prove in the following lemma that Rule 3 is safe.

**Lemma 2** Let  $G = (V_B \cup V_R, E)$  be a graph *reduced under Rules 1 and 2* and let  $v \in V_B$ . If  $(G', k-1)$  is the *instance* obtained from  $(G, k)$  by the application of Rule 3 on a vertex  $v$ , then there is a rbds in  $G$  of size at most  $k$  if and only if there is one in  $G'$  of size at most  $k-1$ .

*Proof.* Let  $D$  be a rbds in  $G$  with  $|D| \leq k$ . Since  $G$  is reduced, necessarily  $v \in D$  in order to dominate  $P(v)$ . Since  $G'$  does not contain any vertex of  $N(v)$ ,  $D \setminus \{v\}$  is also a rbds of  $G'$  of size at most  $k-1$ . Conversely, let  $D'$  be a rbds in  $G'$  with  $|D'| \leq k'$ . Clearly  $D' \cup \{v\}$  is a rbds of  $G$  of size at most  $k'+1$ .  $\square$

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In the following fact we prove that if we assume that Rules 1 and 2 have been exhaustively applied, then Rule 3 is equivalent to a simpler rule which consists in removing an appropriate connected component of size two and decrease the parameter  $k$  by one.

**Fact 1** Assume that  $G$  is reduced under Rules 1 and 2, and let  $v \in V_B$ . Then  $P(v) \neq \emptyset$  if and only if  $N(v) = P(v) = \{r\}$  and  $N(r) = \{v\}$  for some  $r \in V_R$ .

*Proof.* Obviously if  $P(v) = \{r\}$  then  $P(v) \neq \emptyset$ . Conversely, assume for contradiction that  $|N(v)| \geq 2$  with  $r \in P(v)$ , then  $N(r) \supseteq \{v, b\}$  for some  $b \in V_B$  (since  $r$  have incomparable neighborhood with other red vertices in  $N(v)$ , by Rule 2), finally  $N(b) \subseteq N(v)$  (since  $r$  is private), a contradiction with Rule 1.  $\square$

### 2.3 Rule for a pair of vertices

We now provide a rule for either reducing the size of the neighborhood of a pair of blue vertices, or for removing some blue vertices together with their neighborhood. For this, we first define the private neighborhood of a pair of blue vertices.

**Definition 4** Let  $G = (V_B \cup V_R, E)$  be a graph. The private neighborhood of a pair of blue vertices  $v, w \in V_B$  is the set  $P(v, w) = \{r \in N(w, v) : N(N(r)) \subseteq N(v, w)\}$ .

We would like to note that the definition of private neighborhood is similar to that of the third subset of neighbors defined for (classical) DOMINATING SET [2].

**Rule 4** Let  $v, w$  be two distinct blue vertices. If  $|P(v, w)| > 1$  and there is no blue vertex  $d \neq v, w$  which dominates  $P(v, w)$ :

1. if  $P(v, w) \not\subseteq N(v)$  and  $P(v, w) \not\subseteq N(w)$ :
  - remove  $v, w$ , and  $N(v, w)$  from  $G$ ,
  - decrease the parameter  $k$  by 2;
2. if  $P(v, w) \subseteq N(v)$  and  $P(v, w) \subseteq N(w)$ :
  - remove  $P(v, w)$  from  $G$ ,
  - add a new red vertex  $r$  and the edges  $\{v, r\}, \{w, r\}$ ;
3. if  $P(v, w) \subseteq N(v)$  and  $P(v, w) \not\subseteq N(w)$ :
  - remove  $v$  and  $N(v)$  from  $G$ ,
  - decrease the parameter  $k$  by 1;
4. if  $P(v, w) \not\subseteq N(v)$  and  $P(v, w) \subseteq N(w)$ :

- symmetrically to Case 3.

Again, our Rule 4 corresponds to Rule 2 for (classical) DOMINATING SET [2]. When it is possible, we also remove vertices instead of adding gadgets, which will result in a better kernel bound. In this way, we can also expect to obtain better experimental results in practical applications. We prove in the following lemma that Rule 4 is safe.

**Lemma 3** *Let  $G = (V_B \cup V_R, E)$  be a graph reduced under Rules 1 and 2 and let  $v, w$  be two distinct blue vertices. If  $(G', k')$  is the instance obtained from  $(G, k)$  by the application of Rule 4 on  $v$  and  $w$ , then there is a rbds in  $G$  of size at most  $k$  if and only if there is one in  $G'$  of size at most  $k'$ .*

*Proof.* Assume for simplicity in the whole proof that  $D$  is a minimal rbds in  $G$  of size exactly  $k$ , and that  $D'$  is a rbds in  $G'$  of size exactly  $k'$ . We distinguish the four possible cases of Rule 4:

1. Since we are in Case 1, there is no single vertex which dominates  $P(v, w)$ , we need at least two vertices in  $N(P(v, w))$  in order to dominate  $P(v, w)$ . Hence  $D \setminus N(P(v, w))$  is a rbds in  $G'$  of size  $k - 2$ . Conversely, we have that  $D' \cup \{v, w\}$  is a rbds in  $G$  of size  $k' + 2$ .
2. Since we are in Case 2 and because  $G$  is reduced, necessarily  $v \in D$  or  $w \in D$  (otherwise, two vertices are needed in order to dominate  $P(v, w)$ , contradicting minimality). Hence  $D$  is a rbds in  $G'$  of size  $k$ . Conversely, since  $r$  needs to be dominated, we have that  $v \in D'$  or  $w \in D'$ . Hence  $D'$  is a rbds in  $G$  of size  $k'$ .
3. Since we are in Case 3 and because  $G$  is reduced, necessarily  $v \in D$  (otherwise, there is contradiction with minimality). Hence  $D \setminus \{v\}$  is a rbds in  $G'$  of size  $k - 1$ . Conversely, we have that  $D' \cup \{v\}$  is a rbds in  $G$  of size  $k' + 1$ .
4. Symmetrically to Case 3. □

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### 3 Analysis of the kernel size

We will show that a graph reduced under our rules has size linear in  $|D|$ , the size of a solution. To this aim we assume that the graph is plane (that is, given with a fixed embedding). We recall that an embedding of a graph  $G = (V, E)$  in the plane  $\mathbb{R}^2$  is a function  $\pi : V \cup E \mapsto \mathcal{P}(\mathbb{R}^2)$ , which maps each vertex to a point of the plane and each edge to a simple curve of the plane, in such a way that the vertex images are pairwise disjoint, and each edge image corresponding to an edge  $\{u, v\}$  has as endpoints the vertex images of  $u$  and  $v$ , and does not contain any other vertex image. An embedding is planar if any two edge images may intersect only at their endpoints. In the following, for simplicity, we may identify vertices and edges with their images in the plane. Following Albert *et al.*, we will define a notion of region (in an embedded graph) adapted to our definition of neighborhood, and we will show that, given a solution  $D$ , there is a maximal region decomposition  $\mathfrak{R}$  such that:

- $\mathfrak{R}$  has  $O(|D|)$  regions,
- $\mathfrak{R}$  covers all vertices, and
- each region of  $\mathfrak{R}$  contains  $O(1)$  vertices.

The three following propositions treat respectively each of the above claims.

We now define our notion of region.

**Definition 5** Let  $G = (V_B \cup V_R, E)$  be a plane graph and let  $v, w \in V_B$  be a pair of distinct blue vertices. A region  $R(v, w)$  between  $v$  and  $w$  is a closed subset of the plane such that:

- the boundary of  $R(v, w)$  is formed by two simple paths connecting  $v$  and  $w$ , each of them having at most 4 edges,
- all vertices (strictly) inside  $R(v, w)$  belong to  $N(v, w)$  or  $N(N(v, w))$ ,
- the complementary  $\overline{R(v, w)}$  is connected.

We denote by  $\partial R(v, w)$  the boundary of  $R(v, w)$  and by  $V(R(v, w))$  the set of vertices in the region (that is, vertices strictly inside, on the boundary, and the two extremities  $v, w$ ). The size of a region is  $|V(R(v, w))|$ . We denote by  $R_1(v_1, w_1) \cup R_2(v_2, w_2)$  is the union of the two closed sets and by  $R_1(v, w) \uplus R_2(v, w)$  the special case where the union is a region; that can occur only if the two region share both extremities and one path, the boundary of  $R_1(v, w) \uplus R_2(v, w)$  is then the two other paths.

Note that a subgraph defining a region has diameter at most 4, to be compared with diameter at most 3 in [2]. We would like to point out that the assumption that vertices are distinct is not necessary, but it makes the proofs easier. In the following whenever we speak about a pair of vertices we assume them to be distinct. Note also that we do not assume that the two paths of the boundary of a region are edge-disjoint or distinct, hence in particular a path corresponds to a degenerated region.

We want to decompose a reduced graph into a set of regions which do not overlap each other. To formalize this, we provide the following definition of *crossing* regions, which can be seen as a more formal and precise version of the corresponding definition given in [2].

Recall that we are considering a plane graph, hence for each vertex  $v$ , the embedding induces a circular ordering on the edges incident to  $v$ . We first need the (recursive) definition of *confluent* paths.

**Definition 6** Two simple paths  $p_1, p_2$  are confluent if:

- they are vertex-disjoint, or
- they are edge-disjoint and for each vertex  $v \in p_1 \cap p_2$  distinct from the extremities, among the four edges of  $p_1, p_2$  containing  $v$ , the two edges in  $p_1$  are consecutive in the circular ordering given by the embedding (hence, the two edges in  $p_2$  are consecutive as well), or
- the two paths obtained by contracting common edges are confluent.

Note that a path is confluent with itself.

**Definition 7** Two distinct regions  $R_1, R_2$  do not cross if:

- $(R_1 \setminus \partial R_1) \cap R_2 = (R_2 \setminus \partial R_2) \cap R_1 = \emptyset$  (i.e., the interiors of the regions are disjoint), and
- any path  $p_1$  in  $\partial R_1$  is confluent with any path  $p_2$  in  $\partial R_2$ .

Otherwise, we say that  $R_1, R_2$  cross. If two regions cross because of two paths  $p_1 \in \partial R_1$  and  $p_2 \in \partial R_2$  that are not confluent, we say that these regions cross on  $v \in p_1 \cap p_2$  if:

- $v$  does not satisfy the ordering condition of Definition 6, or
- $v$  is an extremity of an edge  $e$  such that in  $G/e$  (i.e., the graph obtained from  $G$  by contracting  $e$ ),  $R_1$  and  $R_2$  cross on the vertex resulting from the contraction.

Of course, two regions can cross on many vertices. We use the latter condition of Definition 7 in the case of degenerated regions (that is, paths), where only this condition may hold.

We now have all the material to define what a region decomposition is. Compared to Alber *et al.* [2, Definition 3], we have two additional conditions to be satisfied by a *maximal* decomposition, which are in fact conditions satisfied by the region decomposition constructed by the greedy algorithm in [2].

**Definition 8** Let  $G = (V_B \cup V_R, E)$  be a plane graph and let  $D \subseteq V_B$ . A  $D$ -decomposition of  $G$  is a set of regions  $\mathfrak{R}$  between pairs of vertices in  $D$  such that:

- any region  $R \in \mathfrak{R}$  between two vertices  $v, w$  is such that  $V(R) \cap D = \{v, w\}$ , and
- any two regions in  $\mathfrak{R}$  do not cross.

We denote  $V(\mathfrak{R}) = \bigcup_{R \in \mathfrak{R}} V(R)$ . A  $D$ -decomposition is maximal if there are no regions

- $R \notin \mathfrak{R}$  such that  $\mathfrak{R} \cup \{R\}$  is a  $D$ -decomposition with  $V(\mathfrak{R}) \subsetneq V(\mathfrak{R} \cup \{R\})$ ,
- $R \in \mathfrak{R}$  and  $R' \notin \mathfrak{R}$  with  $V(R) \subsetneq V(R')$  such that  $\mathfrak{R} \cup \{R\} \setminus \{R'\}$  is a  $D$ -decomposition, or
- $R_1, R_2 \in \mathfrak{R}$  such that  $\mathfrak{R} \cup \{R_1 \uplus R_2\} \setminus \{R_1, R_2\}$  is a  $D$ -decomposition.

In order to bound the number of regions in a decomposition, we need the following definition. **note that we consider multi graph without loop.**

**Definition 9** A planar multigraph  $G$  is thin if there is a planar embedding of  $G$  such that for any two edges  $e_1, e_2$  with identical *extremities*, there is a vertex image inside the two areas enclosed by the edge images of  $e_1$  and  $e_2$ . In other words, no two edges are homotopic.

In [2, Lemma 5], the bound on the number of regions in a decomposition relies on applying Euler's formula to a thin graph. Since it appears that some arguments are missing in the original proof, we provide here for completeness an alternative proof.

**Lemma 4** If  $G = (V, E)$  is a thin planar multigraph with  $|V| \geq 3$ , then  $|E| \leq 3|V| - 6$ .

*Proof.* Recall that a triangulated (simple) graph is a maximal planar graph, that is, a graph with all faces bounded by 3 edges. For a triangulated (simple) graph  $H = (V_H, E_H)$  the Euler's formula claim that  $|E_H| = 3|V_H| - 6$ . We will show that the notion of triangulation can be extended to thin multigraphs and that the Euler's formula is still true. Consider a multigraph, we triangulated it on the following way: the degree of a face is the length of a facial walk; for each face of degree 4 or more, we add arbitrarily an edge between two non-consecutive vertices; for each face of degree 2, we know that there is a vertex inside because the multigraph is thin, and we add the two edges between that vertex and the two ones on the face boundary. Now, given a triangulated multigraph  $G = (V, E)$ , we transform it into a triangulated (simple) graph  $H = (V_H, E_H)$  on the following way: for each multiple edge  $e$  between  $u, v$ , we know that  $e$  is in two triangle faces containing vertices  $x, u, v$  and  $y, v, u$  where  $x, y \in V$ , we subdivide  $e$  into  $\{u, w_e\}, \{v, w_e\}$  where  $w_e$  is a new vertex, and we add edges  $\{x, w_e\}, \{y, w_e\}$ . It holds that  $|E| + 3\alpha = |E_H| = 3|V_H| - 6 = 3(|V| + \alpha) - 6$  where  $\alpha$  is the number of multiple edges.  $\square$

We would like to point out that it is possible to prove that any vertex in a reduced graph is on a path on at most 4 edges connecting two dominating vertices. Since in what follows we will use several restricted variants of this property, we will provide an ad-hoc proof for each case.

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**Proposition 1** *Let  $G$  be a reduced plane graph and let  $D$  be a rbds in  $G$  with  $|D| \geq 3$ . There is a maximal  $D$ -decomposition of  $G$  such that  $|\mathfrak{R}| \leq 3 \cdot |D| - 6$ .*

*Proof.* The proof strongly follows the one of Alber *et al.* [2, Lemma 5 and Proposition 1]. Even if our definition of region is different, we shall show that the same algorithm can be used to construct such a  $D$ -decomposition.

We consider the algorithm which, for each vertex  $u$ , adds greedily to the decomposition  $\mathfrak{R}$  a region  $R$  between any two vertices  $v, w \in D$ , containing  $u$ , not containing any vertex of  $D \setminus \{v, w\}$ , not crossing any region of  $\mathfrak{R}$ , and of maximal size, if it exists. By definition,  $\mathfrak{R} \cup \{R\}$  is a region decomposition, and by greediness and because regions are chosen of maximal size, the decomposition is maximal according to Definition 8.

In order to apply Lemma 4, we proceed to define a multigraph, and then we will prove that it is thin. Let  $G_{\mathfrak{R}} = (D, E_{\mathfrak{R}})$  be the multigraph with vertex set  $D$  and with an edge  $\{v, w\}$  for each region in  $\mathfrak{R}$  between two dominating vertices  $v$  and  $w$ . Let  $\pi$  be the embedding of the plane graph  $G$ , and we consider the embedding  $\pi_{\mathfrak{R}}$  of  $G_{\mathfrak{R}}$  such that for  $v \in D$ ,  $\pi_{\mathfrak{R}}(v) = \pi(v)$ , and for  $e \in E_{\mathfrak{R}}$  corresponding to a region  $R \in \mathfrak{R}$  with  $p$  an arbitrary boundary path,  $\pi_{\mathfrak{R}}(e) = \bigcup_{f \in p} \pi(f)$  (note that such a path does not contain inner dominating vertices, hence  $\pi_{\mathfrak{R}}(e)$  does not contain vertex images). For an edge set  $F \subseteq E_{\mathfrak{R}}$ , we denote  $\pi_{\mathfrak{R}}(F) = \bigcap_{e \in F} \pi_{\mathfrak{R}}(e)$ . If the constructed embedding is not planar, we proceed to modify it in order to make it planar. Given  $F, F' \subseteq E_{\mathfrak{R}}$ , observe that if  $F \subseteq F'$  then  $\pi_{\mathfrak{R}}(F) \supseteq \pi_{\mathfrak{R}}(F')$ . As far as there exists an edge set  $F \subseteq E_{\mathfrak{R}}$  such that  $\pi_{\mathfrak{R}}(F) \neq \emptyset$ ,  $\pi_{\mathfrak{R}}(F) \neq \pi_{\mathfrak{R}}(v)$  and  $\pi_{\mathfrak{R}}(F) \neq \pi_{\mathfrak{R}}(v) \cup \pi_{\mathfrak{R}}(w)$  for  $v, w \in D$  (that is, as far there is edges intersecting elsewhere than on a extremity), we apply the following procedure in the inclusion order of such edge sets, starting with a maximal set of edges. Consider the subset of the plane  $C_{\epsilon}(F)$  which contains for each  $x \in \pi_{\mathfrak{R}}(F)$ , except possibly endpoints of  $\pi_{\mathfrak{R}}(F)$ , a closed boole of center  $x$  and radius  $\epsilon$  for some sufficiently small real number  $\epsilon > 0$ , such that  $C_{\epsilon}(F)$  intersects only curves corresponding to edges in  $F$  (note that since we proceed in the inclusion order, there exists such a subset of the plane). Since the paths defining the borders of regions in  $\mathfrak{R}$  are confluent (that is, edges of a path are consecutive around a intersection vertex), it is easy to see that for each edge  $e \in F$  there is a connected curve  $C_e$  inside  $C_{\epsilon}(F)$  disjoint from all the edge images of  $\pi_{\mathfrak{R}}$  except for the two endpoints of  $\pi_{\mathfrak{R}}(e) \cap C_{\epsilon}(F)$ . Hence we can replace, for each edge  $e$  in such a set  $F$ , the edge image  $\pi_{\mathfrak{R}}(e)$  in  $C_{\epsilon}(F)$  with the curve  $C_e$ . When there does not exist such an edge set  $F$  any more, the obtained embedding of  $G_{\mathfrak{R}}$  is planar, since in that case any two edge images may intersect only at their endpoints.

We will now prove that the multigraph  $G_{\mathfrak{R}}$  is thin, that is, for each pair  $e_1, e_2 \in E_{\mathfrak{R}}$  between  $v, w$  (corresponding to regions  $R_1, R_2$ ) there is a vertex of  $D$  in both open subsets  $O_{\mathfrak{R}}$  of the plane enclosed by  $e_1, e_2$ . This property allows to apply Lemma 4, implying that the constructed decomposition has at most  $3|D| - 6$  regions. In fact, instead of  $O_{\mathfrak{R}}$  we consider  $O = O_{\mathfrak{R}} \setminus (R_1 \cup R_2 \cup \{C_{\epsilon}(F) \mid F \ni e_1 \text{ ou } F \ni e_2\})$ . Note that neither  $R_1, R_2$ , nor any  $C_{\epsilon}(F)$  contains a dominating vertex, therefore any dominating vertex in  $O_{\mathfrak{R}}$  is in  $O$ . Note also that  $O \subset \overline{R_1 \cup R_2}$  is not empty as otherwise  $R_1$  and  $R_2$  share an entire path, which contradict the maximality of the decomposition, since  $R_1, R_2$  could be replaced with  $R_1 \uplus R_2$ .

Let us assume for the sake of contradiction that there is no vertex of  $D$  in  $O$ . We distinguish three cases:

- If  $O$  intersects a region  $R_3 \in \mathfrak{R}$ , then  $R_3$  is necessarily between  $v$  and  $w$ , as otherwise  $R_3$  would cross  $R_1$  or  $R_2$ . In this case, we can recursively apply the same argument to  $R_1, R_3$  and  $R_2, R_3$ . The recursion must be finite as  $u_1 < u_3 < u_2$ , with  $u_i \in V(R_i) \cap N(v)$   $i \in [1, 3]$ , in the circular order around  $v$  (similarly around  $w$ ), and the degree of  $v$  and is finite.

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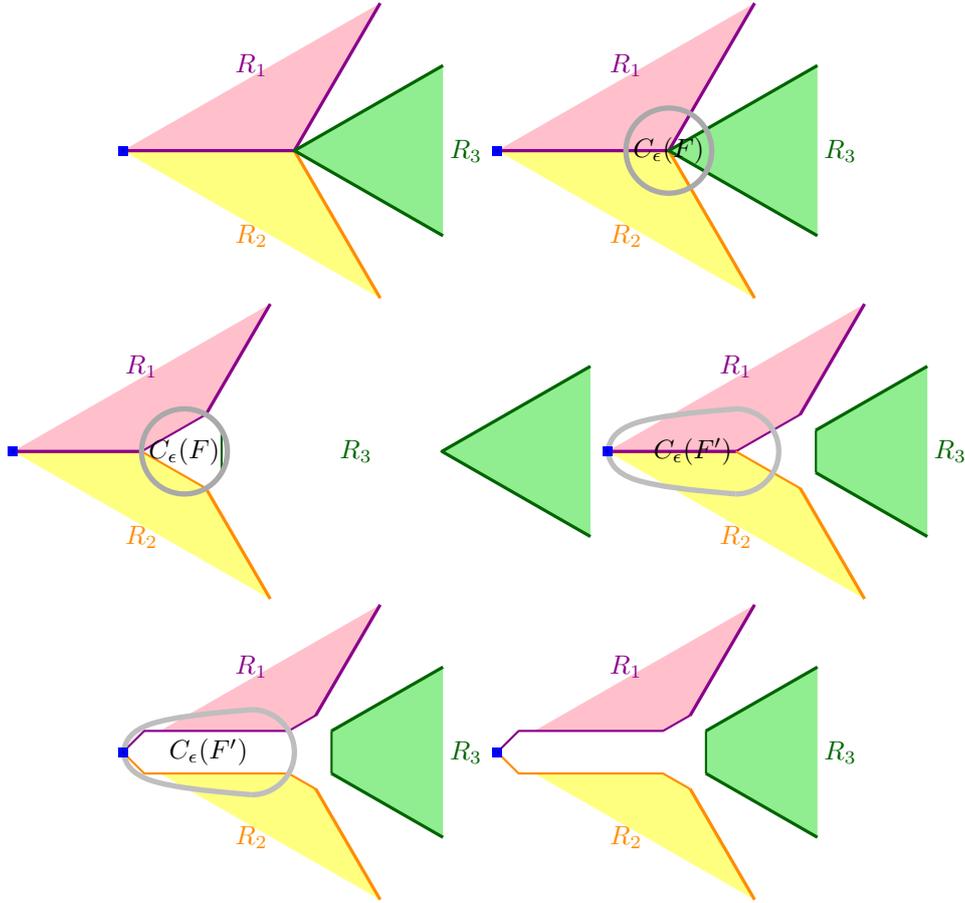


Figure 1: An example of the re-embedding procedure. We consider three regions  $R_1, R_2, R_3$  (light color), the edges of  $G_{\mathfrak{R}}$  (dark color) are initially embedded on the boundary of the regions. First, we modify the intersection of  $F$  the set of three edges (which is a point). Then, we modify the intersection of  $F'$  the set of edges corresponding to  $R_1, R_2$  (which is a segment). The edges are separated, unless on their extremity.

- Otherwise, assume that  $O$  does not contain any blue vertex. Then the red vertices in  $O$  (if any) must be dominated by  $v$  or  $w$ . Hence  $R_1 \cup O \cup \partial(O)$  is a larger region enclosed by a path of  $R_1$  and a path of  $R_2$ . We have a contradiction with the maximality of  $\mathfrak{R}$ .
- Otherwise, if  $O$  contains at least one blue vertex  $b \notin D$ , we shall show that  $b$  lies on a path  $P = \{v, r, b, r', w\}$  for some vertices  $r, r'$ . Indeed, since Rule 1 cannot be applied on  $b$ ,  $N(b) \neq \emptyset$ , there is some vertex  $r \in N(b)$  that is dominated, without loss of generality, by  $v$  and not by  $w$ . Again, by Rule 1,  $b$  and  $v$  have incomparable neighborhoods, so there is some  $r' \in N(b) \setminus N(v)$  that is dominated by  $w$  and not by  $v$ . Notice that  $r, r'$  are in  $O$  or in its boundary, hence  $P = \{v, r, b, r', w\}$  is a region which does not cross  $R_1, R_2$ , a contradiction with the maximality of  $\mathfrak{R}$ .  $\square$

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**Proposition 2** Let  $G = (V_B \cup V_R, E)$  be a reduced plane graph and let  $D$  be a rbds in  $G$  with  $|D| \geq 3$ . If  $\mathfrak{R}$  is a maximal  $D$ -decomposition, then  $V_B \cup V_R \subseteq V(\mathfrak{R})$ .

*Proof.* The proof again follows that of Alber *et al.* [2, Lemma 6 and Proposition 2], where similar arguments are used to bound the number of vertices which are not included in a maximal

region decomposition. We have to show that all vertices are included in a region of  $\mathfrak{R}$ , that is,  $V_R \cup V_B \subseteq V(\mathfrak{R})$ .

Since  $N(D)$  covers  $V_R$ , it holds that  $V_R = \bigcup_{v \in D} N(v)$ . We proceed to prove that  $N(v) \subseteq V(\mathfrak{R})$  for all  $v \in D$ . Let  $v \in D$  and let  $r \in N(v)$ . We now show that there is a path  $P = \{v, r, \dots, w\}$  with  $w \in D$  and with at most four edges. Indeed, since Rule 3 cannot be applied on  $v$ , then  $r \notin P(v)$ , and by definition of private neighborhood there are two vertices  $b \in N(r)$  and  $r' \in N(b) \setminus N(v)$ . If  $b \in D$ , then  $P = \{v, r, b = w\}$  is the desired path. Otherwise,  $r'$  is dominated by some vertex  $w \in D$ , and then  $P = \{v, r, b, r', w\}$ .

**Assume for contradiction that**  $r \notin V(\mathfrak{R})$ , then  $P \not\subseteq R$  and  $P$  does not cross  $R$  on  $r$  for any  $R \in \mathfrak{R}$ . We distinguish two cases depending on the length of  $P$ :

- If  $P = \{v, r, w\}$  for some vertex  $w$ , then  $P$  can be added to  $\mathfrak{R}$ , which contradicts the maximality of  $\mathfrak{R}$ .
- If  $P = \{v, r, b, r', w\}$  for some vertices  $b, r', w$  with  $w \neq v$ , then either  $P$  can be added, which again contradicts the maximality of  $\mathfrak{R}$ , or  $P$  crosses some region  $R(x, y)$  of  $\mathfrak{R}$ . **Recall we assume  $r \in R(x, y)$** . We distinguish two cases (see Figure 2 for an illustration):
  - If  $P$  and  $R(x, y)$  cross on  $b$ , then  $b$  is on  $\partial R(x, y)$ . Let  $r''$  be a vertex on  $\partial R(x, y)$  such that the edge  $\{b, r''\}$  is the successor of the edge  $\{b, r\}$  in the circular order defined by the embedding. In this case, we consider the path  $P' = \{v, r, b, r'', x\}$ .
  - Otherwise, necessarily  $P$  crosses a region  $R(x, y) \in \mathfrak{R}$  on  $r'$ , and then  $r'$  is on  $\partial R(x, y)$ . Assume without loss of generality that  $r' \in N(x)$ . In this case, we consider the path  $P' = \{v, u, b, r', x\}$ .

In both cases, either  $P'$  can be added to  $\mathfrak{R}$ , which contradicts the maximality of  $\mathfrak{R}$ , or  $P'$  crosses another region and we can apply recursively the same argument. **The recursion must be finite as  $r < r'' < r'$  and  $b < x < w$  in the circular order around  $b$  and  $r'$  respectively, and the degree of  $b$  and  $r'$  are finite.**

So  $\bigcup_{v \in D} N(v) \subseteq V(\mathfrak{R})$ , as we wanted to prove.

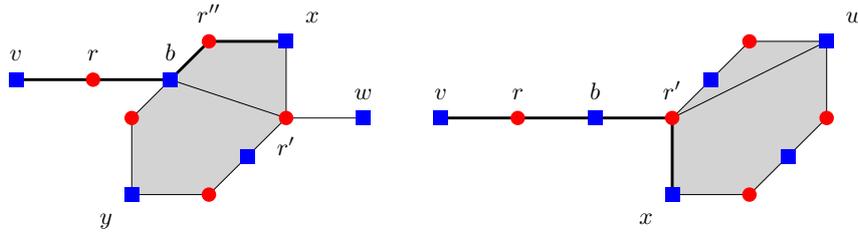


Figure 2: Illustration of the two ways that the path  $\{v, u, b, r, w\}$ , as defined in Proposition 2, can cross a region. Blue (resp. red) vertices are depicted with  $\blacksquare$  (resp.  $\bullet$ ).

We finally show that  $V_B \subseteq V(\mathfrak{R})$ . Recall that we assume that  $|D| > 2$ . Let first  $b \in V_B \setminus D$ . Since  $G$  is reduced, by Rule 1,  $b$  is neighbor of two red vertices  $r'$  and  $r''$  dominated respectively by  $v$  and  $w$  with  $v \neq w$  (as otherwise vertex  $b$  could be removed by Rule 1). We consider the (degenerated) region  $\{v, r', b, r'', w\}$ , and with an argument similar to the one given above, if we assume that  $b \notin V(\mathfrak{R})$  we obtain a contradiction. Let then  $v \in D$ . By Rule 3,  $v$  cannot be a single dominating vertex in a connected component. Hence there is a vertex  $w \in D$  at distance at most 4 from  $v$ . We consider a path between  $v$  and  $w$  as a region, and once again we obtain a contradiction using similar arguments. So  $V_B \subseteq V(\mathfrak{R})$ .

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Therefore, all the vertices of  $G$  belong to the decomposition  $\mathfrak{R}$ , as we wanted to prove.  $\square$

**Proposition 3** *Let  $G = (V_B \cup V_R, E)$  be a reduced plane graph, let  $D$  be a rbd in  $G$ , and let  $v, w \in D$ . A region  $R$  between  $v$  and  $w$  contains at most 15 vertices distinct from  $v$  and  $w$ .*

*Proof.*

Since the graph is reduced, Rule 4 can not be applied on  $v, w$ , then either  $P(v, w) \leq 1$ , or  $P(v, w) \subseteq N(u)$ . First, assume that  $P(v, w) \leq 1$ , then the region  $R$  contains at most 8 vertices on  $\partial R$ , 1 red vertex in  $P(v, w)$  and 2 blue vertices (see Figure 3(a)); it clearly does not exceed the claimed bound. Now, assume that  $P(v, w) \subseteq N(u)$  in the rest of the proof.

We bound separately the number of red neighbors of  $v$  and  $w$  the number of blue vertices in the region  $R$ . It will become clear from the proof that the worst bound is given by the case where  $\partial R$  contains 8 vertices, which will be henceforth denoted by  $v, r_v, b, r_w, w, r'_w, b', r'_v$ .

In order to bound the total number of vertices, we need the following remark. If there is any blue vertex  $c$  in  $R$  distinct from  $v, b, w, b', u$ , then we can show that there is a path  $P = \{v, r, c, r', w\}$ . Indeed, since Rule 1 cannot be applied on  $c$ ,  $N(c) \neq \emptyset$ , and there is some vertex  $r \in N(c)$  that is dominated by a vertex  $v$ . Again, by Rule 1,  $c$  and  $v$  have incomparable neighborhoods, so there is some  $r' \in N(c) \setminus N(v)$  that is dominated  $w$ . Hence  $P = \{v, r, c, r', w\}$  is the desired path. Such a path disconnects  $u$  from either  $b$  or  $b'$ . Hence, considering two blue vertices, we can subdivide  $R$  into at most 3 subregions. We consider vertices  $\tilde{b}, \tilde{b}'$  (possibly  $b = \tilde{b}$  or  $b' = \tilde{b}'$ ) which define a subregion containing exactly  $\tilde{b}, \tilde{b}', u$  as blue vertices (this does not depend on the chosen path). Notice that any red vertex in  $P(v, w)$  should belong to the subregion containing  $u$ .

By definition, the red vertices in  $\partial R$  are  $r_v, r_w, r'_w, r'_v$ . Hence  $|(N(v, w) \setminus P(v, w)) \cap V(R)| \leq 4$ . Recall that red vertices in  $R \setminus \partial R$  are necessarily private and  $P(v, w) \subseteq N(u)$  for some blue vertex  $u$ , so for any vertex  $r \in P(v, w)$  it holds that  $r \in N(u) \cap (N(v) \cup N(w))$ . According to the remark above, we can divide  $R$  into at most 3 subregions and all vertices in  $P(v, w)$  are in the subregion containing  $u$ .

Out of the possible neighborhoods of a red vertex in  $P(v, w) \cap V(R)$ , we claim that while preserving adjacency with  $u$ , adjacency with  $v$  or  $w$ , planarity, and the incomparability of neighborhoods (given by Rule 2), there can be at most 4 private vertices, and that this case is attained when each red vertex in the region has degree 3, with respective neighborhoods  $\{u, v, \tilde{b}\}, \{u, v, \tilde{b}'\}, \{u, w, \tilde{b}\}$ , and  $\{u, w, \tilde{b}'\}$  for two vertices  $\tilde{b}, \tilde{b}'$  as defined above. Let us now sketch how to prove this claim, distinguishing the maximum degree of a red vertex  $r \in P(v, w) \cap V(R)$  in the subregion containing  $u$ , denoted  $d(r)$ :

- If  $d(r) = 1$ , there is a contradiction since  $r$  should be adjacent to  $u$  and  $v$  or  $w$ .
- If  $d(r) = 2$ , then there can be at most two red vertices in  $P(v, w) \cap V(R)$  with neighborhoods  $\{v, u\}$  and  $\{w, u\}$ .
- If  $d(r) = 3$ , the reader can easily check that there are at most 4 red vertices (see Figure 3(b-d), where the considered subregion containing  $u$  is the darker one).
- If  $d(r) = 4$ , then the worst configuration is given by 3 vertices, the other two having degree 3.
- If  $d(r) = 5$ , then there is at most 1 vertex by incomparability of neighborhoods.

Hence  $|P(v, w)| \leq 4$ , as we wanted to prove.

It just remains to bound the number of blue vertices distinct from  $v, w$ . Since  $G$  is reduced by Rule 1, blue vertices have incomparable neighborhoods, in particular with  $N(v)$  and  $N(w)$ , so for any blue vertex  $b$  it holds that  $N(b) \cap N(v) \neq \emptyset$  and  $N(b) \cap N(w) \neq \emptyset$ . Recall that

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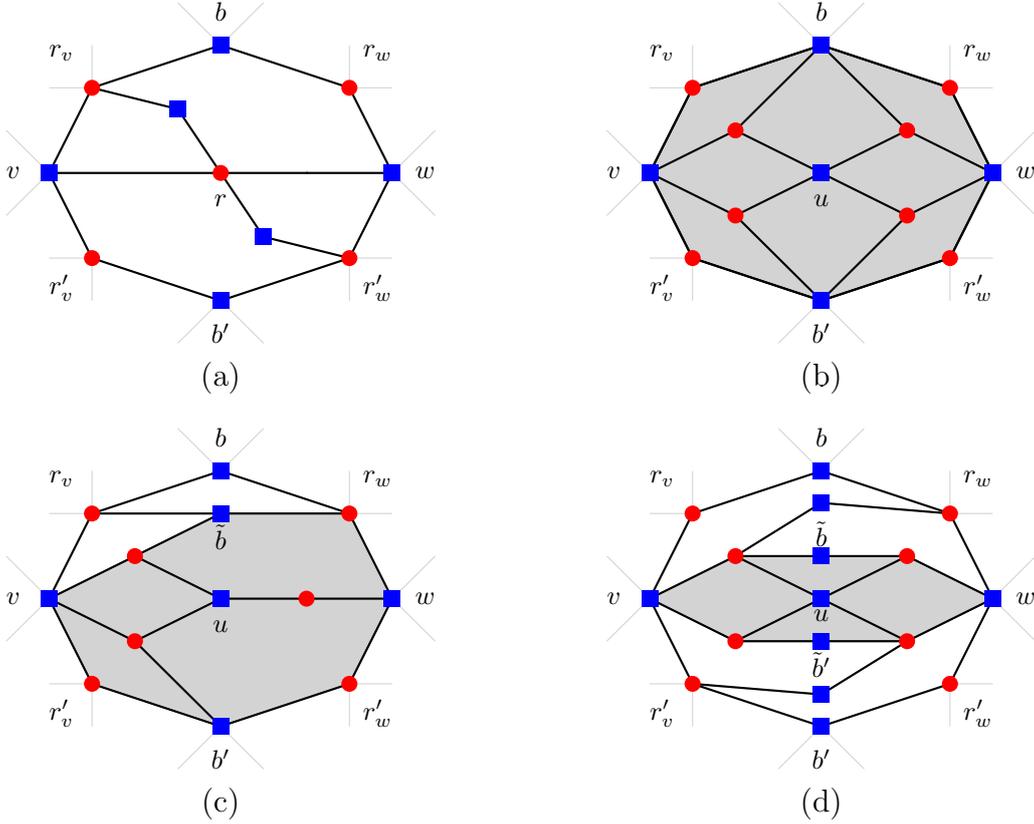


Figure 3: Examples in the proof of Proposition 3. Blue (resp. red) vertices are depicted with  $\blacksquare$  (resp.  $\bullet$ ). In (a), there is only one red vertex in  $P(v, w)$ . In (b)-(c)-(d), the red vertices in  $P(v, w)$  are dominated by a blue vertex  $u$ , which is contained in the darker subregion. The worst case is given by the configuration depicted in (b).

$P(v, w) \subseteq N(u)$  for some blue vertex  $u$ . According to the remark above, the region  $R$  can be split into at most 3 subregions (see Figure 3(b) (resp. (c), (d)) for an example with 0 (resp. 1, 2) separating paths). Note that, by construction, the subregion containing  $u$  (the darker one in Figure 3) cannot contain any other blue vertex strictly inside. We now have to count blue vertices strictly inside the (at most) 2 symmetric subregions not containing  $u$  (the white subregions in Figure 3(c-d)). Without loss of generality, we consider the one defined by the blue vertices  $v, b, w, \tilde{b}$ .

We now claim that while preserving planarity and the incomparability of neighborhoods (given by Rule 1; note that we may assume that each of  $v, w$ , and  $b$  has an incomparable neighborhood with any other vertex, since it can have neighbors out of  $R$ ), there can be at most 1 blue vertex distinct from  $v, w, b, \tilde{b}$ . Again, let us sketch how to prove this claim, distinguishing the maximum degree  $d(\ell)$  of a blue vertex  $\ell$  inside the subregion:

- If  $d(\ell) = 1$ , there is a contradiction since every red vertex is a neighbor of  $v$  or  $w$ , which implies that  $N(\ell)$  is contained in  $N(v)$  or  $N(w)$ .
- If  $d(\ell) = 2$ , for preserving incomparability the vertex  $\ell$  has, without loss of generality, a neighbor in  $N(v) \cap N(\tilde{b})$  and another neighbor  $r_w \in N(w) \cap N(b)$  (see Figure 3(d)). Then preserving planarity, there is no other possible blue vertex.
- If  $d(\ell) = 3$ , similarly it can be checked that there is at most another blue vertex.
- Finally, if  $d(\ell) = 4$ , there is a contraction since in this case  $N(\tilde{b}) \subseteq N(\ell)$ .

Summarizing, we can deduce that the blue vertices distinct from  $v, w$  are  $b, b', \tilde{b}, \tilde{b}', u$  and one vertex strictly inside each of the two subregions, that is,  $|V_B \cap V(R) \setminus \{v, w\}| \leq 5 + 2$ . Thus,

in the worst case the region  $R$  contains at most  $4 + 4 + 7 = 15$  vertices distinct from  $v, w$ .  $\square$

We are finally ready to piece everything together and prove Theorem 1.

*Proof of Theorem 1.* Let  $G$  be the plane input graph and let  $G'$  be the reduced graph obtained from  $G$ . According to Lemmas 1, 2, and 3,  $G$  admits a rbds with size at most  $k$  if and only if  $G'$  admits one with size at most  $k' \leq k$ . It is easy to see that the same time analysis of [2] implies that our reduction rules can be applied in time  $O(|V(G)|^3)$ . Let  $D$  be a rbds of  $G'$ . Note that  $|D| = 0$  if and only if  $G'$  is the graph is empty or with only one blue vertex, that is,  $G'$  has constant size. Moreover,  $|D| \neq 1$ , since the unique dominating vertex should have been removed by Rule 3. Also,  $|D| \neq 2$ , since the pair of dominating vertices should have been removed by Rule 4. Therefore, we may assume that  $|D| \geq 3$ , and then, according to Propositions 1, 2, and 3, if  $G'$  admits a rbds with size at most  $k$ , then  $G'$  has order at most  $15 \cdot (3k - 6) + k \leq 46k$ .  $\square$

## 4 Conclusion

We have presented an explicit linear kernel for the PLANAR RED-BLUE DOMINATING SET problem of size at most  $46k$ . A natural direction for further research is to improve the constant and the running time of our kernelization algorithm (we did not focus on optimizing the latter in this work), as well as proving lower bounds on the size of the kernel. It would also be interesting to extend our result to larger classes of sparse graphs. In particular, does RED-BLUE DOMINATING SET fit into the recent framework introduced in [9] for obtaining explicit and constructive linear kernels on sparse graph classes via dynamic programming?

Note also that a bikernel in  $H$ -minor-free can be derived from the linear kernel of DOMINATING SET in  $H'$ -minor-free proved by Fomin *et al.* [19] combined with the following reduction from RBDS to DOMINATING SET proposed by an anonymous referees. Given an RBDS instance  $(G = (V_B \cup V_R, k))$ , create a Dominating Set instance whose parameter is  $k + 1$ , and whose graph  $G'$  is obtained from  $G$  by adding a new vertex that is adjacent to all blue vertices, and to a new degree-1 vertex. The degree-1 vertex ensures that this new vertex can always be chosen in an optimal solution, thereby dominating all blue vertices. Hence to dominate the entire graph it suffices to dominate the red ones. Optimal solutions can avoid red vertices because they only dominate themselves and blue vertices. The minor  $H'$  is obtained from  $H$  by adding a universal vertex. Nevertheless such a bikernel is not constructive and have huge constant.

**Acknowledgement.** We would like to thank the anonymous referees for helpful and thorough remarks that improved the presentation of the manuscript, and which allowed us to slightly improve the constant of our kernel. We also thanks them for pointing out the imprecise part of the proof of Albert *et al.* [2] and for helping us to write that new proof.

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