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A linear kernel for planar red-blue dominating set*

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Abstract

In the Red-Blue Dominating Set problem, we are given a bipartite graph $G = (V_B \cup V_R, E)$ and an integer k, and asked whether G has a subset $D \subseteq V_B$ of at most k "blue" vertices such that each "red" vertex from V_R is adjacent to a vertex in D. We provide the first explicit linear kernel for this problem on planar graphs, of size at most $45k$.

Keywords: parameterized complexity, planar graphs, linear kernels, red-blue domination.

1 Introduction

Motivation. The field of parameterized complexity (see [6, 7, 16]) deals with algorithms for decision problems whose instances consist of a pair (x, k), where k is known as the *parameter*. A fundamental concept in this area is that of *kernelization*. A kernelization algorithm, or *kernel*, for a parameterized problem takes an instance (x, k) of the problem and, in time polynomial in $|x| + k$, outputs an equivalent instance (x', k') such that $|x'|, k' \leq g(k)$ for some function g. The function g is called the *size* of the kernel and may be viewed as a measure of the "compressibility" of a problem using polynomial-time preprocessing rules. A natural problem in this context is to find polynomial or linear kernels for problems that admit such kernelization algorithms.

A celebrated result in this area is the linear kernel for Dominating Set on planar graphs by Alber *et al.* [2], which gave rise to an explosion of (meta-)results on linear kernels on planar graphs [12] and other sparse graph classes [3, 8, 13]. Although of great theoretical importance, these meta-theorems have two important drawbacks from a practical point of view. On the one hand, these results rely on a problem property called *Finite Integer Index*, which guarantees the *existence* of a linear kernel, but nowadays it is still not clear how and when such a kernel can be effectively *constructed*. On the other hand, at the price of generality one cannot hope that general results of this type may directly provide explicit reduction rules and small constants for particular graph problems. Summarizing, as mentioned explicitly by Bodlaender *et al.* [3], these meta-theorems provide simple criteria to decide whether a problem admits a linear kernel on a graph class, but finding linear kernels with reasonably small constant factors for concrete problems remains a worthy investigation topic.

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Our result. In this article we follow this research avenue and focus on the RED-BLUE DOMINATING SET problem (RBDS for short) on planar graphs. In the RED-BLUE DOMINATING SET problem, we are given a bipartite[1] graph $G = (V_B \cup V_R, E)$ and an integer k, and asked whether G has a subset $D \subseteq V_B$ of at most k "blue" vertices such that each "red" vertex from V_R is adjacent to a vertex in D. This problem appeared in the context of the European railroad network [17]. From a (classical) complexity point of view, finding a red-blue dominating set (or rbds for short) of minimum size is NP-complete on planar graphs [1]. From a parameterized complexity perspective, RBDS parameterized by the size of the solution is $W[2]$-complete on general graphs and FPT on planar graphs [6]. It is worth mentioning that RBDS plays an important role in the theory of non-existence of polynomial kernels for parameterized problems [5].

The fact that RBDS involves a *coloring* of the vertices of the input graph makes it unclear how to make the problem fit into the general frameworks of [3, 8, 12, 13]. In this article we provide the first explicit (and quite simple) polynomial-time data reduction rules for RED-BLUE DOMINATING SET on planar graphs, which lead to a linear kernel for the problem.

Theorem 1 RED-BLUE DOMINATING SET *parameterized by the solution size has a linear kernel on planar graphs. More precisely, there exists a poly-time algorithm that for each positive planar instance (G, k) returns an equivalence instance (G', k) such that $|V(G')| \leq 45 \cdot k$.*

This result complements several explicit linear kernels on planar graphs for other domination problems such as DOMINATING SET [2], EDGE DOMINATING SET [12], EFFICIENT DOMINATING SET [12], CONNECTED DOMINATING SET [11,15], or TOTAL DOMINATING SET [10]. It is worth mentioning that our constant is considerable smaller than most of the constants provided by these results. Since one can easily reduce the FACE COVER problem on a planar graph to RBDS (without changing the parameter)[2], the result of Theorem 1 also provides a linear *bikernel* for FACE COVER (i.e., a polynomial-time algorithm that given an input of FACE COVER, outputs an equivalent instance of RBDS with a graph whose size is linear in k). To the best of our knowledge, the best existing kernel for FACE COVER is quadratic [14]. Our techniques are much inspired by those of Alber *et al.* [2] for DOMINATING SET, although our reduction rules and analysis are slightly simpler. Namely, we start by describing in Section 2 our reduction rules for RED-BLUE DOMINATING SET when the input graph is embedded in the plane, and in Section 3 we prove that the size of a reduced plane YES-instance is linear in the size of the desired red-blue dominating set, thus proving Theorem 1. Finally, we conclude with some directions for further research in Section 4.

2 Reduction rules

In this section we propose reduction rules for RED-BLUE DOMINATING SET, which are largely inspired by the rules that yielded the first linear kernel for DOMINATING SET on planar graphs [2]. The idea is to replace the neighborhood of some blue vertices by appropriate gadgets. We would like to point out that our rules have also some points in common with the ones for the current best kernel for DOMINATING SET [4]. In Subsection 2.1 we present two easy elementary rules that turn out to be helpful in simplifying the instance, and then in Subsections 2.2 and 2.3 we present the rules for a single vertex and a pair of vertices, respectively.

[1]In fact, this assumption is not necessary, as if the input graph G is not bipartite, we can safely remove all edges between vertices of the same color.

[2]Just consider the *radial graph* corresponding to the input graph G and its dual G^*, and color the vertices of G (resp. G^*) as red (resp. blue).

2.1 Elementary rules

The following two simple rules enable us to simplify an instance of RBDS. We would like to point out that similar rules have been provided by Weihe [17] in a more applied setting. We first need the definition of neighborhood.

Definition 1 *Let $G = (V_B \cup V_R, E)$ be a graph. The* neighborhood *of a vertex $v \in V_B \cup V_R$ is the set $N(v) = \{u : \{v, u\} \in E\}$. The* neighborhood *of a pair of vertices $v, w \in V_B$ is the set $N(v, w) = N(v) \cup N(w)$.*

Rule 1 *Remove blue vertices whose neighborhood is included into the neighborhood of another blue vertex.*

Rule 2 *Remove red vertices whose neighborhood includes the neighborhood of another red vertex.*

Lemma 1 *Let $G = (V_B \cup V_R, E)$ be a graph. If G' is the graph obtained from G by the application of Rule 1 or 2, then there is a rbds in G of size at most k if and only if there is one in G'.*

Proof. For Rule 1, if $N(b) \subseteq N(b')$ for two blue vertices b and b', then any solution containing b can be transformed to a solution containing b' in which the set of dominated red vertices may have only increased. For Rule 2, if $N(r') \subseteq N(r)$ for two red vertices r and r', then any blue vertex dominating r' dominates also r. □

2.2 Rule for a single vertex

We present a rule for reducing the size of the neighborhood of a blue vertex. For this we need the definition of private neighborhood.

Definition 2 *Let $G = (V_B \cup V_R, E)$ be a graph. The* private neighborhood *of a blue vertex b is the set $P(b) = \{r \in N(b) : N(N(r)) \subseteq N(b)\}$.*

Let us remark that for (classical) Dominating Set, each neighborhood is split into three subsets [2]. The third one corresponds to our private neighborhood, but since non-private neighbors can be used to dominate the private ones, an intermediary set is necessary for (classical) Dominating Set. In our problem it does not occur because non-private vertices are red and thus cannot belong to a rbds. This is one of the reasons why our rules are simpler. Notice also that if Rules 1 and 2 have be applied respectively on $N(P(v))$ and $P(v)$, then $N(P(v))$ is empty and $P(v)$ consists of a single vertex.

Rule 3 *Let $v \in V_B$ be a blue vertex. If $|P(v)| > 1$:*

- *remove v and $N(v)$ from G,*
- *decrease the parameter k by 1.*

Our Rule 3 corresponds to Rule 1 for (classical) Dominating Set [2]. In both rules we know that v is in the dominating set, but for RBDS we can remove it and its neighborhood (and decrease the parameter accordingly); this is not possible for Dominating Set, since vertices of the neighborhood possibly belong to the dominating set, and hence they cannot be removed and a gadget is added to enforce v to be dominating. We prove in the following lemma that Rule 3 is safe.

Lemma 2 *Let $G = (V_B \cup V_R, E)$ be a graph and let $v \in V_B$. If G' is the graph obtained from G by the application of Rule 3 on a vertex v, then there is a rbds in G of size at most k if and only if there is one in G' of size at most $k-1$.*

Proof. Let D be a rbds in G with $|D| \leq k$. By Lemma 1, there exists a rbds in G of size at most k if and only if there is one in G'', where G'' is the graph obtained from G by exhaustively applying Rules 1 and 2. Therefore, we may deal with G'' instead of G. Since the (only) vertex in $P(v)$ needs to be dominated in G'', we can assume that $v \in D$. But G' does not contain neither v nor vertices in $N(v)$, hence $D \setminus \{v\}$ is also a rbds of G' of size at most $k-1$. Conversely, let D' be a rbds in G' with $|D'| \leq k'$. Clearly $D' \cup \{v\}$ is a rbds of G of size at most $k'+1$. □

2.3 Rule for a pair of vertices

We now provide a rule for reducing the size of the neighborhood of a pair of blue vertices. For this, we first define the neighborhood and the private neighborhood of a pair of blue vertices.

Definition 3 *Let $G = (V_B \cup V_R, E)$ be a graph. The* private neighborhood *of a pair of blue vertices $v, w \in V_B$ is the set $P(v,w) = \{r \in N(w,v) : N(N(r)) \subseteq N(v,w)\}$.*

We would like to note that the definition of private neighborhood is similar to that of the third subset of neighbors defined for (classical) Dominating Set [2].

Rule 4 *Let b, c be two distinct blue vertices. If $|P(b,c)| > 2$ and there is no blue vertex $d \neq b, c$ which dominates $P(b,c)$:*

1. *if $P(b,c) \nsubseteq N(b)$ and $P(b,c) \nsubseteq N(c)$:*
   - *remove b, c, and $N(b,c)$ from G,*
   - *decrease the parameter k by 2;*
2. *if $P(b,c) \subseteq N(b)$ and $P(b,c) \subseteq N(c)$:*
   - *remove $P(b,c)$ from G,*
   - *add a new red vertex r and the edges $\{b,r\}, \{c,r\}$;*
3. *if $P(b,c) \subseteq N(b)$ and $P(b,c) \nsubseteq N(c)$:*
   - *remove b and $N(b)$ from G,*
   - *decrease the parameter k by 1;*
4. *if $P(b,c) \nsubseteq N(b)$ and $P(b,c) \subseteq N(c)$:*
   - *symmetrically to Case 3.*

Again, our Rule 4 corresponds to Rule 2 for (classical) Dominating Set [2]. When it is possible, we also remove vertices instead of adding gadgets, which will result in a better kernel bound. In this way, we can also expect to obtain better experimental results in practical applications. We prove in the following lemma that Rule 4 is safe.

Lemma 3 *Let $G = (V_B \cup V_R, E)$ be a graph and let $b, c \in V_B$. If G' is the graph obtained from G by the application of Rule 4 on two distinct blue vertices b and c, then there is a rbds in G of size at most k if and only if there is one in G' of size at most k, $k-1$ or $k-2$, depending on whether Rule 4 has removed 0, 1, or 2 vertices from G.*

Proof. Assume for simplicity in the whole proof that D be a rbds in G of size exactly k, and that D' is a rbds in G' of size exactly k'. By the equivalence given by Lemma 1. we can assume that Rule 1 has been applied on vertices in $N(P(b,c)) \setminus \{b,c\}$ and that such vertices have been removed, by definition of $P(b,c)$. Then it holds that $N(P(b,c)) = \{b,c\}$. We distinguish the four possible cases of Rule 4:

1. Since there is no single vertex which dominates $P(b,c)$, we need at least two vertices in order to dominate $P(b,c)$. Using the equivalence given by Lemma 1 as in the proof of Lemma 2, and since we are in Case 1, necessarily $b, c \in D$. Hence $D \setminus \{b,c\}$ is a rbds in G' of size $k-2$. Conversely, we have that $D' \cup \{b,c\}$ is a rbds in G of size $k'+2$.
2. By Lemma 1 and since we are in Case 2, necessarily $b \in D$ or $c \in D$. Hence D is a rbds in G' of size k. Conversely, since r needs to be dominated, we have that $b \in D'$ or $c \in D'$. Hence D' is a rbds in G of size k'.
3. By Lemma 1 and since we are in Case 3, necessarily $b \in D$. Hence $D \setminus \{b\}$ is a rbds in G' of size $k-1$. Conversely, we have that $D' \cup \{b\}$ is a rbds in G of size $k'+1$.
4. Symmetrically to Case 3. □

3 Analysis of the kernel size

We will show that a graph *reduced* under our rules has size linear in $|D|$, the size of a solution. To this aim we assume that the graph is *plane* (that is, given with a fixed embedding). In the following, for simplicity, we identify vertices and edges with their images in the plane. Then we will define a notion of region adapted to our definition of neighborhood. Then we will show that, given a solution D, there is a maximal region decomposition $\Re$ such that:

- $\Re$ has $O(|D|)$ regions,
- $\Re$ covers all vertices but $O(|D|)$ of them, and
- each region of $\Re$ contains $O(1)$ vertices.

The three following propositions treat respectively each of the above claims.

Definition 4 *A graph G is* reduced *if none of the reduction Rules 1, 2, 3, or 4 can be applied to G.*

Definition 5 *Let $G = (V_B \cup V_R, E)$ be a plane graph and let $v, w \in V_B$ be a pair of distinct blue vertices. A* region *$R(v,w)$ between v and w is a closed subset of the plane such that:*

- *the boundary of $R(v,w)$ is formed by two (not necessarily disjoint) simple paths connecting v and w, each of them having at most 4 edges,*
- *all vertices (strictly) inside $R(v,w)$ belong to $N(v,w)$ or $N(N(v,w))$.*

We denote by $\partial R(v,w)$ the boundary of $R(v,w)$ and by $V(R(v,w))$ the set of vertices in the region (that is, vertices strictly inside, on the boundary, and the two extremities v, w). The size *of a region is $|V(R(v,w))|$.*

We would like to point out that the assumption that vertices are distinct is not necessary, but it makes the proofs easier. In the following whenever we speak about a pair of vertices we assume them to be always distinct.

We want to decompose a reduced graph into a set of regions which do not overlap each other. To formalize this, we provide the following definition of *crossing* regions, which can be seen as a more formal and precise version of the corresponding definition given in [2].

Recall that we are considering a plane graph, hence for each vertex v, the embedding induces a circular ordering on the edges incident to v. We first need the (recursive) definition of *confluent* paths.

Definition 6 *Two simple paths p_1, p_2 are* confluent *if:*

- *they are vertex-disjoint, or*
- *they are edge-disjoint and for each vertex $v \in p_1 \cap p_2$ distinct from the four extremities, the two edges of p_1 containing v are consecutive in the circular ordering given by the embedding (hence, the two of p_2 are consecutive as well), or*
- *the two paths obtained by contracting common edges are confluent.*

Definition 7 *Two regions R_1, R_2* do not cross *if:*

- $(R_1 \setminus \partial R_1) \cap R_2 = (R_2 \setminus \partial R_2) \cap R_1 = \emptyset$ *(i.e., the interiors of the regions are disjoint), and*
- *the paths in $\partial R_1 \cup \partial R_2$ are pairwise confluent.*

Otherwise, we say that R_1, R_2 cross. *If two regions cross because of two paths $p_1 \in \partial R_1$ and $p_2 \in \partial R_2$ that are not confluent, we say that these regions* cross on $v \in p_1 \cap p_2$ *if:*

- *v does not satisfy the ordering condition of Definition 6, or*
- *v is an extremity of an edge e such that in G/e (i.e., the graph obtained from G by contracting e), R_1 and R_2 cross on the vertex resulting from the contraction.*

Of course, two regions can cross on many vertices. We use the latter condition of Definition 7 in the case of degenerated regions (that is, paths), where only this condition may hold.

Definition 8 *Let $G = (V_B \cup V_R, E)$ be a plane graph and let $D \subseteq V_B$. A D*-decomposition *of G is a set of regions $\Re$ between pairs of vertices in D such that:*

- *any region in $\Re$ between two vertices v, w does not contain vertices in $D \setminus \{v, w\}$, and*
- *any two regions in $\Re$ do not cross.*

We note $V(\Re) = \bigcup_{R \in \Re} V(R)$. A D-decomposition is maximal *if there is no region $R \notin \Re$ such that $\Re \cup \{R\}$ is a D-decomposition with $V(\Re) \subsetneq V(\Re \cup \{R\})$.*

Fact 1 *Let G be a reduced graph and let D be a rbds in G. Each vertex of G sits on a path of at most four edges connecting two vertices in D.*

Proof. Note first that a blue vertex cannot be alone in its connected component, as otherwise it could be removed by Rule 3. We distinguish red vertices, blue dominating vertices, and blue non-dominating vertices.

- Let $r \in V_R$. There is some vertex $v \in D$ that dominates r. Since Rule 3 cannot be applied on v, then $r \notin P(v)$, and by definition of private neighborhood there are two vertices $b \in N(r)$ and $r' \in N(b)$ such that $r' \notin N(v)$. If $b \in D$, then $P = \{v, r, b\}$ is the desired path. Otherwise, r' is dominated by some vertex $w \in D$, and $P = \{v, r, b, r', w\}$ is the desired path.

- Let $b \in V_B \setminus D$. Since Rule 1 cannot be applied on b, $N(b) \neq \emptyset$, and there is some vertex $r \in N(b)$ that is dominated by a vertex v. Again by Rule 1, b and v have incomparable neighborhoods, so there is some $r' \in N(b)$ that is dominated by a vertex $w \neq v$. In this case, $P = \{v, r, b, r', w\}$ is the desired path.
- Let $v \in D$. Since Rule 1 cannot be applied on v, there is some vertex $r \in N(v)$. Since Rule 3 cannot be applied on v, then $r \notin P(v)$, hence there are two vertices $b \in N(r)$ and $r' \in N(b)$ such that $r' \notin N(v)$. If $b \in D$, then $P = \{v, r, b\}$ is the desired path. Otherwise, r' is dominated by some vertex $w \in D$, and $P = \{v, r, b, r', w\}$ is the desired path. □

Proposition 1 *Let G be a reduced plane graph and let D be a rbds in G. There is a maximal D-decomposition of G such that $|\Re| \leq 3 \cdot |D| - 6$.*

Proof. The proof strongly follows the one of Alber *et al.* [2, Lemma 5 and Proposition 1]. Even if our definition of region is different, we shall show that the same algorithm can be used to construct such a D-decomposition. Note first that $|D| \neq 1$, since the unique dominating vertex should be removed by Rule 3. Also, $|D| \neq 2$, since the pair of dominating vertices should be removed by Rule 4. Therefore, we may assume that $|D| \geq 3$.

We consider the algorithm which, for each vertex u, adds greedily to the decomposition $\Re$ a region R between any two vertices $v, w \in D$, containing u, not containing any vertex of $D \setminus \{v, w\}$, not crossing any region of $\Re$, and of maximal size, if it exists. By definition, $\Re \cup \{R\}$ is a region decomposition, and by greediness it is maximal.

Note that in Definition 8 we ask the decomposition to be *maximal* with respect to vertex inclusion, and not *maximum* in terms of number of vertices; moreover we have no constraint about the regions in the decomposition. But, since the algorithm chooses regions which are maximal for vertex inclusion, we know that the returned decomposition is *minimal* in terms of number of regions.

We will now prove that for each pair of distinct regions $R_1(v, w), R_2(v, w) \in \Re$ between an *identical* pair of vertices v, w, there is a vertex of D in both open sets defined by the complement of the union of the two regions in the plane. This property allows to apply [2, Lemma 5], implying that the constructed decomposition has at most $3 \cdot |D| - 6$ regions.

Indeed, let $R_1, R_2 \in \Re$ be two regions and let O be one of the two maximal connected open sets in $\overline{R_1 \cup R_2}$. Let us assume, for the sake of contradiction, that there is no vertex of D in O. We distinguish two cases:

- If O does not contain any blue vertex, then the red vertices in O (if any) must be dominated by v or w. Hence $R_1 \cup R_2 \cup O$ is a larger region which must have been chosen by the algorithm. We have a contradiction with the maximality of the regions R_1 and R_2.
- Otherwise, if O contains at least one blue vertex $b \notin D$, by Fact 1 b is on a path $P = \{v, r, b, r', w\}$ for some vertices r, r' (recall that there is no dominating vertex in O except for v, w). Such a path can be built for any blue vertex, hence again $R_1 \cup R_2 \cup O$ is a larger region, which contradicts the maximality of the regions R_1 and R_2. □

Proposition 2 *Let $G = (V_B \cup V_R, E)$ be a reduced plane graph and let D be a rbds in G. If $\Re$ is a maximal D-decomposition, then $|V \setminus V(\Re)| = 0$.*

Proof. The proof again follows that of Alber *et al.* [2, Lemma 6 and Proposition 2], where similar arguments are used to bound the number of vertices which are not included in a maximal region decomposition. We have to show that all vertices are included in a region of $\Re$, that is, $V_R \cup V_B \subseteq V(\Re)$).

Since $N(D)$ covers V_R, it holds that $V_R = \bigcup_{v \in D} N(v)$. We proceed to show that $N(v) \subseteq V(\Re)$ for all $v \in D$. Let $v \in D$ and let $u \in N(v)$. By Fact 1 there is a path P of at most four edges containing v and u. We distinguish two cases depending on the length of P:

- If $P = \{v, u, w\}$ for some vertex w, then P crosses some region R of $\Re$ (since $\Re$ is maximal). Hence either P is in the interior of R, or P and R cross on u; in both cases $u \in V(\Re)$.
- If $P = \{v, u, b, r, w\}$ for some vertices b, r, w with $w \neq v$, then P crosses some region $R(x, y)$ of $\Re$ (since $\Re$ is maximal). We distinguish three cases:
    - If P is in the interior of $R(x, y)$, or P and $R(x, y)$ cross on u, then $u \in V(\Re)$.
    - Otherwise, if P and $R(x, y)$ cross on b, then b is on $\partial R(x, y)$. Let r' be a vertex on $\partial R(x, y)$ such that $r' \in N(b) \cap N(x)$. Assume for contradiction that $u \notin V(\Re)$. Then the (degenerated) region defined by the path $\{v, u, b, r', x\}$ could be added to $\Re$, which contradicts the maximality of $\Re$.
    - Otherwise, necessarily P crosses a region $R(x, y) \in \Re$ on r, and then r is on $\partial R(x, y)$. Assume without loss of generality that $r \in N(x)$, and assume for contradiction that $u \notin V(\Re)$. Then the (degenerated) region defined by the path $\{v, u, b, r, x\}$ could be added to $\Re$, which contradicts again the maximality of $\Re$ (see Figure 1).

So $\bigcup_{v \in D} N(v) \subseteq V(\Re)$, as we wanted to prove.

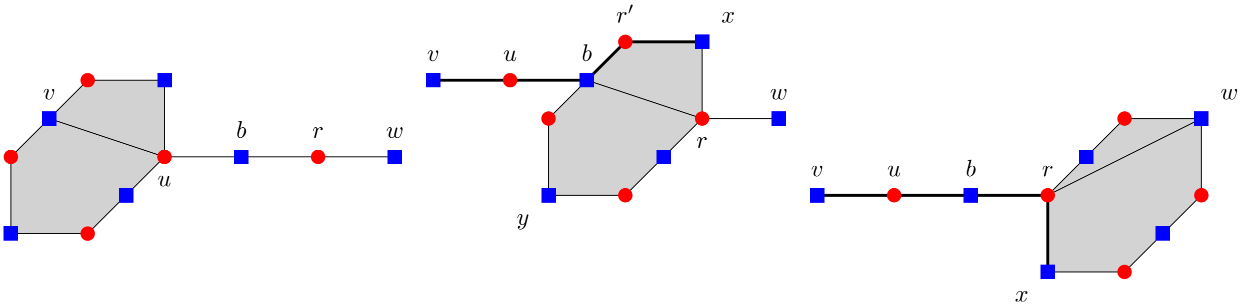

Figure 1: Illustration of the 3 ways that the path $\{v, u, b, r, w\}$, as defined in Proposition 2, can cross a region. Blue (resp. red) vertices are depicted with $\blacksquare$ (resp. $\bullet$).

We finally show that $V_B \subseteq V(\Re)$. Let first $b \in V_B \setminus D$. Since G is reduced, by Rule 1 and Rule 3, b is neighbor of two red vertices r' and r'' dominated respectively by v and w with $v \neq w$ (as otherwise vertex b could be removed by Rule 1). We consider the (degenerated) region $\{v, r', b, r'', w\}$, and with an argument similar to the previous one, if we assume that $b \notin V(\Re)$ we obtain a contradiction. Let then $v \in D$. By Rule 3 v cannot be a single dominating vertex in a connected component. Hence let $w \in D$ be at distance at most 4 from v. We consider this path as a region, and once again we obtain a contradiction using similar arguments. So $V_B \subseteq V(\Re)$).

Therefore, all the vertices of G belong to the decomposition $\Re$, as we wanted to prove. $\square$

Proposition 3 *Let $G = (V_B \cup V_R, E)$ be a reduced plane graph, let D be a rbds in G, and let $v, w \in D$. A region R between v and w contains at most 15 vertices distinct from v and w.*

Proof. We bound separately the number of red neighbors of v and w in ∂R, red neighbors in $R \setminus \partial R$, and blue vertices in the region R. It will become clear from the proof that the worst bound is given by the case where ∂R contains 8 vertices, which will be henceforth denoted by $v, r_v, b, r_w, w, r'_w, b', r'_v$.

In order to bound the total number of vertices, we need the following remark. Assume that Rule 3 has not been applied and that $P(v,w) \neq \emptyset$ (clearly $P(v,w) = \emptyset$ cannot give the worst bound). Then by the condition of Rule 3, $P(v,w) \subseteq N(u)$ for some blue vertex u. If there is any blue vertex in R distinct from v, b, w, b', u, by Fact 1 this vertex is contained in a path of at most 4 edges, and by definition of region we can choose the extremities to be v and w. Such a path disconnects u from either b or b'. Hence, considering two blue vertices, we can subdivide R into at most 3 subregions. We consider vertices $\tilde{b}, \tilde{b}'$ (possibly $b = \tilde{b}$ or $b' = \tilde{b}'$) which define a subregion containing exactly $\tilde{b}, \tilde{b}, u$ as blue vertices (this does not depend on the chosen path). Notice that any red vertex in $P(v,w)$ should belong to the subregion containing u.

By definition the red vertices in ∂R are r_v, r_w, r'_w, r'_v. Hence $|(N(v,w) \setminus P(v,w)) \cap V(R)| \leq 4$.

The number of red vertices strictly in R depends on whether Case 2 of Rule 4 has been applied on the pair v, w or not (it is impossible that Cases 1, 3, or 4 have been applied, since otherwise v or w would have been removed):

- Assume first that Rule 4 has been applied on the pair v, w. Therefore the only red vertex in $R \setminus \partial R$ is the newly added vertex, and thus $|P(v,w)| \leq 1$ (see Figure 2(a)).
- Otherwise, red vertices in $R \setminus \partial R$ are necessarily private and $P(v,w) \subseteq N(u)$ for some blue vertex u, so for any vertex $r \in P(v,w)$ it holds that $r \in N(u) \cap (N(v) \cup N(w))$.

  According to the remark above, we can divide R into at most 3 subregions and all vertices in $P(v,w)$ are in the subregion containing u.

  Out of the possible neighborhoods of a red vertex in $P(v,w) \cap V(R)$, we claim that while preserving adjacency with u, adjacency with v or w, planarity, and the incomparability of neighborhoods (given by Rule 2), there can be at most 4 private vertices, and that this case is attained when each red vertex in the region has degree 3, with respective neighborhoods $\{u, v, \tilde{b}\}, \{u, v, \tilde{b}'\}, \{u, w, \tilde{b}\}$, and $\{u, w, \tilde{b}'\}$ for two vertices $\tilde{b}, \tilde{b}'$ as defined above. Let us now sketch how to prove this claim, distinguishing the maximum degree $d(r)$ of a red vertex $r \in P(v,w) \cap V(R)$:

  - If $d(r) = 1$, there is a contradiction since r should be adjacent to u and v or w.
  - If $d(r) = 2$, then there can be at most two red vertices in $P(v,w) \cap V(R)$ with neighborhoods $\{v, u\}$ and $\{w, u\}$.
  - If $d(r) = 3$, the reader can easily check that there are at most 4 red vertices (see Figure 2(b-c-d), where the considered subregion containing u is the darker one).
  - If $d(r) = 4$, then the worst configuration is given by 3 vertices, the other two having degree 3.
  - If $d(r) = 5$, then there is at most 1 vertex by incomparability of neighborhoods.

  Hence $|P(v,w)| \leq 4$, as we wanted to prove.

It just remains to bound the number of blue vertices distinct from v, w). Since G is reduced by Rule 1, blue vertices have incomparable neighborhoods, in particular with $N(v)$ and $N(w)$, so for any blue vertex b it holds that $N(b) \cap N(v) \neq \emptyset$ and $N(b) \cap N(w) \neq \emptyset$. We also distinguish whether Case 2 of Rule 4 has been applied to the pair v, w or not:

- Assume that Rule 4 has been applied. Recall that in this case R contains 5 red vertices $r_v, r'_v \in N(v)$, $r_w, r'_w \in N(w)$, and r, the newly added vertex. Note that $N(r) = \{v, w\}$, hence the path v, r, w splits R into 2 subregions, and note also that b and b' are necessarily

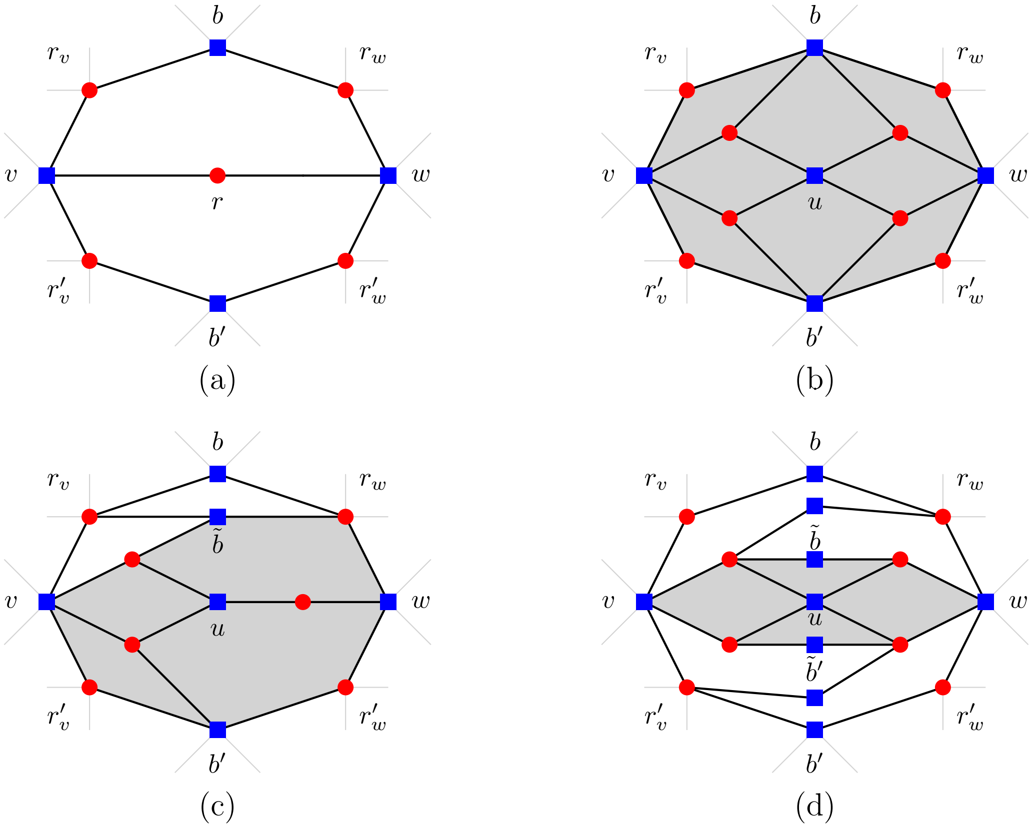

Figure 2: Examples in the proof of Proposition 3. Blue (resp. red) vertices are depicted with ■ (resp. ●). In (a), Rule 4 has been applied on the pair v, w. In (b)-(c)-(d), the red vertices in $P(v, w)$ are dominated by a blue vertex u, which is contained in the darker subregion. The worst case is given by the configuration depicted in (b).

neighbors of r_v, r_w and $r_{v'}, r_{w'}$, respectively. Since blue vertices have incomparable neighborhoods by Rule 1, there is no other possible blue vertex, as the neighborhood of such a vertex would be included in $N(b)$. Hence $|V_B \cap V(R) \setminus \{v, w\}| \leq 2$ (see again Figure 2(a)).

- Otherwise, $P(v, w) \subseteq N(u)$ for some blue vertex u. According to the remark above, the region R can be split into at most 3 subregions (see Figure 2(b) (resp. (c), (d)) for an example with 0 (resp. 1, 2) separating paths). Note that, by construction, the subregion containing u (the darker one in Figure 2) cannot contain any other blue vertex strictly inside. We now have to count blue vertices strictly inside the (at most) 2 symmetric subregions not containing u (the white subregions in Figure 2(c-d)).

  We now claim that while preserving planarity and the incomparability of neighborhoods (by Rule 1, note that we may assume that b has an incomparable neighborhood with any other vertex, since it can have neighbors out of R), there can be at most 1 blue vertex distinct from $v, w, b, \tilde{b}$. Again, let us sketch how to prove this claim, distinguishing the maximum degree $d(\ell)$ of a blue vertex ℓ inside the subregion:

  - If $d(\ell) = 1$, there is a contradiction since every red vertex is a neighbor of v or w, which implies that $N(\ell)$ is contained in $N(v)$ or $N(w)$.
  - If $d(\ell) = 2$, for preserving incomparability the vertex ℓ has, without loss of generality, a neighbor in $N(v) \cap N(\tilde{b})$ and another in $N(w) \cap N(b)$ (see Figure 2(d)). Then. preserving planarity, there is no other possible blue vertex.
  - If $d(\ell) = 3$, similarly it can be checked that there is at most 1 other blue vertex.
  - Finally, if $d(\ell) = 4$, there is a contraction since in this case $N(\tilde{b}) \subseteq N(\ell)$.

Summarizing, we can deduce that the blue vertices distinct from v, w are $b, b', \tilde{b}, \tilde{b}', u$ and vertices strictly in the two subregions, that is, $|V_B \cap V(R) \setminus \{v, w\}| \leq 5 + 2$, where we have also counted the blue vertex u.

Thus, the region R contains at most $4 + \max(1 + 2, 4 + 7) = 15$ vertices distinct from v, w. □

We are finally ready to piece everything together and prove Theorem 1.

Proof of Theorem 1. Let G be the plane input graph and let G' be the reduced graph obtained from G. According to Lemmas 1, 2, and 3, G admits a rbds with size at most k if and only if G' admits one. It is easy to see that the same time analysis of [2] implies that our reduction rules can be applied in time $O(|V(G)|^3)$. According to Propositions 1, 2, and 3, if G' admits a rbds with size at most k, then G' has order at most $15 \cdot (3k - 6) \leq 45k$. □

4 Conclusion

We have presented an explicit linear kernel for the Planar Red-Blue Dominating Set problem of size at most $45k$. A natural direction for further research is to improve the constant and the running time of our kernelization algorithm (we did not focus on optimizing the latter in this work), as well as proving lower bounds on the size of the kernel. It would also be interesting to extend our result to larger classes of sparse graphs. In particular, does Red-Blue Dominating Set fit into the recent framework introduced in [9] for obtaining explicit and constructive linear kernels on sparse graph classes via dynamic programming?

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