

Minors in graphs of large θ_r -girth*

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Abstract

For every $r \in \mathbb{N}$, let θ_r denote the graph with two vertices and r parallel edges. The θ_r -girth of a graph G is the minimum number of edges of a subgraph of G that can be contracted to θ_r . This notion generalizes the usual concept of girth which corresponds to the case $r = 2$. In [Minors in graphs of large girth, *Random Structures & Algorithms*, 22(2):213–225, 2003], Kühn and Osthus showed that graphs of sufficiently large minimum degree contain clique-minors whose order is an exponential function of their girth. We extend this result for the case of θ_r -girth and we show that the minimum degree can be replaced by some connectivity measurement. As an application of our results, we prove that, for every fixed r , graphs excluding as a minor the disjoint union of k θ_r 's have treewidth $O(k \cdot \log k)$.

Keywords: girth, clique minors, tree-partitions, unavoidable minors, exclusion theorems.

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1 Introduction

A classic result in graph theory asserts that if a graph has minimum degree $ck\sqrt{\log k}$, then it can be transformed to a complete graph on at least k vertices by applying edge

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contractions and vertex deletions (i.e., it contains a k -clique minor). This result has been proven by Kostochka in [20] and Thomason in [33] and a precise estimation of the constant c has been given by Thomason in [34]. For recent results related to conditions that force a clique minor see [13, 15, 19, 22, 23].

The *girth* of a graph G is the minimum length of a cycle in G . Interestingly, it follows that graphs of large minimum degree contain clique-minors whose order is an *exponential* function of their girth. In particular, it follows by the main result of Kühn and Osthus in [21] that there is a constant c such that, if a graph has minimum degree $d \geq 3$ and girth z , then it contains as a minor a clique of size k , where

$$k \geq \frac{d^{cz}}{\sqrt{z \cdot \log d}}.$$

In this paper we provide conditions, alternative to the above one, that can force the existence of a clique-minor whose size is exponential.

H -girth. We say that a graph H is a *minor* of a graph G , if H can be obtained from G by using the operations of vertex removal, edge removal, and edge contraction. An H -*model* in G is a subgraph of G that contains H as a minor. Given two graphs G and H , we define the H -*girth* of G as the minimum number of edges of an H -model in G . If G does not contain H as a minor, we will say that its H -girth is equal to infinity. For every $r \in \mathbb{N}$, let θ_r denote the graph with two vertices and r parallel edges, e.g. in Fig. 1 the graph θ_5 with 5 parallel edges. Clearly, the girth of a graph is its θ_2 -girth and, for every $r_1 \leq r_2$, the θ_{r_1} -girth of a graph is at most its θ_{r_2} -girth.

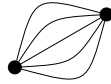


Figure 1: The graph θ_5 .

Our first result is the following extension of the result of Kühn and Osthus in [21] for the case of θ_r -girth.

Theorem 1.1. *There is a constant c such that, for every $r \geq 2$, $d \geq 3r$, and $z \geq 2r$, if a graph has minimum degree d and θ_r -girth at least z , then it contains as a minor a clique of size k , where*

$$k \geq \frac{\left(\frac{d}{r}\right)^{c\frac{z}{r}}}{\sqrt{\frac{z}{r} \cdot \log d}}.$$

In the formula above, a lower bound on the minimum degree as a function of r is necessary. An easy computation shows that when applying Theorem 1.1 for $r = 2$, we can get the aforementioned formula of Kühn and Osthus, where the constant in the exponent is one fourth of the constant of Theorem 1.1.

Our second finding is that this degree condition can be replaced by some “loose connectivity” requirement.

Loose connectivity. For two integers $\alpha, \beta \in \mathbb{N}$, a graph G is called (α, β) -loosely connected if for every $A, B \subseteq V(G)$ such that $V(G) = A \cup B$ and G has no edge between $A \setminus B$ and $B \setminus A$, we have that $|A \cap B| < \beta \Rightarrow \min(|A \setminus B|, |B \setminus A|) \leq \alpha$. Intuitively, this means that a *small* separator (i.e., on less than β vertices) cannot “split” the graph into two *large* parts (that is, with more than α vertices each).

Our second result indicates that the requirement on the minimum degree in [Theorem 1.1](#) can be replaced by the loose connectivity condition as follows.

Theorem 1.2. *There is a constant $c > 0$ such that, for every $r \geq 2$, $\alpha \geq 1$, and $z \geq 168 \cdot \alpha \cdot r \log r$, it holds that if a connected graph with at least z edges is $(\alpha, 2r - 1)$ -loosely connected and has θ_r -girth at least z , then it contains as a minor a clique of size k , where*

$$k \geq \frac{2^{c \cdot \frac{z}{r\alpha}}}{\sqrt{rz}}.$$

Both [Theorem 1.1](#) and [Theorem 1.2](#) are derived from two more general results, namely [Theorem 3.1](#) and [Theorem 3.2](#), respectively. [Theorem 3.1](#) asserts that graphs with large θ_r -girth and sufficiently large minimum degree contain as a minor a graph whose minimum degree is exponential in the girth. [Theorem 3.2](#) replaces the minimum degree condition with the absence of sufficiently large “edge-protrusions”, that are roughly tree-like structured subgraphs with small boundary to the rest of the graph (see [Section 2](#) for the detailed definitions).

Treewidth. A *tree-decomposition* of a graph G is a pair (T, \mathcal{X}) where T is a tree and \mathcal{X} is a family of subsets of $V(G)$, called *bags*, indexed by the vertices of T and such that:

- (i) for each edge $e = (x, y) \in E(G)$ there is a vertex $t \in V(T)$ such that $\{x, y\} \subseteq X_t$;
- (ii) for each vertex $u \in V(G)$ the subgraph of T induced by $\{t \in V(T) \mid u \in X_t\}$ is connected; and
- (iii) $\bigcup_{t \in V(T)} X_t = V(G)$.

The *width* of a tree-decomposition (T, \mathcal{X}) is the maximum size of its bags minus one. The *treewidth* of a graph G , denoted $\mathbf{tw}(G)$, is defined as the minimum width over all tree-decompositions of G .

Treewidth has been introduced in the Graph Minors Series of Robertson and Seymour [\[28\]](#) and is an important parameter in both combinatorics and algorithms. In [\[28\]](#), Robertson and Seymour proved that for every planar graph H , there exists a constant c_H such that every graph excluding H as a minor has treewidth at most c_H . This result has several applications in algorithms and a lot of research has been devoted to optimizing the constant c_H in general or for specific instantiations of H (see [\[11, 30\]](#)). In this direction, Chekuri and Chuzhoy proved in [\[9, 10\]](#) that c_H is bounded by a polynomial

in the size of H . Specific results for particular H 's such that c_H is a low polynomial function have been derived in [2, 3, 6, 27].

Given a graph J , we denote by $k \cdot J$ the disjoint union of k copies of J . A consequence of the general results of Chekuri and Chuzhoy in [8] is that for every planar graph J , it holds that $c_{k \cdot J} = k \cdot \text{polylog } k$, where $\text{polylog } k$ denotes some polynomial in $\log k$. Prior to this, a quadratic (in k) upper bound was derived for the case where $J = \theta_r$ [2, 14]. As an application of our results, we prove that for every fixed r , $c_{k \cdot \theta_r} = O(k \cdot \log k)$ (Theorem 5.1). We also argue that this bound is *tight* in the sense that it cannot be improved to $o(k \cdot \log k)$. Our proof is based on Theorem 3.2 and the results of Geelen, Gerards, Robertson, and Whittle on the excluded minors for the matroids of branch-width k [16].

Organization of the paper. The main notions used in this paper are defined in Section 2. Then, we show in Section 3 that the proofs of Theorem 1.1 and Theorem 1.2 can be derived from Theorem 3.1 and Theorem 3.2, which are proved in Section 4. Finally, in Section 5, we prove our tight bound on the minor-exclusion of $k \cdot \theta_r$.

2 Definitions

Given a function $\phi : A \rightarrow B$ and a set $C \subseteq A$, we define $\phi(C) = \{\phi(x) \mid x \in C\}$. Let $\chi, \psi : \mathbb{N} \rightarrow \mathbb{N}$. We say that $\chi(n) = O_r(\psi(n))$ if there exists a function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that, for every $r \in \mathbb{N}$, $\chi(n) = O(\phi(r) \cdot \psi(n))$. This notation indicates that the contribution of r is hidden in the constant of the big-O notation. If \mathcal{X} is a set of sets, we denote by $\bigcup \mathcal{X}$ the union $\bigcup_{X \in \mathcal{X}} X$. Unless otherwise specified, logarithms are binary.

Graphs. All graphs in this paper are finite, undirected, loopless, and may have multiple edges. For this reason, a graph is represented by a pair $G = (V, E)$ where V is its vertex set, denoted by $V(G)$ and E is its edge multi-set, denoted by $E(G)$. In this paper, when giving the running time of an algorithm involving some graph G , we agree that $n = |V(G)|$ and $m = |E(G)|$. Given a vertex v of a graph G , the set of vertices of G that are adjacent to v is denoted by $N_G(v)$ and the *degree* of v in G is $|N_G(v)|$. Observe that since multiple edges are allowed, the degree of a vertex may differ from the number of incident edges. For every subset $S \subseteq V(G)$, we set $N_G(S) = \bigcup_{v \in S} N_G(v) \setminus S$ (all vertices of $V(G) \setminus S$ that have a neighbor in S). The minimum degree over all vertices of a graph G is denoted by $\delta(G)$. For a given graph G and two vertices $u, v \in V(G)$, $\mathbf{dist}_G(u, v)$ denotes *the distance between u and v* , which is the number of edges on a shortest path between u and v , and $\mathbf{diam}(G)$ denotes $\max\{\mathbf{dist}_G(u, v) \mid u, v \in V(G)\}$. For a set $S \subseteq V(G)$ and a vertex $w \in V$, $\mathbf{dist}_G(S, w)$ denotes $\min\{\mathbf{dist}_G(v, w) \mid v \in S\}$. Also, for a given vertex $u \in V(G)$, $\mathbf{ecc}_G(u)$ denotes the *eccentricity* of the vertex u , that is, $\max\{\mathbf{dist}_G(u, v) \mid v \in V(G)\}$.

Rooted trees. A *rooted tree* is a pair (T, s) such that T is a tree and s , which we call the *root*, belongs to $V(T)$. Given a vertex $x \in V(T)$, the *descendants* of x in (T, s) are the elements of $\mathbf{des}_{(T,s)}(x)$, which is defined as the set containing each vertex w such that the unique path from w to s in T contains x . Given a rooted tree (T, s) and a vertex $x \in V(G)$, the *height* of x in (T, s) is the maximum distance between x and a vertex in $\mathbf{des}_{(T,s)}(x)$. The *height* of (T, s) is the height of s in (T, s) . The *children* of a vertex $x \in V(T)$ are the vertices in $\mathbf{des}_{(T,s)}(x)$ that are adjacent to x . A *leaf* of (T, s) is a vertex of T without children. Notice that, according to this definition, s is not a leaf unless $|V(T)| = 1$. The *parent* of a vertex $x \in V(T) \setminus \{s\}$, denoted by $\mathbf{p}(x)$, is the unique vertex of T that has x as a child.

Partitions and protrusions. A *rooted tree-partition* of a graph G is a triple $\mathcal{D} = (\mathcal{X}, T, s)$ where (T, s) is a rooted tree and $\mathcal{X} = \{X_t\}_{t \in V(T)}$ is a partition of $V(G)$ where either $|V(T)| = 1$ or for every $\{x, y\} \in E(G)$, there exists an edge $\{t, t'\} \in E(T)$ such that $\{x, y\} \subseteq X_t \cup X_{t'}$ (see also [12, 17, 31]). The elements of \mathcal{X} are called *bags*. In other words, the endpoints of every edge of G either belong to the same bag, or they belong to bags of adjacent vertices of T . Given an edge $f = \{t, t'\} \in E(T)$, we define E_f as the set of edges with one endpoint in X_t and the other in $X_{t'}$. The *width* of \mathcal{D} is defined as $\max\{|X_t|\}_{t \in V(T)} \cup \{|E_f|\}_{f \in E(T)}$.

In order to decompose graphs along edge cuts, we introduce the following edge-counterpart of the notion of (*vertex*-)protrusion used in [4, 5] (among others). A subset $Y \subseteq V(G)$ is a *t-edge-protrusion* of G with *extension* w (for some positive integer w) if the graph $G[Y \cup N_G(Y)]$ has a rooted tree-partition $\mathcal{D} = (\mathcal{X}, T, s)$ of width at most t and such that $N_G(Y) = X_s$ and $|V(T)| \geq w$. The protrusion Y is said to be *connected* whenever $Y \cup N_G(Y)$ induces a connected subgraph in G .

Distance-decompositions. A *distance-decomposition* of a connected graph G is a rooted tree-partition $\mathcal{D} = (\mathcal{X}, T, s)$ of G , where the following additional requirements are met (see also [35]):

- (i) X_s contains only one vertex, we shall call it u , referred to as the *origin* of \mathcal{D} ;
- (ii) for every $t \in V(T)$ and every $x \in X_t$, $\mathbf{dist}_G(x, u) = \mathbf{dist}_T(t, s)$;
- (iii) for every $t \in V(T)$, the graph $G_t = G \left[\bigcup_{t' \in \mathbf{des}_{(T,s)}(t)} X_{t'} \right]$ is connected; and
- (iv) if C is the set of children of a vertex $t \in V(T)$, then the graphs $\{G_{t'}\}_{t' \in C}$ are the connected components of $G_t \setminus X_t$.

An example of distance-decomposition is given in Fig. 2. For every vertex u of a graph on m edges, a distance-decomposition (\mathcal{X}, T, s) with origin u can be constructed in $O(m)$ steps by breadth-first search.

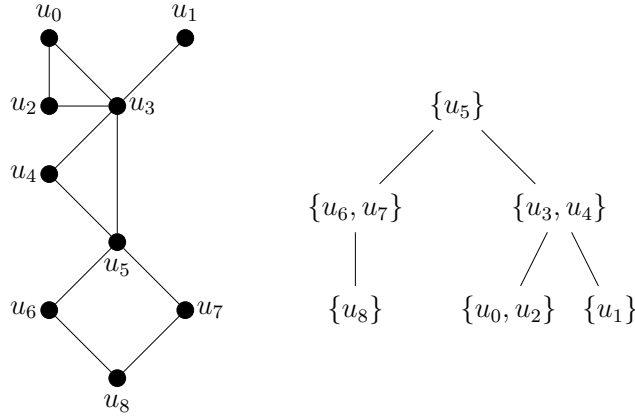


Figure 2: A graph (left) and a distance-decomposition with origin u_5 of it (right).

For every $t \in V(T) \setminus \{s\}$, we define $E^{(t)}$ as the set of edges that have one endpoint in X_t and the other in $X_{\mathbf{p}(t)}$.

Let P be a path in G and $\mathcal{D} = (\mathcal{X}, T, s)$ a distance-decomposition of P . We say that P is a *straight path* if the heights, in (T, s) , of the indices of the bags in \mathcal{D} that contain vertices of P are pairwise distinct. Obviously, in that case, the sequence of the heights of the bags that contain each subsequent vertex of the path is strictly monotone.

Grouped partitions. Let G be a connected graph and let $d \in \mathbb{N}$. A *d-grouped partition* of G is a partition $\mathcal{R} = \{R_1, \dots, R_l\}$ of $V(G)$ (for some positive integer l) such that for each $i \in \{1, \dots, l\}$, the graph $G[R_i]$ is connected and there is a vertex $s_i \in R_i$ with the following properties:

- (i) $\mathbf{ecc}_{G[R_i]}(s_i) \leq 2d$ and
- (ii) for each edge $e = \{x, y\} \in E(G)$ where $x \in R_i$ and $y \in R_j$ for some distinct integers $i, j \in \{1, \dots, l\}$, it holds that $\mathbf{dist}_G(x, s_i) \geq d$ and $\mathbf{dist}_G(y, s_j) \geq d$.

A set $S = \{s_1, \dots, s_l\}$ as above is a *set of centers* of \mathcal{R} where s_i is the *center* of R_i for $i \in \{1, \dots, l\}$.

Given a graph G , we define a *d-scattered set* W of G as follows:

- $W \subseteq V(G)$ and
- $\forall u, v \in W, \mathbf{dist}_G(u, v) > d$.

If W is inclusion-maximal, it will be called a *maximal d-scattered set* of G .

Frontiers and ports. Let G be a graph, let $\mathcal{R} = \{R_1, \dots, R_l\}$ be a d -grouped partition of G , and let $S = \{s_1, \dots, s_l\}$ be a set of centers of \mathcal{R} . For every $i \in \{1, \dots, l\}$, we denote by $\mathcal{D}_i = (\mathcal{X}_i, T_i, s_i)$ the unique distance-decomposition with origin s_i of the graph $G[R_i]$ where $\mathcal{X}_i = \{X_t^i\}_{t \in V(T_i)}$. For every $i \in \{1, \dots, l\}$ and every $h \in \{0, \dots, \text{ecc}_{T_i}(s_i)\}$, we denote by I_i^h the vertices of (T_i, s_i) that are at distance h from s_i , and we set $I_i^{<h} = \bigcup_{h'=0}^{h-1} I_i^{h'}$ and $I_i^{\geq h} = \bigcup_{h'=h}^{\text{ecc}_{T_i}(s_i)} I_i^{h'}$. We also set

$$V_i^h = \bigcup_{t \in I_i^h} X_t^i, \quad V_i^{<h} = \bigcup_{t \in I_i^{<h}} X_t^i, \quad \text{and} \quad V_i^{\geq h} = \bigcup_{t \in I_i^{\geq h}} X_t^i.$$

The *vertex-frontier* F_i of R_i is the set of vertices in V_i^{d-1} that are connected in G to a vertex $x \in V(G) \setminus R_i$ via a path, the internal vertices of which belong to $V_i^{\geq d}$. The *node-frontier* of T_i is

$$N_i = \{t \in V(T_i) \mid F_i \cap X_t \neq \emptyset\}. \quad (1)$$

A vertex $t \in I_i^{\geq d-1}$ is called a *port* of T_i if X_t^i contains some vertex that is adjacent in G to a vertex of $V(G) \setminus R_i$.

3 Finding small θ_r -models

3.1 Two intermediate results

The main results of this section are the following.

Theorem 3.1. *There exists an algorithm that, with input three integers r, δ, z , where $r \geq 2$, $\delta \geq 3r$, and $z \geq r$ and an m -edge graph G , outputs one the following:*

- a θ_r -model in G with at most z edges,
- a vertex v of G of degree less than δ , or
- an H -model in G for some graph H where $\delta(H) \geq \frac{\delta-2r+3}{r-1} \cdot \lfloor \frac{\delta}{r-1} - 1 \rfloor^{\frac{z-r}{4r}}$,

in $O_r(m)$ steps.

Theorem 3.2. *There exists an algorithm that, with input three positive integers $r \geq 2, w, z$ and a connected m -edge graph G , where $m \geq z > r \geq 2$, outputs one of the following:*

- a θ_r -model in G with at most z edges,
- a connected $(2r-2)$ -edge-protrusion Y of G with extension more than w , or
- an H -model in G for some graph H where $\delta(H) \geq \frac{1}{r-1} 2^{\frac{z-5r}{4r(2w+1)}}$,

in $O_r(m)$ steps.

The results of Chandran and Subramanian in [7] imply that if G has girth at least z and minimum degree at least δ , then $\mathbf{tw}(G) \geq \delta^{c \cdot z}$, for some constant c . As in the third condition of [Theorem 3.1](#) it holds that $\mathbf{tw}(G) \geq \mathbf{tw}(H) \geq \delta(H)$, [Theorem 3.1](#) can also be seen as a qualitative extension of the results of [7].

The above two results will be used to prove [Theorem 1.1](#) and [Theorem 1.2](#). We will also need the following result of Kostochka [20].

Proposition 3.3 ([20], see also [33, 34]). *There exists a constant $\xi \in \mathbb{R}$ such that for every $d \in \mathbb{N}$, every graph of average degree at least d contains a clique of order k as a minor, for some integer k satisfying*

$$k \geq \xi \cdot \frac{d}{\sqrt{\log d}}.$$

3.2 The proofs of the main theorems

We are now ready to prove [Theorem 1.1](#) and [Theorem 1.2](#) using the intermediate results described in the previous section.

Proof of [Theorem 1.1](#). Observe that since G has no θ_r -model with at most z edges and G has minimum degree $d \geq 3r$, a call to the algorithm of [Theorem 3.1](#) on (r, d, z, G) should return an H -model of G , for some graph H where $\delta(H) \geq \frac{d-2r+3}{r-1} \cdot \lfloor \frac{d}{r-1} - 1 \rfloor^{\frac{z-r}{4r}} =: d'$. Using the fact that $z - r \geq z/2$, it is not hard to check that there is a constant $c' \in \mathbb{R}$ such that

$$\xi \cdot \frac{d'}{\sqrt{\log d'}} \geq \frac{\left(\frac{d}{r}\right)^{c' \cdot \frac{z}{r}}}{\sqrt{\frac{z}{r} \cdot \log d}}.$$

Hence by [Proposition 3.3](#), G has a clique of the desired order as a minor. \square

Proof of [Theorem 1.2](#). As in the proof of [Theorem 1.1](#), the properties that G enjoys will force a minor of large minimum degree. Let us call the algorithm of [Theorem 3.2](#) on $(r, 3\alpha, z, G)$. We assumed that G has no θ_r -model with z edges or less, hence the output of the algorithm cannot be such a model. Let us now assume that the algorithm outputs a $(2r - 2)$ -edge-protrusion Y with extension more than 3α , and let (\mathcal{X}, T, s) be a rooted tree-partition of $G[Y \cup N_G(Y)]$ of width at most $2r - 2$ such that $N_G(Y) = X_s$ and $n(T) > 3\alpha$. It is known that every tree of order n has a vertex, the removal of which partitions the tree into components of size at most $n/2$ each. Hence, there is a vertex $v \in V(T)$ and a partition (Z, Z') of $V(T) \setminus \{v\}$ such that:

- both $Z \cup \{v\}$ and $Z' \cup \{v\}$ induce connected subtrees of T ;
- $\frac{1}{3}n(T) \leq |Z|, |Z'| \leq \frac{2}{3}n(T)$; and

- $s \in Z$ or $v = s$.

Let $A = \bigcup_{t \in Z'} X_t \cup \{X_v\}$ and $B = V(G) \setminus \bigcup_{t \in Z'} X_t$. Notice that $V(G) = A \cup B$ and that no edge of G lies between $A \setminus B$ and $B \setminus A$. As $A \cap B = X_v$, we have $|A \cap B| < 2r - 1$. Last, $Z' \subseteq A \setminus B$ and $Z \subseteq B \setminus A$ give that $|A \setminus B|, |B \setminus A| > \alpha$. The existence of A and B contradicts the fact that G is $(\alpha, 2r - 1)$ -loosely connected. Thus G has no $(2r - 2)$ -edge-protrusion Y of extension more than 3α .

A consequence of this observation is that the only possible output of the algorithm mentioned above is an H -model of G for some graph H , where

$$\delta(H) \geq \frac{1}{r-1} \cdot 2^{\frac{z-5r}{4r(6\alpha+1)}} \geq \frac{1}{r} \cdot 2^{\frac{z}{168 \cdot r\alpha}} =: d.$$

Notice also that $\log d = \frac{z}{168 \cdot r\alpha} - \log r$ which, by the condition of the theorem, is a non-negative number. Moreover, $\log d \leq z/r$. Therefore, there is a constant $c'' \in \mathbb{R}$ such that

$$\xi \cdot \frac{d}{\sqrt{\log d}} \geq \frac{2^{c'' \cdot \frac{z}{r\alpha}}}{\sqrt{z \cdot r}}$$

in order to conclude the proof. □

4 The proofs of **Theorem 3.1** and **Theorem 3.2**

4.1 Preliminary results

Before proving **Theorem 3.2** and **Theorem 3.1** (in **Section 4.2** and **Section 4.3**, respectively) we need some preliminary results. Let us start with some definitions.

Let (T, s) be a rooted tree and let N be a subset of its vertices, such that for every two vertices in N no one is a descendant of the other. We say that a vertex u of T is *N -critical* if either it belongs to N or there are at least two vertices in N that are descendants of two distinct children of u . An *N -unimportant* path in T is a path with at least 2 vertices, with exactly two N -critical vertices, which are its endpoints (see **Fig. 3** for a picture). Notice that an N -unimportant path in T cannot have an internal vertex that belongs to some other N -unimportant path. Also, among the two endpoints of an N -unimportant path there is always one which is a descendant of the other. As we see in the proof of the following lemma, N -unimportant paths are the maximal paths with internal vertices of degree 2 that appear if we repeatedly delete leaves that do not belong to N .

Lemma 4.1. *Let $d, k \in \mathbb{N}$, $k \geq 1$. Let (T, s) be a rooted tree and let N be a set of vertices of (T, s) , each of which is at distance at least d from s . If for some integer k , every N -unimportant path in T has length at most k , then $|N| \geq 2^{d/k}$.*

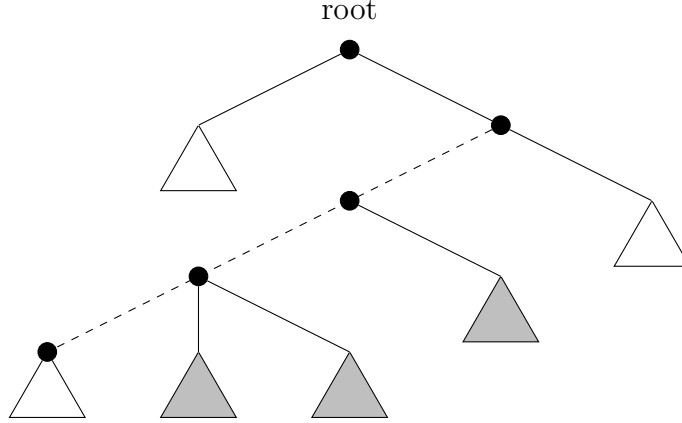


Figure 3: An unimportant path (dashed) in a tree. Gray subtrees are those without vertices from N .

Proof. We consider the subtree T' of T obtained by repeatedly deleting leaves that do not belong to N . For the purposes of this construction, the root of T is considered a leaf if it has degree 1 and, when deleted, the root of the new tree is the previous root's (unique) child. Let s' denote the root of T' . By construction, every leaf of (T', s') belongs to N , hence our goal is then to show that (T', s') has many leaves. Notice that in (T', s') , every vertex of degree at least 3 is N -critical. Therefore, the N -unimportant paths of (T', s') are the maximal paths, the internal vertices of which have degree two. By contracting each of these paths into an edge, we obtain a tree T'' where every internal vertex has degree at least 3. Observe that every edge on a root-leaf path of T'' is originated from the contraction of a path on at most k edges, as we assume that every N -unimportant path in T has length at most k . We deduce that T'' has height at least d/k , hence it has at least $2^{d/k}$ leaves. Consequently, T' has at least $2^{d/k}$ leaves, and then $|N| \geq 2^{d/k}$. \square

Recall that if (\mathcal{X}, T, s) is a distance-decomposition of a graph and $t \in V(T) \setminus \{s\}$, $E^{(t)}$ denotes as the set of edges that have one endpoint in X_t and the other in $X_{p(t)}$.

Lemma 4.2. *Let G be an n -vertex graph, let r be a positive integer, let $\mathcal{D} = (\mathcal{X}, T, s)$ be a distance-decomposition of G , and let $d > 1$ be the height of (T, s) . Then either G contains a θ_r -model with at most $2 \cdot r \cdot d$ edges or for every vertex $i \in V(T) \setminus s$, it holds that $|E^{(i)}| \leq r - 1$. Moreover there exists an algorithm that, in $O_r(m)$ steps, either finds such a model, or asserts that $|E^{(i)}| \leq r - 1$ for every $i \in V(T) \setminus s$.*

Proof. We consider the non-trivial case where $r \geq 2$. Suppose that there exists a node t of (T, s) such that $|E^{(t)}| \geq r$. Clearly, such a t can be found in $O(m)$ steps. We will prove that G contains a θ_r -model. Let k be the height of t in T .

We need first the following claim.

Claim 4.3. *Given a non-empty proper subset U of X_t , we can find in G_t a path of length at most $2k$ from a vertex of U to a vertex of $X_t \setminus U$, in $O(m)$ steps.*

Proof of Claim 4.3. We can compute a shortest path P from a vertex of U to a vertex of $X_t \setminus U$, in $O(m)$ steps using a BFS. Let us show that P has length at most $2k$. Let $u \in U$ and $v \in X_t \setminus U$ be the endpoints of P , and let w be a vertex of P of minimum height h ($0 \leq h \leq k$). Then it holds that $\mathbf{dist}_{G_t}(v, u) = \mathbf{dist}_{G_t}(U, v)$. We examine the non-trivial case where P has more than one edge. By minimality of P we have $w \notin X_t$.

Our next step is to prove that if P has more than one edge, then both the subpaths of P from u to w and from v to w are straight. Suppose now, without loss of generality, that the subpath from u to w is not straight and let z be the first vertex of it (starting from u) which is contained in a bag of height greater than or equal to the height of the bag of its predecessor in P . By definition of a distance-decomposition (in particular items (ii) and (iii)), there is at least one vertex $x \in X_t$ which is connected by a straight path P' to z in G . Then there are two possibilities:

- either $x \in U$, and then the union of the path P' and the portion of P between z and v is a path that is shorter than P ;
- or $x \in X_t \setminus U$, and in this case the union of the path P' and the portion of P between u and z is a path that is shorter than P .

As, in both cases, the occurring paths contradict the construction of P , we conclude that both the subpath of P from u to w and the one from v to w are straight. This implies that P has length at most $2 \cdot (k - h) \leq 2 \cdot k$ and the claim follows. \diamond

Our next step is to construct a vertex set U and a set of paths \mathcal{P} as follows. We set $\mathcal{P} = \emptyset$, $U = \emptyset$, and we start by adding in U an arbitrarily chosen vertex $u \in X_t$. Using the procedure of Claim 4.3, we repeatedly find a path from a vertex of U to a vertex of $X_t \setminus U$, add this second vertex to U and the path to \mathcal{P} , until there are at least r edges in $E^{(t)}$ that have endpoints in U .

The construction of U requires at most r repetitions of the procedure of Claim 4.3, and therefore $O(r \cdot m)$ steps in total. Clearly $|U| \leq r$, hence $|\mathcal{P}| \leq r - 1$. Besides, every path in \mathcal{P} has length at most $2k$ according to Claim 4.3. Notice now that $\bigcup \mathcal{P}$ is a connected subgraph of G_t with at most $2k \cdot (r - 1)$ edges.

As there are at least r edges in $E^{(t)}$ with endpoints in U we may consider a subset F of them where $|F| = r$. Since \mathcal{D} is a distance-decomposition (by item (ii) of the definition), each edge $e \in F$ is connected to the origin by a path of length $d - k - 1$ whose edges do not belong to G_t . Let \mathcal{P}' be the collection of these paths. Clearly, the paths in \mathcal{P}' contain, in total, at most $r \cdot (d - k - 1)$ edges.

If we now contract in G all edges in \mathcal{P} and all edges in \mathcal{P}' , except those in F , and then remove all edges not in F , we obtain a graph isomorphic to θ_r . Therefore we found in G a θ_r -model with at most

$$\begin{aligned}
r \cdot (d - k - 1) + 2 \cdot k \cdot (r - 1) + r &\leq r \cdot (d - k - 1) + 2 \cdot k \cdot r + r \\
&= r \cdot (d + k) \\
&\leq 2 \cdot r \cdot d && \text{(since } d \geq k)
\end{aligned}$$

edges in $O(r \cdot m)$ steps. \square

The following result is a direct consequence of [Lemma 4.2](#) and item (ii) of the definition of a distance-decomposition.

Corollary 4.4. *Let G be an n -vertex graph, let r be a positive integer, let $\mathcal{D} = (\mathcal{X}, T, s)$ be a distance-decomposition of G , and let $d > 1$ be the height of (T, s) . If some bag of \mathcal{D} contains at least r vertices, then G contains a θ_r -model with at most $2 \cdot r \cdot d$ edges, which can be found in $O_r(m)$ steps. \square*

The remaining lemmata are related to [grouped partitions](#).

Lemma 4.5. *For every positive integer d and every connected graph G there is a d -grouped partition of G that can be constructed in $O(m)$ steps.*

Proof. If $\mathbf{diam}(G) \leq 2d$, then $\{V(G)\}$ is a d -grouped partition of G . Otherwise, let $R = \{s_1, \dots, s_l\}$ be a maximal $2d$ -scattered set in G . This set can be constructed in $O(m)$ steps by breadth-first search. The sets $\{R_i\}_{i \in \{1, \dots, l\}}$ are constructed by the following procedure:

1. Set $k = 0$ and $R_i^0 = \{s_i\}$ for every $i \in \{1, \dots, l\}$;
2. For every $i \in \{1, \dots, l\}$, every $v \in R_i^k$ and every $u \in N_G(v)$, if u has not been considered so far, add u to R_i^{k+1} ;
3. If $k < 2d$, increment k by 1 and go to step 2;
4. Let $R_i = \bigcup_{k=0}^{2d} R_i^k$ for every $i \in \{1, \dots, l\}$.

Let $\mathcal{R} = \{R_i\}_{i \in \{1, \dots, l\}}$. By construction, each set R_i induces a connected graph in G . It remains to prove that \mathcal{R} is a partition of $V(G)$ and that it has the desired properties.

Notice that in the above construction if a vertex is assigned to the set R_i , then it is not assigned to R_j , for every distinct integers $i, j \in \{1, \dots, l\}$. Let $v \in V(G)$ be a vertex that does not belong to R_i for any $i \in \{1, \dots, l\}$ after the procedure is completed. Then for every $i \in \{1, \dots, l\}$ we have $\mathbf{dist}_G(v, s_i) > 2d$ and $v \notin R$, which contradicts the maximality of R . Therefore \mathcal{R} is a partition of $V(G)$.

Since for each vertex v in R_i it holds that $\mathbf{dist}_G(v, s_i) \leq 2d$, \mathcal{R} obviously satisfies property (i) of the definition.

For property (ii) of the definition, let $e = \{x, y\}$ be an edge in G such that $x \in R_i$, $y \in R_j$, for some distinct integers $i, j \in \{1, \dots, l\}$. Towards a contradiction, we assume

without loss of generality that $\mathbf{dist}_G(x, s_i) < d$. This means that during the construction of R_i , the vertex x was added to the set R_i^k for some $k \leq d - 1$. Also, since the vertex y is adjacent to x but was added to R_j^l for some $l \leq 2d$ instead of R_i^{k+1} , it follows that $l \leq k + 1$, which means that $\mathbf{dist}_G(y, s_j) \leq k + 1$. Hence $\mathbf{dist}_G(s_i, s_j) \leq \mathbf{dist}_G(s_i, x) + \mathbf{dist}_G(x, y) + \mathbf{dist}_G(y, s_j) \leq k + 1 + k + 1 \leq 2d$ again is not possible since R is a $2d$ -scattered set.

Finally, in the procedure above, each edge of the graph is encountered at most once, hence the whole algorithm will take at most $O(m)$ time. This concludes the proof of the lemma. \square

Lemma 4.6. *Let G be a graph, let $\mathcal{R} = \{R_1, \dots, R_l\}$ be a d -grouped partition of G , and let s_i be a center of R_i , for every $i \in \{1, \dots, l\}$. If for some distinct $i, j \in \{1, \dots, l\}$, G has at least r edges from vertices in R_i to vertices in R_j then $G[R_i \cup R_j]$ contains a θ_r -model with at most $4 \cdot r \cdot d + r$ edges, which can be found in $O_r(m)$ steps.*

Proof. Suppose that for some $i \in \{1, \dots, l\}$, G has a set F of at least r edges from vertices in R_i to vertices in R_j . Let $R'_i \subseteq R_i$ and $R'_j \subseteq R_j$ be the sets of the endpoints of those edges. Since \mathcal{R} is a d -grouped partition of G , it holds that, for each $x \in R'_i$ and $y \in R'_j$, $\mathbf{dist}_G(x, s_i) \leq 2d$ and $\mathbf{dist}_G(y, s_j) \leq 2d$. That directly implies that for every $h \in \{i, j\}$, there is a collection \mathcal{P}_h of r paths, each of length at most $2d$ and not necessarily disjoint, in $G[R_h]$ connecting s_h with each vertex in R'_h , which we can find in $O_r(m)$ steps. It is now easy to observe that the graph Q , obtained from $\bigcup \mathcal{P}_i \cup \bigcup \mathcal{P}_j$ by adding all edges of F , is the union of r paths between s_i and s_j , each containing at most $4 \cdot d + 1$ edges. Therefore, Q is a model of θ_r with at most $4 \cdot r \cdot d + r$ edges, as required. As mentioned earlier the construction of \mathcal{P}_i and \mathcal{P}_j takes $O_r(m)$ steps. \square

Lemma 4.7. *Let G be a graph, let $\mathcal{R} = \{R_1, \dots, R_l\}$ be a d -grouped partition of G , and let $S = \{s_1, \dots, s_l\}$ be a set of centers of \mathcal{R} . For every $i \in \{1, \dots, l\}$, let $\mathcal{D}_i = (\mathcal{X}_i, T_i, r_i)$ be the distance-decomposition with origin s_i of the graph $G[R_i]$. Let also r be a positive integer such that for every $i \in \{1, \dots, l\}$ and for every vertex $t \in V(T_i) \setminus \{r_i\}$ it holds that $|E^{(t)}| \leq r - 1$. If for some $i \in \{1, \dots, l\}$ and $w \in \mathbb{N}$, the tree T_i , with node-frontier N_i , has an N_i -unimportant path of length at least $2(w + 1)$, then G has a connected $(2r - 2)$ -edge-protrusion Y with extension more than w , which can be constructed in $O_r(m)$ steps.*

Proof. Let $P = t_0 \dots t_p$ be an N_i -unimportant path of length $p \geq 2(w + 1)$ in T_i . We assume without loss of generality that $t_p \in \mathbf{des}_{(T_i, r_i)}(t_0)$. Due to the definition of distance-decompositions, the vertices in $X_{t_0}^i$ or $X_{t_p}^i$ form a vertex-separator of G . Let $Z \subseteq E(G)$ be the set containing all edges between $X_{t_0}^i$ and $X_{t_1}^i$ and all edges between $X_{t_{p-1}}^i$ and $X_{t_p}^i$ in G . Clearly, Z is an edge-separator of G with at most $2r - 2$ edges. Let T'_i be the subtree of T_i that we obtain if we remove the descendants of t_p that are distinct from this vertex and any vertex that is not t_0 or a descendant of t_1 . Let $Y = \bigcup_{t \in V(T'_i) \setminus \{t_0, t_p\}} X_t^i$. In other words, Y consists of the vertices in the bags of T'_i excluding $X_{t_0}^i$ and $X_{t_p}^i$. Obviously, $N_G(Y) = X_{t_0}^i \cup X_{t_p}^i$.

We will now construct a rooted tree-partition $\mathcal{F} = (\mathcal{X}_{\mathcal{F}}, T_{\mathcal{F}}, r_{\mathcal{F}})$ of $G[Y \cup N_G(Y)]$ of width at most $2r - 2$ and such that $|V(T_{\mathcal{F}})| > w$. Let $T_{\mathcal{F}}$ be the tree obtained from T'_i by identifying, for every $j \in \{0, \dots, \lfloor (p-1)/2 \rfloor\}$, the vertex t_j with the vertex t_{p-j} . If multiple edges are created during this identification, we replace them with simple ones. We also delete loops that may be created. Let us define the elements of $\mathcal{X}^{\mathcal{F}} = \{X_t^{\mathcal{F}}\}_{t \in V(T_{\mathcal{F}})}$ as follows. If $t \in V(T_{\mathcal{F}})$ is the result of the identification of t_j and t_{p-j} for some $j \in \{0, \dots, \lfloor (p-1)/2 \rfloor\}$, then we set $X_t^{\mathcal{F}} = X_{t_j} \cup X_{t_{p-j}}$. On the other hand, if $t \in V(T_{\mathcal{F}})$ is a vertex of T'_i that has not been identified with some other vertex, then $X_t^{\mathcal{F}} = X_t$. The construction of \mathcal{F} is completed by setting $r_{\mathcal{F}}$ to be the result of the identification of t_0 and t_p , the endpoints of P .

It is easy to verify that \mathcal{F} is a rooted tree-partition of $G[Y \cup N_G(Y)]$ of width at most $2r - 2$. Notice also that the identification of the antipodal vertices of the path P creates a path in $T_{\mathcal{F}}$ of length $\lfloor (p-1)/2 \rfloor$. This implies that the extension of \mathcal{F} is at least $\lfloor (p-1)/2 \rfloor \geq w + 1$. Besides, all the operations performed to construct \mathcal{F} can be implemented in $O_r(m)$ steps. This completes the proof. \square

We conclude this section with two easy lemmata related to **ports and frontiers**.

Lemma 4.8. *Let G be a graph, let $\mathcal{R} = \{R_1, \dots, R_l\}$ be a d -grouped partition of G , and let $S = \{s_1, \dots, s_l\}$ be a set of centers of \mathcal{R} . For every $i \in \{1, \dots, l\}$, let $\mathcal{D}_i = (\mathcal{X}_i, T_i, r_i)$ be the distance-decomposition with origin s_i of the graph $G[R_i]$, and let N_i be the node-frontier of T_i . Then, for every $i \in \{1, \dots, l\}$, there are at least $|N_i|$ ports in T_i .*

Proof. Let $i \in \{1, \dots, l\}$. We will show that every vertex in the node-frontier of T_i has a descendant which is a port. For every vertex $t \in N_i \subseteq V(T_i)$, there is, by definition, a path from t to a vertex in $G \setminus R_i$, the internal vertices of which belong to $V_i^{\geq d}$. Let v be the last vertex of this path (starting from t) which belongs to R_i and let $t' \in V(T)$ be the vertex such that $v \in X_{t'}^i$. Then t' is a port of T_i . Observe that t' cannot be the descendant of any other vertex of N_i . Therefore there are at least $|N_i|$ ports in T_i . \square

Corollary 4.9. *Let G be a graph, let $\mathcal{R} = \{R_1, \dots, R_l\}$ be a d -grouped partition of G , and let $S = \{s_1, \dots, s_l\}$ be a set of centers of \mathcal{R} . For every $i \in \{1, \dots, l\}$, let $\mathcal{D}_i = (\mathcal{X}_i, T_i, r_i)$ be the distance-decomposition with origin s_i of the graph $G[R_i]$, and let N_i be the node-frontier of T_i . If for some integer k , every N_i -unimportant path in T_i has length at most k , then T_i contains at least $2^{d/k}$ ports.* \square

Proof. Let $i \in \{1, \dots, l\}$. From **Lemma 4.8**, it is enough to prove that $|N_i| \geq 2^{d/k}$. Then the result follows by applying **Lemma 4.1** for (T_i, s_i) , d , N_i , and k . \square

4.2 Proof of **Theorem 3.2**

Proof. Let $d = \frac{z-r}{4r}$. According to **Lemma 4.5**, we can construct in $O(m)$ steps a d -grouped partition $\mathcal{R} = \{R_1, \dots, R_l\}$ of $V(G)$, with a set of centers $S = \{s_1, \dots, s_l\}$, and also, for every $i \in \{1, \dots, l\}$, the distance-decompositions $\mathcal{D}_i = (\mathcal{X}_i, T_i, r_i)$ with origins

s_i of the graphs $G[R_i]$. For every $i \in \{1, \dots, l\}$, we use the notation $\mathcal{X}_i = \{X_t^i\}_{t \in V(T_i)}$ and denote by N_i the node-frontiers of T_i .

By applying the algorithm of [Lemma 4.6](#), in $O_r(m)$ steps, we either find a θ_r -model in G with at most $z = 4 \cdot r \cdot d + r$ edges or we know that for every two distinct $i, j \in \{1, \dots, l\}$ there are at most $r - 1$ edges of G with one endpoint in R_i and one in R_j .

Similarly, by applying the algorithm of [Lemma 4.2](#), in $O_r(m)$ steps we either find a θ_r -model in G with at most $2 \cdot r \cdot d \leq z$ edges (in which case we are done) or we know that for every $i \in \{1, \dots, l\}$ and every $t \in V(T_i) \setminus s_i$, it holds that $|E^{(i)}| \leq r - 1$.

In the second case, by using the algorithm of [Lemma 4.7](#), in $O_r(m)$ steps we either find a connected $(2r - 2)$ -edge-protrusion with extension more than w , or we know that for every $i \in \{1, \dots, l\}$, all N_i -unimportant paths of T_i have length at most $2w + 1$.

We may now assume that none of the above algorithms provided a θ_r -model with z edges, or a $(2r - 2)$ -edge-protrusion.

From [Corollary 4.9](#), for every $i \in \{1, \dots, l\}$ the tree T_i contains at least $2^{\frac{d-1}{2w+1}} = \frac{z-5r}{2^{4r \cdot (2w+1)}}$ ports, which by definition means that there are at least $2^{\frac{z-5r}{4r \cdot (2w+1)}}$ edges in G with one endpoint in R_i and the other in $V(G) \setminus R_i$. By [Lemma 4.6](#), for every distinct integers $i, j \in \{1, \dots, l\}$ there are at most $r - 1$ edges with one endpoint in R_i and the other in R_j . As a consequence of the two previous implications, for every $i \in \{1, \dots, l\}$ there is a set $Z_i \subseteq \{1, \dots, l\} \setminus \{i\}$, where $|Z_i| \geq \frac{1}{r-1} 2^{\frac{z-5r}{4r(2w+1)}}$, such that for every $j \in Z_i$ there exists an edge with one endpoint in R_i and the other in R_j . Consequently, if we now contract all edges in $G[R_i]$ for every $i \in \{1, \dots, l\}$, the resulting graph H is a minor of G of minimum degree at least $\frac{1}{r-1} 2^{\frac{z-5r}{4r(2w+1)}}$. Therefore, we output G , which is an H -model, as required in this case. \square

4.3 Proof of [Theorem 3.1](#)

Proof. The proof is quite similar to the one of [Theorem 3.2](#). If G contains a vertex v of degree less than δ , we can easily find it in $O(m)$ steps. Hence, from now on we can assume that every vertex has degree at least δ .

Let $d = \frac{z-r}{4r}$. From [Lemma 4.5](#), in $O(m)$ steps, we can construct a d -grouped partition $\mathcal{R} = \{R_1, \dots, R_l\}$ of G , with a set of centers $S = \{s_1, \dots, s_l\}$, and also the distance-decomposition $\mathcal{D}_i = (\mathcal{X}_i, T_i, r_i)$ with origins s_i of the graphs $G[R_i]$, for every $i \in \{1, \dots, l\}$. We use again the notation $\mathcal{X}_i = \{X_t^i\}_{t \in V(T_i)}$.

As in the proof of [Theorem 3.2](#), in $O_r(m)$ steps, we can either find a θ_r -model in G with at most $z = 4 \cdot r \cdot d + r$ edges or we know that for every distinct integers $i, j \in [l]$ there are at most $r - 1$ edges of G with one endpoint in R_i and one in R_j (*cf.* [Lemma 4.6](#)).

Using [Corollary 4.4](#), we can in $O_r(m)$ steps either find a θ_r -model in G with at most z edges or we know that every bag of \mathcal{D}_i has less than r vertices, for every $i \in \{1, \dots, l\}$. Let $i \in \{1, \dots, l\}$ and let $u \in R_i$ be a vertex at distance less than d from s_i . As u has degree at least $3r$, it must have neighbors in at least 3 different bags of \mathcal{D}_i , apart from the one containing it. This means that every vertex in T_i of distance less than d from r_i has

degree at least $\lfloor \frac{\delta}{r-1} \rfloor \geq 3$ and therefore T_i has at least $\lfloor \frac{\delta}{r-1} - 1 \rfloor^d$ leaves. Notice also that if t is a leaf of T_i , then each vertex in X_t^i can have at most $r-1$ neighbors in $X_{\mathbf{p}(t)}^i$ and at most $r-2$ neighbors in X_t^i . Therefore there are at least $\delta - (r-1) - (r-2) = \delta - 2r + 3$ edges in G with one endpoint in X_t^i and the other in $V(G) \setminus R_i$. This means that for every $i \in \{1, \dots, l\}$ there are at least $(\delta - 2r + 3) \cdot \lfloor \frac{\delta}{r-1} - 1 \rfloor^d$ edges with one endpoint in R_i and the other $V(G) \setminus R_i$.

Similarly to the proof of [Theorem 3.2](#), we deduce that, for each $i \in \{1, \dots, l\}$, there is a set $Z_i \subseteq \{1, \dots, l\} \setminus \{i\}$ where $|Z_i| \geq \frac{\delta - 2r + 3}{r-1} \cdot \lfloor \frac{\delta}{r-1} - 1 \rfloor^d$ such that, for every $j \in Z_i$, there exists an edge with one endpoint in R_i and the other in R_j . This implies the existence of an H -model in G for some H with $\delta(H) \geq \frac{\delta - 2r + 3}{r-1} \cdot \lfloor \frac{\delta}{r-1} - 1 \rfloor^{\frac{z-r}{4r}}$. We then output G , which, in this case, is an H -model. \square

5 Excluding k copies of θ_r as a minor

This section is devoted to the proof of the following theorem.

Theorem 5.1. *For every graph G , $r \geq 2$, and $k \geq 1$, if $\mathbf{tw}(G) \geq 2^{6r} \cdot k \cdot \log(k+1)$, then G contains $k \cdot \theta_r$ as a minor.*

For the proof, we need to introduce some definitions and related results.

5.1 Preliminaries

Let G be a graph and G_1, G_2 two non-empty subgraphs of G . We say that (G_1, G_2) is a *separation* of G if:

- $V(G_1) \cup V(G_2) = V(G)$; and
- $(E(G_1), E(G_2))$ is a partition of $E(G)$.

Let G be a graph. Given a set $E \subseteq E(G)$, we define V_E as the set of all endpoints of the edges in E . Given a partition (E_1, E_2) of $E(G)$ we define $\delta(E_1, E_2) = |V_{E_1} \cap V_{E_2}|$.

A *cut* $C = (X, Y)$ of G is a partition of $V(G)$ into two subsets X and Y . We define the *cut-set* of C as $E_C = \{\{x, y\} \in E(G) \mid x \in X \text{ and } y \in Y\}$ and call $|E_C|$ the *order* of the cut. Also, given a graph G , we denote by $\sigma(G)$ the number of connected components of G .

The branchwidth of a graph. A *branch-decomposition* of a graph G is a pair (T, τ) where T is a ternary tree and τ a bijection from the edges of G to the leaves of T . Deleting any edge e of T partitions the leaves of T into two sets, and thus the edges of G into two subsets E_1^e and E_2^e . The *width* of a branch-decomposition (T, τ) is equal to $\max_{e \in E(T)} \{\delta(E_1^e, E_2^e)\}$. The *branchwidth* of a graph G , denoted $\mathbf{bw}(G)$, is defined as the minimum width over all branch-decompositions of G .

The branchwidth of a matroid. We assume that the reader is familiar with the basic notions of matroid theory. We will use the standard notation from Oxley’s book [25]. The branchwidth of a matroid is defined very similarly to that of a graph. Let \mathcal{M} be a matroid with finite ground set $E(\mathcal{M})$ and rank function r . The *order* of a non-trivial partition (E_1, E_2) of $E(\mathcal{M})$ is defined as $\lambda(E_1, E_2) = r(E_1) + r(E_2) - r(E) + 1$. A *branch-decomposition* of a matroid \mathcal{M} is a pair (T, μ) where T is a ternary tree and μ is a bijection from the elements of $E(\mathcal{M})$ to the leaves of T . Deleting any edge e of T partitions the leaves of T into two sets, and thus the elements of $E(\mathcal{M})$ into two subsets E_1^e and E_2^e . The *width* of a branch-decomposition (T, μ) is equal to $\max_{e \in E(T)} \{\lambda(E_1^e, E_2^e)\}$. The *branchwidth* of a matroid \mathcal{M} , denoted $\mathbf{bw}(\mathcal{M})$, is again defined as the minimum width over all branch-decompositions of \mathcal{M} . The *cycle matroid of a graph* G denoted \mathcal{M}_G , has ground set $E(\mathcal{M}_G) = E(G)$ and the cycles of G as the circuits of \mathcal{M}_G . Let G be a graph, \mathcal{M}_G its cycle matroid and (G_1, G_2) a separation of G . Then clearly $(E(G_1), E(G_2))$ is a partition of $E(\mathcal{M}_G)$, but to avoid confusion we will henceforth denote it (E_1, E_2) and we will call it *the partition of \mathcal{M}_G that corresponds to the separation (G_1, G_2) of G* . Observe that the order of this partition is:

$$\lambda(E_1, E_2) = \delta(E(G_1), E(G_2)) - \sigma(G_1) - \sigma(G_2) + \sigma(G) + 1. \quad (\star)$$

Minor obstructions. Let \mathcal{G} be a graph class. We denote by $\mathbf{obs}(\mathcal{G})$ the set of all minor-minimal graphs H such that $H \notin \mathcal{G}$ and we will call it *the minor obstruction set for \mathcal{G}* . Clearly, if \mathcal{G} is closed under minors, the minor obstruction set for \mathcal{G} provides a complete characterization for \mathcal{G} : a graph G belongs in \mathcal{G} if and only if none of the graphs in $\mathbf{obs}(\mathcal{G})$ is a minor of G .

Given a class of matroids \mathbf{M} , *the minor obstruction set for \mathbf{M}* , denoted by $\mathbf{obs}(\mathbf{M})$, is defined very similarly to its graph-counterpart: it is simply the set of all minor-minimal matroids \mathcal{M} such that $\mathcal{M} \notin \mathbf{M}$.

We will need the following results.

Proposition 5.2 ([29, Theorem 5.1]). *Let G be a graph of branchwidth at least 2. Then, $\mathbf{bw}(G) \leq \mathbf{tw}(G) + 1 \leq \lfloor \frac{3}{2} \mathbf{bw}(G) \rfloor$.*

Proposition 5.3 ([6]). *Let $r \in \mathbb{N}_{\geq 1}$ and let G be a graph. If $\mathbf{bw}(G) \geq 2r + 1$, then G contains a θ_r -model.*

Proposition 5.4 ([18, Theorem 4]). *Let G be a graph that contains a cycle and \mathcal{M}_G be its cycle matroid. Then, $\mathbf{bw}(G) = \mathbf{bw}(\mathcal{M}_G)$.*

Proposition 5.5 ([16, Lemma 4.1]). *Let a matroid \mathcal{M} be a minor obstruction for the class of matroids of branchwidth at most k and let $g(n) = (6^{n-1} - 1)/5$. Then, for every partition (X, Y) of \mathcal{M} with $\lambda(X, Y) \leq k$, either $|X| \leq g(\lambda(X, Y))$ or $|Y| \leq g(\lambda(X, Y))$.*

The following observations are also crucial.

Observation 5.6. *Let \mathcal{G} be a graph class that is closed under minors and let $\mathcal{M}_{\mathcal{G}} = \{\mathcal{M}_G \mid G \in \mathcal{G}\}$. \mathcal{G} is minor closed if and only if $\mathcal{M}_{\mathcal{G}}$ is minor closed. Moreover, for every $H \in \mathbf{obs}(\mathcal{G})$ it holds that $\mathcal{M}_H \in \mathbf{obs}(\mathcal{M}_{\mathcal{G}})$.*

The above observation is a direct consequence of the definition of matroid removal/contraction, e.g., see Proposition 4.9 of [26].

Lemma 5.7. *There is a constant $c \in \mathbb{R}_{\geq 2}$, such that for any integer $k \geq r \geq 2$, if $g(n) = (6^{n-1} - 1)/5$, then $\frac{1}{r-1} 2^{\frac{c^r \log k - 5r}{4r(2g(2r-2)+1)}} \geq k(r+1) - 1$. Moreover, this holds for $c = 6^3$.*

Proof. We only present here the main steps of the computation. Let $c = 6^3$. If $r \geq 4$, then we have:

$$\begin{aligned}
c^r &\geq 6^{3r-3} \\
&\geq \frac{8r \cdot 6^{2r-3}(r^2 - 1) \log k}{\log k} \\
&\geq \frac{4r[1 + 2(6^{2r-3} - 1)] \log[k(r^2 - 1) - (r - 1)] + 5r}{\log k} \\
c^r \log k - 5r &\geq 4r[1 + \frac{2}{5}(6^{2r-3} - 1)] \log[k(r^2 - 1) - (r - 1)] \\
2^{\frac{c^r \log k - 5r}{4r(1+2g(2r-2))}} &\geq k(r^2 - 1) - (r - 1) \\
\frac{1}{r-1} 2^{\frac{c^r \log k - 5r}{4r(1+2g(2r-2))}} &\geq k(r+1) - 1.
\end{aligned}$$

Finally, for $c = 6^3$ and either $r = 2$ or $r = 3$, one can easily check that the inequality holds for any $k \geq r$. □

5.2 Graphs with large minimum degree

In this subsection we show that every graph of large minimum degree contains $k \cdot \theta_r$ as minor. Our proof relies on the following result.

Proposition 5.8 ([32, Corollary 3]). *For every $k, r \in \mathbb{N}_{\geq 1}$, every graph G with $\delta(G) \geq k(r+1) - 1$ has a partition (V_1, \dots, V_k) of its vertex set satisfying $\delta(G[V_i]) \geq r$ for every $i \in \{1, \dots, k\}$.*

Lemma 5.9. *For every integer $r \in \mathbb{N}_{\geq 1}$, every graph of minimum degree at least r contains a θ_r -model.*

Proof. Starting from any vertex u , we grow a maximal path P in G by iteratively adding to P a vertex that is adjacent to the previously added vertex but does not belong to P .

Since $\delta(G) \geq r$, any such path will have length at least $r + 1$. At the end, all the neighbors of the last vertex v of P belong to P (otherwise P could be extended). Since v has degree at least r , v has at least r neighbors in P . Therefore P is a θ_r -model in G . \square

Corollary 5.10. *For every $k, r \in \mathbb{N}_{\geq 1}$, every graph G with $\delta(G) \geq k(r + 1) - 1$ contains a $k \cdot \theta_r$ -model.*

Proof. According to [Proposition 5.8](#), $V(G)$ has a partition (V_1, \dots, V_k) such that $\delta(G[V_i]) \geq r$ for every $i \in \{1, \dots, k\}$. Therefore, by [Lemma 5.9](#), for every $i \in \{1, \dots, k\}$ the graph $G[V_i]$ has a θ_r -model M_i . Clearly $M_1 \cup \dots \cup M_k$ is a $k \cdot \theta_r$ -model in G , as desired. \square

Now we are ready to prove the main result of this section.

5.3 Proof of [Theorem 5.1](#)

For every $r \in \mathbb{N}$, we define $f(r) = \frac{2}{3}2^{6r}$. By [Proposition 5.2](#), it is enough to prove that if $\mathbf{bw}(G) \geq f(r) \cdot k \cdot \log(k + 1)$, then G contains $k \cdot \theta_r$ as a minor. To prove this we use induction on k .

The case where $k = 1$ follows from [Proposition 5.3](#) and the fact that $f(r) \geq 2r + 1$. We now examine the case where $k > 1$, assuming that the proposition holds for smaller values of k . As $\mathbf{bw}(G) \geq f(r) \cdot k \cdot \log(k + 1)$, G contains a minor obstruction H for the class of graphs of branchwidth at most $f(r) \cdot k \cdot \log(k + 1) - 1$. We will in fact prove that H contains $k \cdot \theta_r$ as a minor.

Claim 5.11. *Any $(2r - 2)$ -edge-protrusion of H has extension at most $g(2r - 2)$.*

Proof of Claim 5.11. Let $C = (X, Y)$ be a cut in H of order at most $2r - 2$ and let H_X be the subgraph of H with $V(H_X) = X \cup N_H(X)$ and let $E(H_X) = E(H[X]) \cup E_C$. Clearly the pair $(H_X, H[Y])$ is a separation of H . Let \mathcal{M}_H be the cycle matroid of H and (E_X, E_Y) be the partition of \mathcal{M}_H that corresponds to the aforementioned separation. By [Proposition 5.4](#), $\mathbf{bw}(\mathcal{M}_H) = \mathbf{bw}(H) \geq f(r) \cdot k \cdot \log(k + 1)$ (as $\mathbf{bw}(H) \geq 3$, H is not acyclic). Therefore, by [Observation 5.6](#), \mathcal{M}_H is a minor obstruction for the class of matroids of branchwidth $f(r) \cdot k \cdot \log(k + 1) - 1$. We set $\lambda = \lambda(E_X, E_Y)$. From [\(★\)](#), we have:

$$\begin{aligned} \lambda &= r(E_X) + r(E_Y) - r(\mathcal{M}_H) + 1 \\ &= \delta(E(H_X), E(H[Y])) - \sigma(H_X) - \sigma(H[Y]) + \sigma(H) + 1 \\ &\leq \delta(E(H_X), E(H[Y])) \\ &\leq |E_C| = 2r - 2 \\ &\leq f(r) \cdot k \cdot \log(k + 1) - 1. \end{aligned}$$

Thus, by [Proposition 5.5](#), either $|E_X| \leq g(\lambda)$ or $|E_Y| \leq g(\lambda)$. Since g is non-decreasing, either $|E(H_X)| \leq g(2r - 2)$ or $|E(H[Y])| \leq g(2r - 2)$. This directly implies

that for any $(2r-2)$ -edge-protrusion Z of H , $H[Z \cup N_H(Z)]$ has at most $g(2r-2)$ edges. Therefore Z 's extension is also at most $g(2r-2)$ and the claim follows. \diamond

Combining the above claim, [Lemma 5.7](#), and [Theorem 3.2](#), we infer that either H contains a θ_r -model with at most $f(r) \cdot \log k$ edges, or it contains a minor with minimum degree at least $\frac{1}{r-1} \cdot 2^{\frac{f(r) \log k - 5r}{4r(2g(2r-2)+1)}} \geq k(r+1) - 1$. If the second case is true, then by [Corollary 5.10](#), H contains $k \cdot \theta_r$ as a minor. We immediately deduce that G contains $k \cdot \theta_r$ as a minor, which proves the inductive step. We now consider the first case. Let M be a θ_r -model with a minimum number of edges. Because we are in the first case, $|E(M)| \leq f(r) \cdot \log k$. Observe that, because θ_r is 2-connected (as we assume $r \geq 2$), there is some 2-connected component of M which contains θ_r as a minor. We deduce from the minimality of M that M is 2-connected. Therefore, $|V(M)| \leq |E(M)| \leq f(r) \cdot \log k$ and we can bound the treewidth of the graph $H' = H \setminus V(M)$ as follows:

$$\begin{aligned} \mathbf{tw}(H') &\geq \mathbf{tw}(H) - |V(M)| \\ &\geq f(r) \cdot k \cdot \log(k+1) - f(r) \cdot \log k \\ &\geq f(r) \cdot k \cdot \log k - f(r) \cdot \log k \\ &= f(r) \cdot (k-1) \cdot \log k. \end{aligned}$$

Then, from the induction hypothesis, H' contains a $(k-1) \cdot \theta_r$ -model M' and obviously $M \cup M'$ is a $k \cdot \theta_r$ -model in H . Again, this implies that G contains $k \cdot \theta_r$ as a minor, which concludes the induction step. \square

[Theorem 5.1](#) implies that for every fixed r , it holds that every graph excluding $k \cdot \theta_r$ as a minor has treewidth $O(k \cdot \log k)$. We conclude with a lemma indicating that this bound is tight up to the constants hidden in the O -notation.

Lemma 5.12. *There is an increasing sequence of integers $(k_i)_{i \in \mathbb{N}}$ and an infinite sequence of graphs $(G_i)_{i \in \mathbb{N}}$ such that $\mathbf{tw}(G_i) = \Omega(k_i \log k_i)$ and G_i does not contain $k_i \cdot \theta_r$ as a minor, for every $r \in \mathbb{N}_{\geq 2}$.*

Proof. According to [[24](#), Theorem 5.13], there is an infinite family $\{G_i\}_{i \in \mathbb{N}}$ of 3-regular Ramanujan graphs G_i such that $i \mapsto |G_i|$ is an increasing function. Furthermore, for every $i \in \mathbb{N}$, the graph G_i has girth at least $\frac{2}{3} \log |V(G_i)|$ ([[24](#), Theorem 5.13]) and satisfies $\mathbf{tw}(G_i) = \Omega(|V(G_i)|)$ (see [[1](#), Corollary 1]). For every $i \in \mathbb{N}$, let k_i be the minimum integer such that $|V(G_i)| < k_i \cdot \frac{2}{3} \log |V(G_i)|$. Observe that $(k_i)_{i \in \mathbb{N}}$ is increasing. Notice that $|V(G_i)| = \Omega(k_i \cdot \log k_i)$, and thus $\mathbf{tw}(G_i) = \Omega(k_i \cdot \log k_i)$. We will show that G_i does not contain k_i vertex-disjoint cycles, which implies that $k_i \cdot \theta_r$ is not a minor of G_i , for every $r \in \mathbb{N}_{\geq 2}$. Suppose for contradiction that G_i contains k_i vertex-disjoint cycles. As the girth of G_i is at least $\frac{2}{3} \log |V(G_i)|$, each of these cycles has at least $\frac{2}{3} \log |V(G_i)|$ vertices. Therefore G should contain at least $k \cdot \frac{2}{3} \log |V(G_i)|$ vertices. This implies that $|V(G)| \geq k \cdot \frac{2}{3} \log |V(G_i)| > |V(G_i)|$, a contradiction. Therefore $(k_i)_{i \in \mathbb{N}}$ and $(G_i)_{i \in \mathbb{N}}$ satisfy the required properties. \square

6 Concluding remarks

In this paper, we introduced the concept of H -girth and proved that for every $r \in \mathbb{N}_{\geq 2}$, a large θ_r -girth forces an exponentially large clique minor. This extends the results of Kühn and Osthus related to the usual notion of girth. We also gave a variant of our result where the minimum degree is replaced by a connectivity measure. As an application of our result, we optimally improved (up to a constant factor) the upper-bound on the treewidth of graphs excluding $k \cdot \theta_r$ as a minor. A first question is whether our lower-bound on the clique minor size can be improved.

Let us now state more general questions spawned by this work. A natural line of research is to investigate the H -girth parameter for different instantiations of H . An interesting problem in this direction could be to characterize the graphs H for which our results ([Theorem 1.1](#) and [Theorem 1.2](#)) can be extended.

From its definition, the H -girth is related to the minor relation. An other direction of research would be to extend the parameter of H -girth to other containment relations. One could consider, for a fixed graph H , the minimum size of an induced subgraph that can be contracted to H , or the minimum size of a subdivision of H in a graph. The first one of these parameters is related to induced minors and the second one to topological minors.

As the usual notion of girth appears in various contexts in graph theory, we wonder for which graphs H the results related to girth can be extended to the H -girth or to the two aforementioned variants.

References

- [1] S. Bezrukov, R. Elsässer, B. Monien, R. Preis, and J.-P. Tillich. New spectral lower bounds on the bisection width of graphs. *Theoretical Computer Science*, 320(2-3):155–174, 2004.
- [2] E. Birmelé, A. Bondy, and B. Reed. Brambles, prisms and grids. In A. Bondy, J. Fonlupt, J.-L. Fouquet, J.-C. Fournier, and J. Ramírez Alfonsín, editors, *Graph Theory in Paris*, Trends in Mathematics, pages 37–44. Birkhäuser Basel, 2007.
- [3] H. L. Bodlaender. On linear time minor tests with depth-first search. *Journal of Algorithms*, 14(1):1–23, 1993.
- [4] H. L. Bodlaender, F. V. Fomin, D. Lokshtanov, E. Penninkx, S. Saurabh, and D. M. Thilikos. (Meta) kernelization. In *Proceedings of the 2009 50th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 629–638, Washington, DC, USA, 2009. IEEE Computer Society.
- [5] H. L. Bodlaender, F. V. Fomin, D. Lokshtanov, E. Penninkx, S. Saurabh, and D. M. Thilikos. (Meta) kernelization. *CoRR*, abs/0904.0727, 2009.

- [6] H. L. Bodlaender, J. van Leeuwen, R. Tan, and D. M. Thilikos. On interval routing schemes and treewidth. *Information and Computation*, 139(1):92–109, 1997.
- [7] L. S. Chandran and C. Subramanian. Girth and treewidth. *Journal of Combinatorial Theory, Series B*, 93(1):23 – 32, 2005.
- [8] C. Chekuri and J. Chuzhoy. Large-treewidth graph decompositions and applications. In *Proceedings of the Forty-fifth Annual ACM Symposium on Theory of Computing (STOC)*, pages 291–300, New York, NY, USA, 2013. ACM.
- [9] C. Chekuri and J. Chuzhoy. Polynomial bounds for the grid-minor theorem. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing (STOC)*, pages 60–69, New York, NY, USA, 2014. ACM.
- [10] J. Chuzhoy. Excluded grid theorem: Improved and simplified. In *Proceedings of the 47th Annual ACM Symposium on Theory of Computing (STOC)*, pages 645–654, New York, NY, USA, 2015. ACM.
- [11] R. Diestel, T. R. Jensen, K. Y. Gorbunov, and C. Thomassen. Highly connected sets and the excluded grid theorem. *Journal of Combinatorial Theory, Series B*, 75(1):61–73, 1999.
- [12] G. Ding and B. Oporowski. On tree-partitions of graphs. *Discrete Mathematics*, 149(1–3):45 – 58, 1996.
- [13] V. Dujmovic, D. J. Harvey, G. Joret, B. A. Reed, and D. R. Wood. A linear-time algorithm for finding a complete graph minor in a dense graph. *SIAM Journal of Discrete Mathematics*, 27(4):1770–1774, 2013.
- [14] F. V. Fomin, D. Lokshtanov, N. Misra, G. Philip, and S. Saurabh. Quadratic upper bounds on the Erdős-Pósa property for a generalization of packing and covering cycles. *Journal of Graph Theory*, 74(4):417–424, 2013.
- [15] N. Fountoulakis, D. Kühn, and D. Osthus. Minors in random regular graphs. *Random Structures & Algorithms*, 35(4):444–463, 2009.
- [16] J. F. Geelen, A. M. H. Gerards, N. Robertson, and G. P. Whittle. On the excluded minors for the matroids of branch-width k . *Journal of Combinatorial Theory, Series B*, 88:261–265, July 2003.
- [17] R. Halin. Tree-partitions of infinite graphs. *Discrete Mathematics*, 97(1–3):203 – 217, 1991.
- [18] I. V. Hicks and N. B. M. Jr. The branchwidth of graphs and their cycle matroids. *Journal of Combinatorial Theory, Series B*, 97(5):681–692, 2007.

- [19] G. Joret and D. R. Wood. Complete graph minors and the graph minor structure theorem. *Journal of Combinatorial Theory, Series B*, 103(1):61–74, 2013.
- [20] A. V. Kostochka. Lower bound of the hadwiger number of graphs by their average degree. *Combinatorica*, 4(4):307–316, 1984.
- [21] D. Kühn and D. Osthus. Minors in graphs of large girth. *Random Structures & Algorithms*, 22(2):213–225, 2003.
- [22] D. Kühn and D. Osthus. Complete minors in $K_{s,s}$ -free graphs. *Combinatorica*, 25(1):49–64, 2004.
- [23] K. Markström. Complete minors in cubic graphs with few short cycles and random cubic graphs. *Ars Combinatorica*, 70, 2004.
- [24] M. Morgenstern. Existence and explicit constructions of $q + 1$ regular ramanujan graphs for every prime power q . *Journal of Combinatorial Theory, Series B*, 62(1):44 – 62, 1994.
- [25] J. G. Oxley. *Matroid Theory*. Oxford University Press, New York, 1992.
- [26] L. S. Pitsoulis. *Topics in matroid theory*. Springer, 2014.
- [27] J.-F. Raymond and D. M. Thilikos. Low polynomial exclusion of planar graph patterns. *CoRR*, abs/1305.7112, 2013.
- [28] N. Robertson and P. D. Seymour. Graph Minors. V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(2):92–114, 1986.
- [29] N. Robertson and P. D. Seymour. Graph Minors. X. Obstructions to Tree-decomposition. *Journal of Combinatorial Theory, Series B*, 52(2):153–190, 1991.
- [30] N. Robertson, P. D. Seymour, and R. Thomas. Quickly excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 62(2):323–348, 1994.
- [31] D. Seese. Tree-partite graphs and the complexity of algorithms. In *Proceedings of Fundamentals of Computation Theory*, volume 199 of *Lecture Notes in Computer Science*, pages 412–421. Springer Berlin Heidelberg, 1985.
- [32] M. Stiebitz. Decomposing graphs under degree constraints. *Journal of Graph Theory*, 23(3):321–324, 1996.
- [33] A. Thomason. An extremal function for contractions of graphs. *Mathematical Proceedings of the Cambridge Philosophical Society*, 95:261, 1983.
- [34] A. Thomason. The extremal function for complete minors. *Journal of Combinatorial Theory, Series B*, 81(2):318 – 338, 2001.

- [35] K. Yamazaki, H. L. Bodlaender, B. de Fluiter, and D. M. Thilikos. Isomorphism for graphs of bounded distance width. *Algorithmica*, 24(2):105–127, 1999.