ICGT 2014, Grenoble, France, June 30–July 4, 2014

Covering and packing pumpkin models*

Dimitris Chatzidimitriou† Jean-Florent Raymond‡ Ignasi Sau§
Dimitrios M. Thilikos†§

Abstract

Let θ_r (the *r-pumpkin*) be the multi-graph containing two vertices and r parallel edges between them. We say that a graph is a *a θ_r-model* if it can be transformed into θ_r after a (possibly empty) sequence of contractions. We prove that there is a function $g : \mathbb{N} \to \mathbb{N}$ such that, for every two positive integers k and q, if G is a K_q-minor-free graph, then either G contains a set of k vertex-disjoint subgraphs (a *θ_r-model-vertex-packing*) each isomorphic to a θ_r-model or a set of $g(r) \cdot \log q \cdot k$ vertices (a *θ_r-model-vertex-cover*) meeting all subgraphs of G that are isomorphic to a θ_r-model. Our results imply a $O(\log OPT)$-approximation for the maximum (minimum) size of a θ_r-model packing (θ_r-model covering) of a graph G.

1 Introduction

The Erdős–Pósa theorem, proved in 1965 [6], revealed the following min-max relation between coverings and packings of cycles in graphs: every graph that does not contain k disjoint cycles, contains a set of $O(k \log k)$ vertices meeting all its cycles. They also proved that this result is tight by giving graph contractions where the $O(k \log k)$ bound is realized. Various extensions of this result, referring to different notions of packing and covering, attracted the attention of many researchers in modern Graph Theory (see [1, 10]).

A *model* of a graph H is any graph that can be contracted to H. Given two graphs H and G, we denote by $\mathbf{pack}_H(G)$ the maximum number of vertex-disjoint models of H in G. We also denote by $\mathbf{cover}_H(G)$ the minimum number of vertices that intersect all minor models of H in G. We are interested in graphs H for which the following relation holds:

$$\text{for every } G, \text{ if } \mathbf{pack}_H(G) \leq k \text{ then } \mathbf{cover}_H(G) = O(k \cdot \log k) \quad (1)$$

Clearly if θ_2 is the graph with two vertices and two edges between them, then Relation (1) holds because of the Erdős–Pósa theorem. In the most general case, Robertson and Seymour proved [13] that if H is planar, then for every graph G, $\mathbf{cover}_H(G)$ is bounded by some function of $\mathbf{pack}_H(G)$. Moreover,

*The third author was co-financed by the E.U. (European Social Fund - ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: "Thales. Investing in knowledge society through the European Social Fund". Emails: hatzisdimitris@gmail.com, jean-florent.raymond@mimuw.edu.pl, ignasi.sau@lirmm.fr, sedthilk@thilikos.info

†Dept. of Mathematics, National & Kapodistrian University of Athens, Athens, Greece.
‡Faculty of Math., Informatics and Mechanics, University of Warsaw, Warsaw, Poland.
§AlGCo project team, CNRS, LIRMM, France.

it also follows that this bound does not hold any more if H is non-planar (see also Diestel's monograph [5, Corollary 12.4.10 and Exercise 40 of Chapter 12]). In [8] Fiorini, Joret, and Wood argued that the $O(k \cdot \log k)$ bound is the best we may expect when H is a planar graph containing a cycle and they proved that if H is acyclic, then $O(k \cdot \log k)$ can be reduced to $O(k)$. This implies that $O(k \cdot \log k)$ bound in Relation (1) is the best we may expect for non-acyclic planar graphs and the question remains whether the same bound can be achieved for every planar graph H. The most general result in this direction concerns the (parameterized) case where H is the multi-graph containing two vertices and r parallel edges between them. This graph is also known as *the r-pumpkin* and is denoted by θ_r. In [7], Fiorini, Joret, and Sau proved that Relation (1) holds for every $H = \theta_r, r \geq 2$.

Another approach towards improving the bound in Relation (1) was to restrict the class of graphs where it applies. In this direction, it was proven by Fomin, Saurabh, and Thilikos in [9] that the $\log k$ factor can be dropped in Relation (1), for all planar H's, in the case when we restrict G to to a graph class that excludes some fixed graph as a minor.

In this paper we provide a common extension of both the results of [7] and [9] that indicates how the effect of excluding a graph as a minor is reflected in the transition from $O(k \cdot \log k)$ to $O(k)$. Our result is the following:

Theorem 1. *There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every two positive integers r, q, and every graph G excluding K_q as a minor, it holds that $\mathbf{cover}_{\theta_r}(G) \leq f(r) \cdot \mathbf{pack}_{\theta_r}(G) \cdot \log q$.*

Notice that if $\mathbf{pack}_{\theta_r}(G) \leq k$ then G does not contain a clique of $(k+1)(r+1)$ vertices as a minor. Therefore, if we set $q = (k + 1)(r + 1)$, we conclude that Relation (1) holds when $H = \theta_r$. This implies the main result of [7].

A consequence of the proof of Theorem 1 is the existence of a polynomial $O(\log OPT)$-approximation algorithm for both $\mathbf{cover}_{\theta_r}$ and $\mathbf{pack}_{\theta_r}(G)$. This improves the $O(\log n)$-approximation algorithm that was given by Joret, Paul, Sau, Saurabh, and Thomassé in [11] for the same parameters.

2 The proof

The proof of Theorem 1 is based on three results that we describe in this section.

Given a graph G, a *separation* of G is a pair (X_1, X_2) such that no edge of G has an endpoint in $X_1 \setminus X_2$ and the other endpoint in $X_2 \setminus X_1$. The *order* of (X_1, X_2) is the cardinality of the set $X_1 \cap X_2$.

We start with an observation in the case where there is no separation (X_1, X_2) of G for which both $G[X_1 \setminus X_2]$ and $G[X_2 \setminus X_1]$ contain some model of H as a subgraph.

Observation 1. *Let H and G be graphs and let (X_1, X_2) be a separation of G such that both $G_1 = G[X_1 \setminus X_2]$ and $G_2 = G[X_2 \setminus X_1]$ contain H as a minor. Then*
- $\mathbf{cover}_H(G) \leq \mathbf{cover}_H(G_1) + \mathbf{cover}_H(G_2) + |X_1 \cap X_2|$
- $\mathbf{pack}_H(G_1) + \mathbf{pack}_H(G_2) \leq \mathbf{pack}_H(G)$
- $\mathbf{cover}_H(G_i) < \mathbf{cover}_H(G)$, $i = 1, 2$.
- $\mathbf{pack}_H(G_i) < \mathbf{pack}_H(G)$, $i = 1, 2$.

The following lemma reduces the problem to the case where for each separation (X_1, X_2) of G of order at most $2r - 2$, if $G[X_1]$ does not contain a model of H, then $|X_1|$ is bounded by some function depending only on H.

Lemma 1. *For every $q \in \mathbb{N}$ and graph H there exists a function $f_H : \mathbb{N} \to \mathbb{N}$ such that for every K_q-minor free graph G there is a K_q-minor free graph G' such that*
• $\mathbf{pack}_H(G) = \mathbf{pack}_H(G')$
• $\mathbf{cover}_H(G) = \mathbf{cover}_H(G')$
• *for every separation (X_1, X_2) of G', if $G'[X_1 \setminus X_2]$ is a H-minor-free graph in G', then $|X_1| \leq f_H(|X_1 \cap X_2|)$.*

The proof of Lemma 1 is too long to fit in this extended abstract. It uses protrusion replacement techniques that have been developed in [2] (see also [3] and [4]) that permit the replacement of the part of graph induced by X_1 by another graph so that none of the parameters $\mathbf{pack}_H$ and $\mathbf{cover}_H$ change in the new graph.

We say that a graph G is (α, β)-*loosely connected* if for every separation (X_1, X_2) of G, $|X_1 \cap X_2| \leq \alpha \Rightarrow \min\{|X_1|, |X_2|\} \leq \beta$.

Observation 1 and Lemma 1, applied for separators of order at most $2r - 2$ reduce the proof of Theorem 1 to the case where G is $(2r - 2, f_{\theta_r}(2r - 2))$-loosely connected. The second lemma departs from this assumption and is able to detect in such a graph a model of θ_r that contains $O(\log q)$ vertices.

Lemma 2. *There is a function $g : \mathbb{N} \to \mathbb{N}$ such that every $(2r - 2, f_{\theta_r}(2r - 2))$-loosely connected K_q-minor free graph G contains a model of θ_r with $\leq g(r) \cdot \log q$ vertices.*

The proof of Lemma 2 is the core of our results and is technical. We enumerate below its main steps.

1. Take into account that every K_q-minor free graph has some vertex of degree $< c \cdot q \cdot \sqrt{\log q}$ (see e.g., [12,14]). Let $d = f_{\theta_r}(2r - 2) \cdot \log((r - 1) \cdot c \cdot q \cdot \sqrt{\log q})$ and our target is to find a model of θ_r in G with at most $4 \cdot r \cdot d + 2$ vertices.
2. Let $S = \{s_1, \ldots, s_l\}$ be a maximal $2d$-scattered set (all its elements are in distance $\geq 2d$ from each other) and consider the s_i-rooted BFS trees $T_1, \ldots, T_l$ each of depth $\leq 2d$ and such that
- $\forall_{i,j} \forall_{u \in T_i} \ \ \mathbf{dist}_G(u, s_i) \leq \mathbf{dist}_G(u, s_j)$
- $V(T_1), \ldots, V(T_l)$ is a partition of $V(G)$

3. Let $G_i^{\text{in}} = G[\mathbf{B}_G^d(s_i)]$ and $G_i = G[V(T_i)]$. ($\mathbf{B}_G^i(x)$ are the vertices that are in distance at most i from x.)
4. Let $\mathcal{D}_i = (\{X_t\}_{t \in V(U_i)}, U_i)$ be an s_i-rooted tree-distance decomposition of G_i (for the definition of a rooted tree-distance decomposition, see [15]).
5. Observe that $\forall_{i,j} \ \ V(G_i) \cap V(G_j) = \emptyset$ and that all edges between G_i and G_j have endpoints from their d-th layer and then.
6. Prove that, unless we are done, all bags of $\mathcal{D}_i$ have $\leq r - 1$ vertices.
7. Prove that, unless we are done, in each $G_{i,j} = G[V(G_i) \cup V(G_j)]$ there are at most $r - 1$ paths from the d-th level of G_i to the d-th level of G_j.
8. Deduce from the previous step and the $(2r-2, f_{\theta_r}(2r-2))$-loose connectivity of G that there are at least $2^{d/f_{\theta_r}(2r-2)}$ pairwise vertex-disjoint paths between vertices in the d-th level of G_i and the union of the vertices of the d-th levels of all other G_j's.

9. Deduce from the previous two steps that there is a collection of pairwise vertex-disjoint paths $\mathcal{P}$ in G, such that
• if for some $i \neq j$ there is a path $P_{i,j} \in \mathcal{P}$ joining a vertex from G_i^{in} with a vertex of G_j^{in} then this is unique path in $\mathcal{P}$ with this property and
• G_i contains the endpoints of at least $\frac{2^{d/f_{\theta_r}(2r-2)}}{r-1}$ paths from $\mathcal{P}$.
10. Contract all edges of every G_i, contract all but one edge of the paths in $\mathcal{P}$ and remove all other edges and isolated vertices. Let H be the resulting minor of G. Prove that, from Step 8, $\delta(H) \geq \frac{2^{d/f_{\theta_r}(2r-2)}}{r-1} = c \cdot q \cdot \sqrt{\log q}$, a contradiction to what we took into account in the 1st step.

We now require the following observation.

Observation 2. *If* $k = \mathbf{pack}_{\theta_r}(G)$ *and* W *is the set of vertices of a model as in Lemma 2, then*
• $\mathbf{pack}_{\theta_r}(G \setminus W) \leq \mathbf{pack}_{\theta_r}(G) - 1 \leq k - 1$
• $\mathbf{cover}_{\theta_r}(G) \leq \mathbf{cover}_{\theta_r}(G \setminus W) + |V(W)| \leq \mathbf{cover}_{\theta_r}(G \setminus W) + g(r) \cdot \log q$.

It is now easy to verify that, taking into account Observation 2, Theorem 1 follows by successively applying Observation 1, and Lemmata 1 and 2.

References

[1] Etienne Birmelé, J. Adrian Bondy, and Bruce A. Reed. The Erdős–Pósa property for long circuits. *Combinatorica*, 27:135–145, 2007.

[2] Hans Bodlaender, Fedor Fomin, Daniel Lokshtanov, Eelko Penninkx, Saket Saurabh, and Dimitrios Thilikos. (Meta) kernelization. In *50th Annual IEEE Symposium on Foundations of Computer Science, (FOCS 2009)*. 2009.

[3] Hans L. Bodlaender and Babette van Antwerpen-de Fluiter. Reduction algorithms for graphs of small treewidth. *Inf. Comput.*, 167:86–119, June 2001.

[4] Richard B. Borie, R. Gary Parker, and Craig A. Tovey. Automatic generation of linear-time algorithms from predicate calculus descriptions of problems on recursively constructed graph families. *Algorithmica*, 7:555–581, 1992.

[5] Reinhard Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Heidelberg, third edition, 2005.

[6] Paul Erdős and Louis Pósa. On independent circuits contained in a graph. *Canadian Journal of Mathematics*, 17:347–352, 1965.

[7] Samuel Fiorini, Gwenaël Joret, and Ignasi Sau. Optimal Erdős–Pósa property for pumpkins. Manuscript, 2013.

[8] Samuel Fiorini, Gwenaël Joret, and David R. Wood. Excluded forest minors and the Erdős-Pósa property. *Combinatorics, Probability & Computing*, 22(5):700–721, 2013.

[9] Fedor V. Fomin, Saket Saurabh, and Dimitrios M. Thilikos. Strengthening Erdős–Pósa property for minor-closed graph classes. *Journal of Graph Theory*, 66(3):235–240, 2011.

[10] Jim Geelen and Kasper Kabell. The Erdős–Pósa property for matroid circuits. *J. Comb. Theory Ser. B*, 99(2):407–419, March 2009.

[11] Gwenaël Joret, Christophe Paul, Ignasi Sau, Saket Saurabh, and Stéphan Thomassé. *Hitting and harvesting pumpkins.* ESA'11. Springer-Verlag, Berlin, Heidelberg, 2011.

[12] A.V. Kostochka. Lower bound of the hadwiger number of graphs by their average degree. *Combinatorica*, 4:307–316, 1984.

[13] Neil Robertson and Paul D. Seymour. Graph Minors. V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(2):92–114, 1986.

[14] Andrew Thomason. The extremal function for complete minors. *J. Combin. Theory Ser. B*, 81(2):318–338, 2001.

[15] K. Yamazaki, H. L. Bodlaender, B. de Fluiter, and D. M. Thilikos. Isomorphism for graphs of bounded distance width. *Algorithmica*, 24(2):105–127, 1999.