New Upper Bounds on the Decomposability of Planar Graphs*

Fedor V. Fomin[1] and Dimitrios M. Thilikos[2]

[1]*DEPARTMENT OF INFORMATICS*
UNIVERSITY OF BERGEN
N-5020 BERGEN, NORWAY
E-mail: fomin@ii.uib.no

[2]*NATIONAL AND CAPODISTRIAN UNIVERSITY OF ATHENS*
PANEPISTIMIOUPOLIS
GR-15784, ATHENS, GREECE
E-mail: sedthilk@math.uoa.gr

Received May 13, 2003; Revised July 7, 2005

Published online 28 September 2005 in Wiley InterScience(www.interscience.wiley.com).
DOI 10.1002/jgt.20121

Abstract: It is known that a planar graph on n vertices has branch-width/tree-width bounded by $\alpha\sqrt{n}$. In many algorithmic applications, it is useful to have a small bound on the constant α. We give a proof of the best, so far, upper bound for the constant α. In particular, for the case of tree-width, $\alpha < 3.182$ and for the case of branch-width, $\alpha < 2.122$. Our proof is based on the planar separation theorem of Alon, Seymour, and Thomas and some min–max theorems of Robertson and Seymour from the graph minors series. We also discuss some algorithmic consequences of this result.

*The first author is supported by Norges forskningsråd projects 162731/V00 and 160778/V30. The second author is supported by the Spanish CICYT project TIN-2004-07925 (GRAMMARS).

Keywords: *Tree-width; branch-width; planar graphs; separation theorems; parameterized algorithms; subexponential algorithms*

1. INTRODUCTION

In this paper, we give an improved upper bound to the branch-width and the tree-width of planar graphs. Both these parameters were introduced (and served) as basic tools by Robertson and Seymour in their Graph Minors series of papers. Tree-width and branch-width are related parameters (See Theorem 2.1) and can be considered as measures of the ''global connectivity'' of a graph. Moreover, they appear to be of a major importance in algorithmic design as many NP-hard problems admit polynomial or even linear time solutions when their inputs are restricted to graphs of bounded tree-width or branch-width [7].

The objective of this paper is to show that every n-vertex planar graph G has branch-width $\leq 2.122\sqrt{n}$ and tree-width $\leq 3.182\sqrt{n}$. To obtain the new upper bounds we use deep 'dual' and 'min–max' theorems from Graph Minors series papers of Robertson, Seymour. We also observe interesting algorithmic consequences following from the new upper bounds.

A. Previous Results and our Contribution

Computation of constants α_t and α_b such that for every planar graph on n vertices $\mathbf{tw}(G) \leq \alpha_t\sqrt{n} + O(1)$ and $\mathbf{bw}(G) \leq \alpha_b\sqrt{n} + O(1)$ is of a great theoretical importance. In [5], Alon, Seymour, and Thomas proved that any K_r-minor free graph on n vertices has tree-width$\leq r^{1.5}\sqrt{n}$. (Here K_r is complete graph on r vertices.) Since no planar graph contains K_5 as a minor, we have that $\alpha_b(G) \leq \alpha_t(G) \leq 6^{1.5} \leq 14.697$. By using deep results of Robertson, Seymour, and Thomas, one can easily prove much better bounds as follows.

Before we proceed, let us remind the notion of a minor. Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. H is a *minor* of G if H is the subgraph of a some contraction of G.

The following is a combination of statements (4.3) in [18] and (6.3) in [21].

Theorem 1.1 ([21]). *Let $k \geq 1$ be an integer. Every planar graph with no $(k \times k)$-grid as a minor has branch-width $\leq 4k - 3$.*

Because a graph on n vertices does not contain a $((\lceil\sqrt{n}\rceil + 1) \times (\lceil\sqrt{n}\rceil + 1))$-grid as a minor, we have that $\alpha_b(G) \leq 4$. Robertson, Seymour, and Thomas showed (unpublished result announced by Thomas [23]) that any planar graph without a $(k \times k)$-grid as a minor has tree-width $\leq 5k - 1$ implying $\alpha_t \leq 5$.

In this paper, we reduce the bound for constant α_b to 2.122 (for the case of branch-width) and for constant α_t to 3.182 (for the case of tree-width).

The paper is organized as follows. In Section 2, we present the basic definitions and well-known facts about decompositions of planar graphs. In Section 3, we give the proof of the main combinatorial result of this paper. The proof is long and we split it into several subsections. Our proof makes strong use of deep graph theoretic results from [6] and [19, 22]. In particular, Alon, Seymour, and Thomas introduced the concept of "majority" in order to study the existence of small separators in planar graphs. On the other side, the results in [19, 22] were strongly based on the notion of "slope." The main idea of our proof is to show that slopes can be transformed to majorities for triangulated planar graphs without multiple edges (in this paper only consider plane triangulations). Then combining this results with the results from [6] and [22], we obtain the claimed upper bound. In Section 4, we observe why our theoretical upper bounds are interesting from the algorithmic point of view. We prove that the running time of many known algorithms on planar graphs (parameterized or exact) can be improved significantly. Finally, in Section 5, we conclude with three open problems related to our results.

2. DEFINITIONS

All graphs in this paper are undirected, loop-less and, unless otherwise mentioned, they may have multiple edges.

A. Tree-Width and Branch-Width

A *tree decomposition* of a graph G is a pair $(\{X_i \mid i \in V(T)\}, T)$, where $\{X_i \mid i \in V(T)\}$ is a collection of subsets of $V(G)$ and T is a tree, such that

- $\bigcup_{i \in V(T)} X_i = V(G)$,
- for each edge $\{v, w\} \in E(G)$, there is an $i \in V(T)$ such that $v, w \in X_i$, and
- for each $v \in V(G)$, the set of nodes $\{i \mid v \in X_i\}$ forms a subtree of T.

The *width* of a tree decomposition $(\{X_i \mid i \in V(T)\}, T)$ equals $\max_{i \in V(T)} (|X_i| - 1)$. The *tree-width* of a graph G, $\mathbf{tw}(G)$, is the minimum width over all tree decompositions of G.

A *branch decomposition* of a graph (or a hyper-graph) G is a pair (T, τ), where T is a tree with vertices of degree 1 or 3, and τ is a bijection from the set of leaves of T to $E(G)$. The *order* of an edge e in T is the number of vertices $v \in V(G)$ such that there are leaves t_1, t_2 in T in different components of $T(V(T), E(T) - e)$ with $\tau(t_1)$ and $\tau(t_2)$ both containing v as an endpoint.

The *width* of (T, τ) is the maximum order over all edges of T, and the *branch-width* of G, $\mathbf{bw}(G)$, is the minimum width over all branch decompositions of G. (In case where $|E(G)| \leq 1$, we define the branch-width to be 0; if $|E(G)| = 0$, then G has no branch decomposition; if $|E(G)| = 1$, then G has a branch decomposition consisting of a tree with one vertex–the width of this branch decomposition is considered to be 0.)

It is easy to see that if H is a subgraph of G, then $\mathbf{bw}(H) \leq \mathbf{bw}(G)$. The following result is due to Robertson and Seymour [(5.1) in [18]].

Theorem 2.1 ([18]). *For any connected graph G where $|E(G)| \geq 3$, $\mathbf{bw}(G) \leq \mathbf{tw}(G) + 1 \leq \frac{3}{2}\mathbf{bw}(G)$.*

From Theorem 2.1, any upper bound on tree-width implies an upper bound on branch-width and vice versa.

B. Planar Graphs: Slopes and Majorities

In this paper, we use the expression Σ-*plane graph* for any planar graph drawn in the sphere Σ. To simplify notations, we do not distinguish between a vertex of a Σ-plane graph and the point of Σ used in the drawing to represent the vertex or between an edge and the *open* line segment representing it. We also consider G as the union of the points corresponding to its vertices and edges. That way, a subgraph H of G can be seen as a graph H where $H \subseteq G$. We call by *face* of G any connected component of $\Sigma - E(G) - V(G)$. (Every face is an open set.) We use the notation $V(G), E(G)$, and $R(G)$ for the set of the vertices, edges, and faces of G respectively. A *path* of G is any connected subgraph P of G with two vertices of degree 1 (we call them *extremes*) and all other vertices (we call them *internal*) of degree 2. A sub-path of a path P is any path $P' \subseteq P$. A *cycle* of G is any connected subgraph C of G with all the vertices of degree 2. The length $|C|$ ($|P|$) of a cycle C (path $|P|$) is the number of its edges.

If $\Delta \subseteq \Sigma$, then $\overline{\Delta}$ denotes the *closure* of Δ, and the boundary of Δ is $\mathbf{bd}(\Delta) = \overline{\Delta} \cap \overline{\Sigma - \Delta}$. An edge e (a vertex v) is incident with a face r if $e \subseteq \mathbf{bd}(r)$ $(v \subseteq \mathbf{bd}(r))$.

We call a Σ-plane graph G *triangulated* if all of its faces are triangles, i.e., for every face r, $\mathbf{bd}(r)$ is a cycle of three edges and three vertices. Given a face r of a triangulated graph G, we call the cycle $\mathbf{bd}(r)$ *triangle* of G. A *triangulation* H of a Σ-plane graph G is any triangulated Σ-plane graph H where $G \subseteq H$. Notice that any Σ-plane graph with all faces of size ≥ 3 has a triangulation. A triangle of a triangulated Σ-plane graph G is a *facial triangle* if it bounds a face of G.

Let G be a Σ-plane graph. A subset of Σ meeting the drawing only in vertices of G is called G-*normal*. A subset of Σ homeomorphic to the closed interval $[0, 1]$ is called I-*arc*. If the ends of a G-normal I-arc L are both vertices of G, then we call it *line* of G. If a simple closed curve $F \subseteq \Sigma$ is G-normal, then we call it *noose*.

The length of a line is the number of its vertices minus 1 and the length of a noose is the number of its vertices. We denote by $|N|(|L|)$ the length of a noose N (line L). $\Delta \subseteq \Sigma$ is an open disc if it is homeomorphic to $\{(x, y) : x^2 + y^2 < 1\}$. We say that a disc D is *bounded* by a noose N if $N = \mathbf{bd}(D)$. From the theorem of

Jordan, any noose N bounds exactly two closed discs Δ_1, Δ_2 in Σ where $\Delta_1 \cap \Delta_2 = N$.

Let $x, y \in \Sigma$ be distinct. We call Θ-*structure* $S = (L_1, L_2, L_3)$ of G the union of three lines L_1, L_2, L_3 between x and y that are otherwise disjoint. If for $i, j, 1 \leq i < j \leq 3$ the noose $L_i \cup L_j$ has size $\leq k$, then we say that S is a Θ-structure of length $\leq k$. We call a Θ-structure *non-trivial* if at least two of its lines have length ≥ 2. We call the 6 closed discs bounded by the nooses $L_i \cup L_j$, $1 \leq i < j \leq 3$ *closed discs bounded* by S.

The *radial graph of* a Σ-plane graph G is the bipartite Σ-plane graph R_G obtained by selecting a point in every face r of G and connecting it to every vertex of G incident to that face. We call the vertices of R_G that are not vertices of G *radial* vertices. For an example of a graph G drawn along with its radial, see Figure 1.

Slopes and majorities are important tools for the proofs of this paper.

Slopes (Robertson and Seymour [19]). *Let G be a Σ-plane graph and let $k \geq 1$ be an integer. A slope in G of order $k/2$ is a function* **ins** *which assigns to every cycle C of G of length $< k$ one of the two closed discs* $\mathbf{ins}(C) \subseteq \Sigma$ *bounded by C such that*

[S1] *If C, C' are cycles of length $< k$ and $C \subseteq \mathbf{ins}(C')$, then $\mathbf{ins}(C) \subseteq \mathbf{ins}(C')$.*
[S2] *If P_1, P_2, P_3 are three paths of G joining the same pair u, v of distinct vertices but otherwise disjoint, and the three cycles $P_1 \cup P_2$, $P_1 \cup P_3$, $P_2 \cup P_3$ all have length $< k$, then*

$$\mathbf{ins}(P_1 \cup P_2) \cup \mathbf{ins}(P_1 \cup P_3) \cup \mathbf{ins}(P_2 \cup P_3) \neq \Sigma.$$

A slope is uniform if for every face $r \in R(G)$ there is a cycle C of G of length $< k$ such that $r \subseteq \mathbf{ins}(C)$.

We need the following deep result proved in the Graph Minors papers by Robertson and Seymour. This result follows from Theorems (6.1) and (6.5) in [19] and Theorem (4.3) in [18]. (See also Theorems (6.2) and (7.1) in [22].)

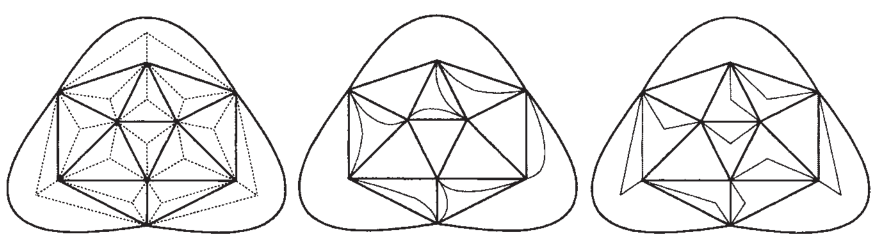

FIGURE 1. An example of a Σ-plane graph G drawn (i) with its radial R_G (ii) with a noose S that is not a cycle of R_G and (iii) with a noose S' that is a cycle of R_G and a vibration of S.

Theorem 2.2 ([19]). *Let G be a connected Σ-plane graph where $|E(G)| \geq 2$ and let $k \geq 1$ be an integer. The radial drawing R_G has a uniform slope of order $\geq k$ if and only if G has branch-width $\geq k$.*

Majorities (Alon, Seymour, and Thomas [6]). Let G be a Σ-plane graph and let $k \geq 0$ be an integer. A *majority of order k* is a function **big** that assigns to every noose N in G of length $\leq k$ a closed disc $\mathbf{big}(N) \subseteq \Sigma$ bounded by N such that

[M1] If P_1, P_2, P_3 is a Θ-structure of G with length $\leq k$ and $P_3 \subseteq \mathbf{big}(P_1 \cup P_2)$, then $\mathbf{big}(P_1 \cup P_3) \subseteq \mathbf{big}(P_1 \cup P_2)$ or $\mathbf{big}(P_2 \cup P_3) \subseteq \mathbf{big}(P_1 \cup P_2)$.

[M2] If N is a noose of length $\leq \min(2, k)$, then either $\mathbf{big}(N) - N$ contains a vertex or $\mathbf{big}(N)$ includes at least two edges of G.

The following result gives an upper bound on the order of a majority (Statement (3.7) of [6]). This is a basic ingredient of our bound for the branch-width of planar graphs.

Theorem 2.3 ([6]). *Any majority of a Σ-plane graph G has order at most $\sqrt{4.5 \cdot |V(G)|} - 1$.*

3. CREATING MAJORITIES FROM SLOPES

Our bounds on branch-width and tree-width follow from the following theorem that is the main technical result of the paper.

Theorem 3.1. *Let G, $|V(G)| \geq 5$, be a triangulated Σ-plane graph without multiple edges, drawn in Σ along with its radial graph, and let $k \geq 2$ be an integer. If there exists a uniform slope of order $k+1$ in R_G, then G contains a majority of order k.*

This section is devoted to the proof of Theorem 3.1 and is organized as follows. We start with the definitions of the notions of variations and vibrations (Subsection 3A). Then we prove that any noose can be transformed, after applying to it a sequence of variations, to a cycle of the radial graph (Subsection 3C). We also prove that the same type of representation via variations applies also to the Θ-structures (Subsection 3D). That way, we are able to "translate" the slope axioms to majority ones. This requires a series of auxiliary results assuring that the basic topological properties involved in the majority axioms are invariants under vibrations (Subsection 3F). With all this knowledge on hands, we proceed with the proof of the main result in Subsection 3G.

A. Variations and Vibrations

If G is a Σ-plane graph without loops or multiple edges, and $S \subseteq \Sigma$ is an I-arc (simple closed curve) in Σ, then we use the notation $\kappa_G(S) = (v_1, \ldots, v_{|S \cap V(G)|})$

for the ordering (cyclic ordering) of the vertex set $F \cap V(G)$ that represents the way the vertices of G are met by S.

Notice that κ can be applied to both cycles and nooses but also to paths and lines. Especially for cycles and paths of graphs without multiple edges, we can directly represent them with the output of the function κ (we will use the same notation for a cycle/path and the (cyclic) ordering of the vertices that it meets).

The basic idea of the proof is to correspond nooses of G to cycles of R_G and try to translate the slope axioms to majority axioms. Corresponding nooses to cycles are not direct as not every noose is a cycle of the radial graph (see Fig. 1). To overcome this problem, we need to introduce the concepts of variations and vibrations of nooses.

Let S be one of the following structures in G: a noose, a line, or a Θ-structure. A *variation* of S is the operation that transforms S to another structure S' of the same type such that $\overline{(S \cup S') - (S \cap S')}$ is a noose of size 2 and one of the closed discs bounded by this noose, we denote this disc by $\mathbf{dif}(S, S')$, has the following two properties:

1. $\mathbf{dif}(S, S') - \mathbf{bd}(\mathbf{dif}(S, S'))$ contains no vertices of G,
2. $\mathbf{dif}(S, S')$ contains at most one edge of G.

If two structures S_1 and S_2 are variations each of the other, we denote it as $S_1 \sim S_2$. If a structure S' is the result of a finite number of consecutive variations with S as starting point, we call S' *vibration* of S and we denote this fact as $S \sim^* S'$. Notice that if $S \sim^* S'$, then $V(G) \cap S = V(G) \cap S'$ and S and S' have the same length. In fact, it is easy to observe that if N, N' are nooses or lines where $N \sim^* N'$, then $\kappa_G(N) = \kappa_G(N')$. Moreover, if $S = (L_1, L_2, L_3)$ and $S' = (L'_1, L'_2, L'_3)$ are Θ-structures with $S \sim^* S'$, then we order the elements of S and S' such that for every i, $1 \leq i < j \leq 3$, $L_i \cup L_j \sim^* L'_i \cup L'_j$. For examples of the notions of variation and vibration, see Figure 2.

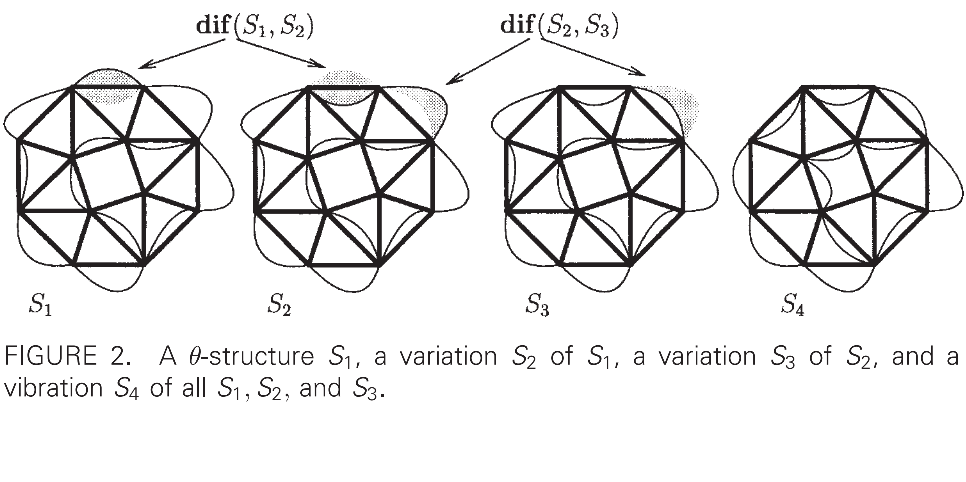

FIGURE 2. A θ-structure S_1, a variation S_2 of S_1, a variation S_3 of S_2, and a vibration S_4 of all S_1, S_2, and S_3.

B. Corresponding Nooses and Lines to Cycles and Paths

The main result of this subsection is Lemma 3.3, claiming that if two nooses of a plane triangulation meet vertices in the same cyclic ordering, then the one should be a vibration of the other.

Lemma 3.2. *Let G be a triangulated Σ-plane graph without multiple edges. If S is a line or a noose of length 2, then exists a unique edge Q in G such that $\kappa_G(S) = \kappa_G(Q)$. If S is a noose of length ≥ 3, then there exists a unique cycle Q in G such that $\kappa_G(S) = \kappa_G(Q)$.*

Proof. Let $\kappa_G(S) = (v_0, \ldots, v_{r-1})$. We prove that for any $i = 0, \ldots, r-2$, the vertices $v_i, v_{i+1} \in \kappa_G(S)$ are adjacent via only one edge (in case S is a noose we take $i = 0, \ldots, r-1$ and indices are taken modulo r). As S is G-normal, the portion of S that is between v_i and v_{i+1} should be a subset of some, say r, of the faces of G (this face is not well defined only if $|V(G)| = 3$ and, in this case, r can be any face of G). Notice that r is a triangle where $v_i, v_{i+1} \in \mathbf{bd}(r)$ and therefore $\{v_i, v_{i+1}\}$ is an edge of G. This edge is unique because G does not have multiple edges (for an example, see the first graph of Fig. 3). ■

Lemma 3.3. *Let G be a triangulated Σ-planar graph without multiple edges and let N_1, N_2 be nooses of G where $|N_1|, |N_2| \geq 3$. Then $\kappa_G(N_1) = \kappa_G(N_2)$ implies $N_1 \sim^* N_2$.*

Proof. Suppose that N_1, N_2 are nooses where $|N_1|, |N_2| \geq 3$ and $\kappa_G(N_1) = \kappa_G(N_2)$. By Lemma 3.2, there is a unique cycle C where $\kappa_G(C) = \kappa_G(N_1)$ and a unique cycle C' where $\kappa_G(C') = \kappa_G(N_2)$. As $\kappa_G(N) = \kappa_G(N')$ we have that $\kappa_G(C) = \kappa_G(C')$ and as G does not have multiple edges, we have that $C = C'$. We use the notation $C = (x_0, \ldots, x_{r-1})$. For $j = 1, 2$, we define the function σ_j corresponding to each edge $e_i = \{x_i, x_{i+1}\}$ of C the unique line, $\sigma_j(e_i)$ in Σ that is a subset of N_i and has endpoints x_i and x_{i+1} (as $|N_1|, |N_2| \geq 3$, σ_j is well defined). Let Δ_1, Δ_2 be the closed discs bounded by C in Σ. We define

$$\mathcal{D}_j = \{i \mid \sigma_j(e_i) \subseteq \Delta_{3-j}\}, j = 1, 2.$$

For $j = 1, 2$ we apply a sequence of variations on N_j as indicated by the following routine. The target of this routine is to put the whole N_i inside the closed disc Δ_i.

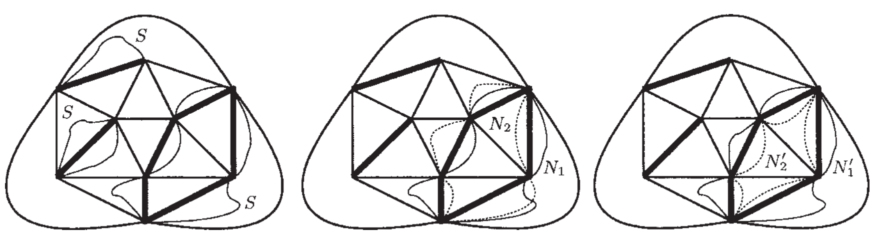

FIGURE 3. Examples of the proofs of Lemmata 3.2 and 3.3.

1. If $\mathcal{D}_j$ is empty, then stop and output N_j.
2. Pick an integer i in $\mathcal{D}_j$.
3. Let L be any line $L \subseteq \Sigma$ where $|L| = 1$, $L \subseteq \Delta_j$, and $L \cap L_j = x_i, x_{i+1}$.
4. Set $N_j \leftarrow N_j - \sigma_j(e_i) \cup L$. (Notice that this is a variation operation on N_j.)
5. Recalculate σ_j and $\mathcal{D}_j$. (Notice that now $i \notin \mathcal{D}_j$.)
6. Go to step 1.

For $j = 1, 2$, we call N'_j the resulting nooses and observe that $N'_j \subseteq \Delta_j$ and $N_j \sim^* N'_j$. We now apply the following sequence of variations on N_1^*: For any $i = 0, \ldots, r-1$, we set $N'_1 = N'_1 - \sigma_1(e_i) \cup \sigma_2(e_i)$. The resulting noose is N_2 and therefore, $N'_1 \sim^* N'_2$. We conclude that $N_1 \sim^* N_2$ and this completes the proof of the lemma (for an example, see the second and the third graph of Fig. 3). ■

C. Representing Nooses by Vibrations

We are now ready to show that any noose in a plane triangulation can be seen as a vibration of a cycle of its radial graph.

Observe that if G is a Σ-plane graph drawn in Σ along with its radial graph R_G, then any cycle of R_G of length $2k$ is a noose of length k. Any path of length $2k$ in R_G with both endpoints in $V(G)$ is a line in G of length k. Notice that if r is a face R_G, then $\mathbf{bd}(r)$ is a cycle of length 4 where $\overline{r}$ contains exactly one edge of G. Every edge e of G is contained in $\overline{r}$ for some face r. From now on, we use the notation $\mathbf{r}_e$ to denote this face. If T is a triangle of G and $|V(G)| \geq 4$, then we use the notation $\mathbf{v}(T)$ for the unique vertex of R_G that is adjacent in R_G with all the vertices of T.

Let G be a triangulated Σ-plane graph and let $F \subseteq E(G)$. We define the graph H_F as the subgraph of a dual graph G^* formed by edges F^*. In other words, its vertices are the triangles of G that contain some edge in F and two such triangles are connected by an edge if they have an edge of F in common. To distinguish the vertices of H_F from the vertices of the original graph we refer to the vertices of H_F as to *triangles.*

Notice that, as G is triangulated, the maximum degree of the vertices of H_F is 3 (in the extreme case where the maximum degree is 3 we have that three of the edges in F induce a triangle in G). This construction will be the basic common ingredient of the proofs of this and the next subsection. We call two triangles of degree 1 in H_F *irrelevant* if they belong to different connected components of H_F.

We call a subgraph P of a Σ-plane graph G *generalized* (x)*-path* if either

- P is a path with an extreme x, or
- it is a cycle of length ≥ 4 passing through x and such that there is no edge connecting the neighbors of x in P.

Notice that the stressed cycle of the graph of Figure 4 is a generalized (x)-path iff x is one of the gray vertices.

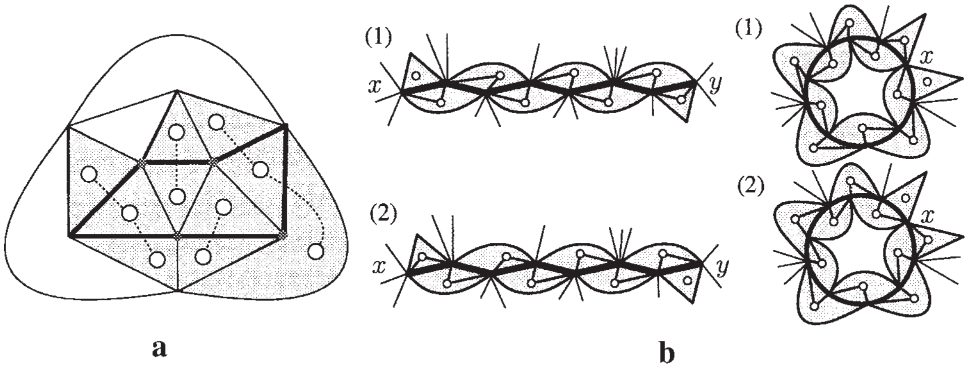

FIGURE 4. **a**: If F contains the edges of the "fat" cycle, then the graph H_F is the one formed by the dotted vertices and the white vertices. **b**: Examples of the constructions (1) and (2) of the proof of Lemma 3.4 when generalized (x)-path is a (x, y)-paths and a cycle.

Lemma 3.4. *Let G be a triangulated Σ-plane graph without multiple edges and where $|V(G)| \geq 4$, drawn in Σ along with its radial graph R_G. Let also P be a generalized (x)-path of G with the property that $H_{E(P)}$ is connected. Let also T be a triangle of degree 1 in $H_{E(P)}$. Then there exists a generalized (x)-path P_R in R_G such that $\kappa_G(P_R) = \kappa_G(P)$ and $\mathbf{v}(T) \notin P_R$.*

Proof. We use the notation $P = (x = v_0, \ldots, v_r = y), r \geq 1$ (in case P is a cycle we have $x = y$). As $|V(G)| \geq 4$ and G does not have multiple edges, the connectivity of $H_{E(P)}$ yields that $H_{E(P)}$ is a path whose extreme vertices are triangles of G. Each of these triangles has only one edge in common with P. Therefore, we can denote them as (a, v_0, v_1) and (v_{r-1}, v_{r-2}, b) for some $a \neq v_0$ and $b \neq v_d$. Notice that, for $j = 2, \ldots, r-2$ the edge $\{v_j, v_{j+1}\}$ is the common edge of the triangles (v_{j-1}, v_j, v_{j+1}) and (v_j, v_{j+1}, v_{j+2}) in $V(H)$. Moreover, $\{v_0, v_1\}$ is the common edge of (a, v_0, v_1) and (v_0, v_1, v_2) and $\{v_{r-1}, v_r\}$ is the common edge of (v_{r-2}, v_{r-1}, v_r) and (v_{r-1}, v_r, b).
If $(b, v_{r-1}, v_r) = T$ we set

$$P_R = (v_0, \mathbf{v}(a, v_0, v_1), v_1, \mathbf{v}(v_0, v_1, v_2), v_2, \mathbf{v}(v_1, v_2, v_3), \ldots \\ \ldots, \mathbf{v}(v_{q-3}, v_{q-2}, v_{q-1}), v_{q-1}, \mathbf{v}(v_{r-2}, v_{r-1}, v_r), v_r). \quad (1)$$

If $(a, v_0, v_1) = T$ we set

$$P_R = (v_0, \mathbf{v}(v_0, v_1, v_2), v_1, \mathbf{v}(v_1, v_2, v_3), v_2, \ldots \\ \ldots, v_{r-2}, \mathbf{v}(v_{r-2}, v_{r-1}, v_r), v_{r-1}, \mathbf{v}(b, v_{r-1}, v_r), v_r). \quad (2)$$

In any case, we guarantee that we can choose a line P_R that does not meet the vertex $\mathbf{v}(T)$. Observe that, by the construction of P_R, $\kappa_G(P_R) = \kappa_G(P)$ and the lemma follows. For examples of the above constructions, see Figure 4. ■

The next Lemma is a generalization of Lemma 3.4 for the general case where $H_{E(P)}$ is not necessarily connected.

Lemma 3.5. *Let G, $|V(G)| \geq 4$, be a triangulated Σ-plane graph without multiple edges drawn in Σ along with its radial graph R_G. Let also P be a generalized x-path of G and let $\mathcal{T}$ be a collection of mutually irrelevant degree one triangles in $V(H_{E(P)})$. Then there exists a generalized x-path P_R in R_G such that $\forall_{T \in \mathcal{T}}, \mathbf{v}(T) \notin P_R$ and $\kappa_G(P_R) = \kappa_G(P)$.*

Proof. Let $P_1, \ldots, P_q$ be the maximal sub-paths of P with the property that $H_{E(P_i)}$ is connected. (When P is a cycle these sub-paths still exist because x belongs into two distinct degree one triangles of $H_{E(P)}$.) Notice that $\{P_i \mid i = 1, \ldots, q\}$ is a partition of P and assume that its indices order it into consecutive segments of P. We assume that the endpoints of P_i are a_i, b_i, $1 \leq i \leq q$ where $x = a_1, b_1 = a_2, \ldots, b_{q-1} = a_q$, and $b_q = y$; the equalities follow from the maximality of each P_i (when P is a cycle, $x = y$). We denote as $H_1, \ldots, H_q$ the connected components of $H_{E(P)}$ indexed in a way that $H_i = H_{E(P_i)}$. Notice that $|\mathcal{T} \cap V(H_i)| \leq 1, i = 1, \ldots, q$ (otherwise we should have two irrelevant degree one triangles in the same component of H). If $|\mathcal{T} \cap V(H_i)|$ is nonempty, then let T_i be the unique triangle in it. Otherwise, let T_i be any of the triangles of $V(H_i)$ with degree 1 in H_i. We now apply Lemma 3.4 for H_i and T_i and we get a path P^i_R connecting a_i and b_i in R_G and such that $\kappa_G(P^i_R) = \kappa_G(P^i)$ and $\mathbf{v}(T_i) \notin P^i_R$. We set $C_R = \bigcup_{i=1,\ldots,q} P^i_R$ and observe that, for any $T \in \mathcal{T}$, $\mathbf{v}(T_i) \notin P^i_R$. As none of the triangles in $H_{E(P)}$ belongs to two different connected components of $H_{E(P)}$, we have that $\kappa_G(P_R) = \kappa_G(P)$ and the lemma follows (for an example, see Figure 5a). ■

Lemma 3.6. *Let G be a triangulated Σ-plane graph with ≥ 4 vertices and without multiple edges, drawn in Σ along with its radial graph R_G. Let also C be a cycle in G and $\mathcal{T}$ be an collection of mutually irrelevant degree one triangles in $H_{E(C)}$. Then there exists a cycle C_R in R_G such that $\kappa_G(C_R) = \kappa_G(C)$ and $\forall_{T \in \mathcal{T}}, \mathbf{v}(T) \notin C_R$.*

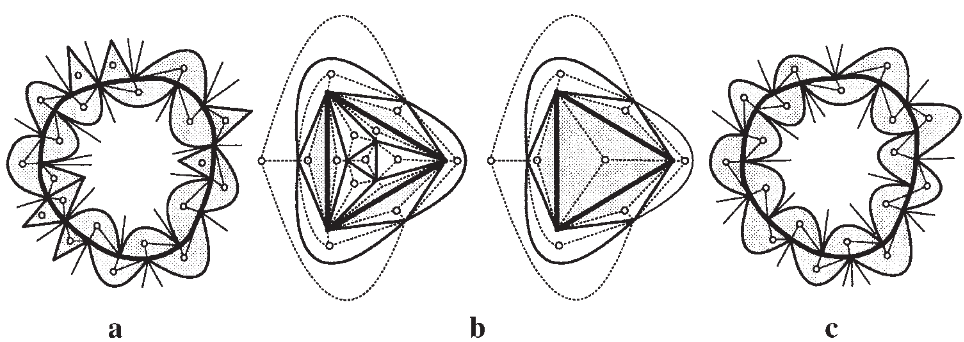

FIGURE 5. **a**: An example of the proof of Lemma 3.5. **b**: Examples of the case $|C| = 3$ of the proof of Lemma 3.6. **c**: Example of the first case of the proof of Lemma 3.6.

Proof. If $|C| = 3$, then we use the notation $C = (x, y, z)$ and we notice that

$$\mathbf{bd}(\bar{\mathbf{r}}_{\{x,y\}\}} \cup \bar{\mathbf{r}}_{\{y,z\}} \cup \bar{\mathbf{r}}_{\{z,x\}})$$

is a subgraph of R_G and contains as a subgraph at least one cycle C_R of length 6 as required (it meets all the vertices of C, otherwise, G should have a multiple edge—see also Fig. 5b).

Suppose now that $C = (x_0, \ldots, x_{r-1}, x_0), r \geq 4$. As $|C| \geq 4$, we have that all the vertices in $H_{E(C)}$ have degree at most 2 (otherwise C is a triangle). We examine two cases:

Case 1. H is a cycle of r vertices. In this case we should have $\mathcal{T} = \emptyset$. Observe that

$$C_R = (x_0, \mathbf{v}(x_0, x_1, x_2), x_2, \mathbf{v}(x_1, x_2, x_3), \ldots, x_{r-1}, \mathbf{v}(x_{r-1}, x_0, x_1), x_0)$$

is the required cycle of R_G (all indices are taken modulo r). For an example of this case, see Figure 5c.

Case 2. All the connected components of H are paths. In this case, there will exist a vertex $x \in C$ such that its neighbors in C are not adjacent. Therefore, C is a generalized (x)-path, it is not a triangle, and by applying Lemma 3.5 for C and $\mathcal{T}$, the result follows. ∎

The following lemma is the main conclusion of this subsection.

Lemma 3.7. *Let G be a triangulated Σ-plane graph with ≥ 4 vertices and without multiple edges, drawn in Σ along with its radial graph R_G. Then any noose N, $|N| \geq 2$, of G is a vibration of some of the cycles of R_G.*

Proof. If $|N| = 2$, then let e be the unique edge connecting the extreme points of N (e is unique because G does not have multiple edges). We directly have that $\mathbf{bd}(\mathbf{r}_e)$ is a cycle of R_G and it is easy to verify that it is also a vibration of N. Therefore, we may assume that $|N| \geq 3$. From Lemma 3.2 there exist a unique cycle C where $\kappa_G(C) = \kappa_G(N)$. From Lemma 3.6, there exist a noose C_R of G where $\kappa_G(C_R) = \kappa_G(C)$. Notice that C_R is a cycle of R_G and, as $\kappa_G(N) = \kappa_G(C_R)$, from Lemma 3.3, we conclude that $N \sim^* C_R$. ∎

D. Representing Θ-Structures by Vibrations

In this section, we extend the results of Subsection 3C to Θ-structures. In particular, we prove Lemma 3.9 claiming that any Θ-structure of a plane triangulation is a vibration of some Θ-structure of its radial graph.

Let N be a noose in Σ and let Q be a continuous subset of Σ such that $N \cap Q = \emptyset$. Then one of the discs bounded by N does not contain points of Q. We call this disc by *Q-avoiding disc bounded by N.*

Lemma 3.8. *Let G be a triangulated Σ-plane graph with ≥ 5 vertices and without multiple edges, drawn in Σ along with its radial graph R_G. Then for any three paths P^1, P^2, P^3 of G that connect two vertices x and y, and are otherwise disjoint, there exist three paths P^1_R, P^2_R, P^3_R in R_G that connect x and y, and are otherwise disjoint, and such that for any $i, 1 \leq i \leq 3$, $\kappa_G(P^i_R) = \kappa_G(P^i)$.*

Proof. We first examine the special case where some of $P_1 \cup P_2$, $P_1 \cup P_3$, or $P_2 \cup P_3$ has length 3. W.l.o.g we assume that $|P_2 \cup P_3| = 3$ and, in particular, we let $P_2 = (x, y)$ and $P_3 = (x, z, y)$. Notice that $|P_1| \geq 2$ because G has no multiple edges. We examine two subcases:

$|P_1| = 2$. We assume that $P_1 = (x, w, y)$. We examine first the case where either x or y is connected with a vertex u of the $\{x, y\}$-avoiding open disc D bounded by (x, z, y, w) (see Fig. 6a). W.l.o.g. assume that x is adjacent to u and let (w, x, u_1) and (z, x, u_2) be the facial triangles containing $\{w, x\}$ and $\{z, x\}$ where $u_1, u_2 \in D$ (each of these two triangles can have $\{x, u\}$ as an edge). Let also (w, y, z') be the facial triangle containing $\{w, y\}$ and such that $z' \in \overline{D}$ (notice that z and z' may be identical). Then we set $P^1_R = (x, \mathbf{v}(x, u_1, w), w, \mathbf{v}(y, w, z'), y)$, $P^2_R = (x, \mathbf{v}(x, w, y), y)$, and $P^3_R = (x, \mathbf{v}(z, u_2, x), z, \mathbf{v}(z, x, y), y)$. Observe that $P^i_R, i = 1, 2, 3$ are paths and that for every i, $1 \leq i \leq 3$, $\kappa_G(P^i_R) = \kappa_G(P^i)$.

In the remaining case, w and z are adjacent, and the triangles (w, x, z) and (w, y, z) are both facial (see Fig. 6b). Then, as $|V(G)| \geq 5$, there exist a vertex u that is adjacent to either x or y and is included into either the w-avoiding open disc bounded by (x, y, z) or into the z-avoiding open disc bounded by (x, y, w). W.l.o.g. we assume that u is adjacent to x and that x is included in the w-avoiding open disc D bounded by (x, y, z). Let (x, u_1, y) and (x, u_2, z) be the facial triangles containing $\{x, y\}$ and $\{x, z\}$ where $u_1, u_2 \in D$ (each of these two triangles can have $\{x, u\}$ as an edge). Let also (w, y, t) be a facial triangle containing $\{w, y\}$ where t belongs in the z-avoiding open disc bounded by (x, w, y). Then we set $P^1_R = (x, \mathbf{v}(x, w, z), w, \mathbf{v}(w, y, t), y)$, $P^2_R = (x, \mathbf{v}(x, u_1, y), y)$, and $P^3_R = (x, \mathbf{v}(x, u_2, z), z, \mathbf{v}(w, z, y), y)$. Observe that for every i, $1 \leq i \leq 3$, $P^i_R, i = 1, 2, 3$ are paths and $\kappa_G(P^i_R) = \kappa_G(P^i)$.

$|P_1| \geq 3$. We assume that $P_1 = (x = v_0, v_1, \ldots, v_{r-2}, v_r = y)$, $r \geq 3$ and observe that $C = (v_0, v_1, \ldots, v_{r-1}, v_r)$ is a cycle of G where $|C| \geq 4$. We call

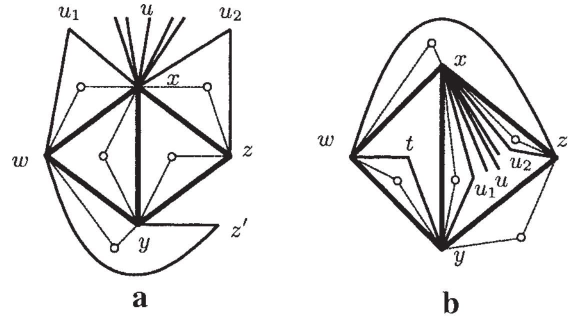

FIGURE 6. The case $|P_2 \cup P_3| = 3$ and $|P_1| = 2$ of the proof of Lemma 3.8.

D the $\{x,y\}$-avoiding closed disc bounded by $P_1 \cup P_3$ in Σ. Let $T_z = (x,y,z)$. Also let $T_x = (x,z,a)$ be the unique facial triangle different than (x,y,z) that contains $\{x,z\}$ and where $a \in D$ and let $T_y = (y,z,b)$ be the unique triangle different than (x,y,z) that contains $\{y,z\}$ and where $b \in D$. We now construct the set $\mathcal{T}$ distinguishing 4 cases (see also Fig. 7).

1. If $a \neq v_1$ and $b \neq v_{r-1}$, then we set $\mathcal{T} = \emptyset$.
2. If $a = v_1$ and $b \neq v_{r-1}$, then we have that T_x is a triangle of degree 1 in $H_{E(C)}$ and we set $\mathcal{T} = \{T_x\}$.
3. If $a \neq v_1$ and $b = v_{r-1}$, then we have that T_y is a triangle of degree 1 in $H_{E(C)}$ and we set $\mathcal{T} = \{T_y\}$.
4. If $a = v_1$ and $b = v_{r-1}$, then we have that both T_x and T_y are triangless of degree 1 in $H_{E(C)}$. As $|C| \geq 4$, any connected component of $H_{E(C)}$ has two triangles of degree 1. This implies that either $\{T_z, T_x\}$ or $\{T_z, T_y\}$ is a collection of mutually irrelevant degree one triangles in $V(H_{E(C)})$. We distinguish two subcases:

   4a. If T_z and T_x are irrelevant we set $\mathcal{T} = \{T_z, T_x\}$.
   4b. If T_z and T_y are irrelevant we set $\mathcal{T} = \{T_z, T_y\}$.

   (If both pairs T_z, T_x and T_z, T_y are irrelevant we make an arbitrary choice.)

For any of the above cases, we apply Lemma 3.6 for C and $\mathcal{T}$ and we get a cycle C_R in R_G where $\kappa_G(C_R) = \kappa_G(C)$. Clearly, C_R is the union of two internally disjoint paths P_R^1 and P_R^2 that connect in R_G the vertices x and y. In cases 1–3, we set $P_R^3 = (x, \mathbf{v}(T_x), z, \mathbf{v}(T_y), y)$. In case 4a, we set $P_R^3 = (x, \mathbf{v}(T_x), z, \mathbf{v}(T_z), y)$. In case 4b, we set $P_R^3 = (x, \mathbf{v}(T_z), z, \mathbf{v}(T_y), y)$. It is now easy to see that, in any case, for all i, $1 \leq i \leq 3$, $\kappa_G(P_R^i) = \kappa_G(P^i)$. This completes the analysis of the special case.

Assume now that for all i,j, $1 \leq i < j \leq 3$, $|P^i \cup P^j| \geq 4$. Let $P_1 = (x, v_1, \ldots, v_{r-2}, y), P_2 = (x, u_1, \ldots, u_{s-2}, y)$, and $P_3 = (x, w_1, \ldots, w_{t-2}, y)$. We consider the cycle $C = P^1 \cup P^2$ and the path $P = P^3$. As $|C| \geq 4$ and $|P| \geq 3$, $V(H_{E(C)})$ and $V(H_{E(P)})$ can have at most 4 triangles in common that can be the triangles $A = (u_1, x, w_1), B = (v_1, x, w_1)$, $C = (u_{s-2}, y, w_{t-2})$, and $D = (v_{r-2}, y, w_{t-2})$. Our target will be to apply Lemmata 3.5 and 3.6 on P and C in order to construct a path P_R and a cycle C_R without common radial vertices. In order not to use the same interior vertices of R_G two times we have to apply them with the restrictions

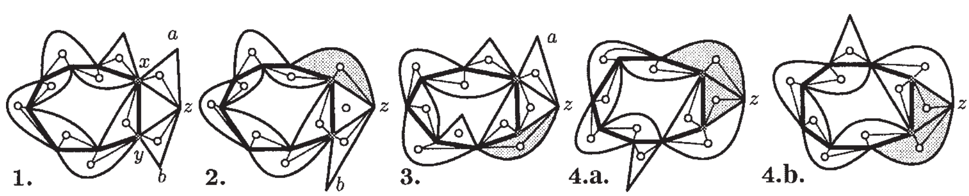

FIGURE 7. Examples of the proof of Lemma 3.8 for the case where $|P_2 \cup P_3| = 3$ and $|P_1| \geq 3$ (subcases 1,2,3,4a, and 4b).

imposed by suitably chosen collections $\mathcal{T}_C$,$\mathcal{T}_P$ of mutually irrelevant degree one triangles in $V(H_{E(C)})$ and $V(H_{E(P)})$, respectively. We set $\mathcal{C} = V(H_{E(C)}) \cap V(H_{E(P)})$ and we distinguish the following cases (for examples, see Figs. 8 and 9).

1. $|\mathcal{C}| = 0$. Then we set $\mathcal{T}_C = \mathcal{T}_P = \emptyset$.
2. $|\mathcal{C}| = 1$. Then we set $\mathcal{T}_C = V(H_{E(C)}) \cap V(H_{E(P)})$ and $\mathcal{T}_P = \emptyset$.
3. $|\mathcal{C}| = 2$. Then we put in $\mathcal{T}_C$ one of the two elements of $\mathcal{C}$ and we put in $\mathcal{T}_P$ the other.
4. $|\mathcal{C}| = 3$. Then we distinguish the following subcases:

   4a. if $\mathcal{C} = \{A, B, C\}$, then $\mathcal{T}_C = \{A\}$ and $\mathcal{T}_P = \{B, C\}$.
   4b. if $\mathcal{C} = \{A, C, D\}$, then $\mathcal{T}_C = \{C\}$ and $\mathcal{T}_P = \{A, D\}$.
   4c. if $\mathcal{C} = \{A, B, D\}$, then $\mathcal{T}_C = \{A\}$ and $\mathcal{T}_P = \{B, D\}$.
   4d. if $\mathcal{C} = \{B, C, D\}$, then $\mathcal{T}_C = \{C\}$ and $\mathcal{T}_P = \{B, D\}$.

5. $|\mathcal{C}| = 4$. Then we set $\mathcal{T}_C = \{A, D\}$ and $\mathcal{T}_P = \{B, C\}$.

Notice that, in any of the above cases, the triangles in $\mathcal{T}_C$ and $\mathcal{T}_P$ are mutually irrelevant degree one triangles of $V(H_{E(C)})$ and $V(H_{E(P)})$, respectively. Therefore, we can apply Lemma 3.5 for P and $\mathcal{T}_P$ and Lemma 3.6 for C and $\mathcal{T}_C$ and construct the cycle C_R and the path P_R where $\kappa_G(C_R) = \kappa_G(C)$ and $\kappa_G(P_R) = \kappa_G(P)$. Notice that, in each case, the choice of $\mathcal{T}_C$ and $\mathcal{T}_P$ does not allow C_R and P_R to have common radial vertices. C_R defines two paths P^1 and P^2 connecting x and y and if we set $P_R^3 = P_R$ we have that $\kappa_G(P_R^i) = \kappa_G(P^i)$ for all $1 \leq i \leq 3$. ■

Let us remind that a Θ-structure is non-trivial if at least two of its lines have length ≥ 2.

Lemma 3.9. *Let G be a triangulated Σ-plane graph with ≥ 5 vertices and without multiple edges, drawn in Σ along with its radial graph R_G. If $S = (L^1, L^2, L^3)$ is a non-trivial Θ-structure of G, then there exists a non-trivial Θ-structure (P_R^1, P_R^2, P_R^3) of G that is a vibration of S, where P_R^1, P_R^2 and P_R^3 are paths of R_G.*

Proof. We apply Lemma 3.2 for the noose $N = L^1 \cup L^2$ and we get a cycle C of G where $\kappa_G(C) = \kappa_G(N)$. This cycle defines two internally disjoint paths P^1

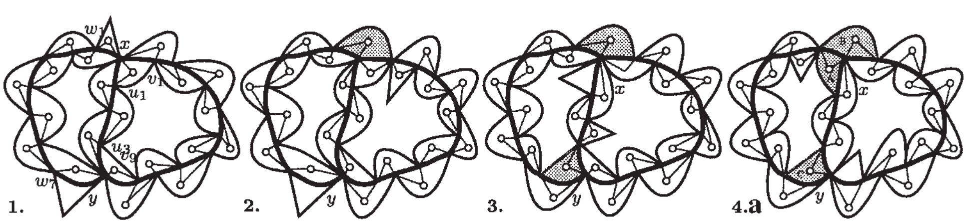

FIGURE 8. Examples of the proof of Lemma 3.8 for the case where $|P_2 \cup P_3| \geq 4$ and $|P_1| \geq 3$ (subcases 1,2,3,4.a).

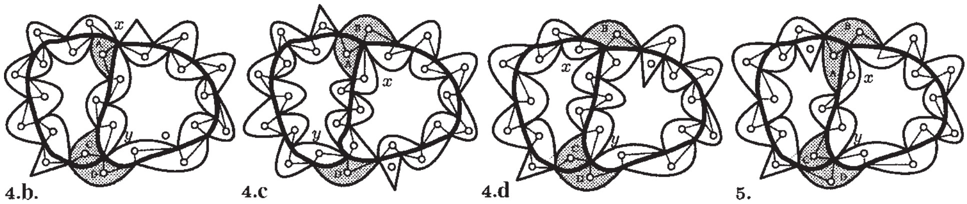

FIGURE 9. Examples of the proof of Lemma 3.8 for the case where $|P_2 \cup P_3| \geq 4$ and $|P_1| \geq 3$ (subcases 4.b,4.c,4.d, and 5).

and P^2 between x and y in G where $\kappa_G(P^i) = \kappa_G(L^i), i = 1, 2$. Applying now again Lemma 3.2 for the line L_3, we get a path P^3 between x and y in G where $\kappa_G(P^3) = \kappa_G(L^3)$. We now apply Lemma 3.8 on $P^i, i = 1, 2, 3$ and get three internally disjoint paths P^1_R, P^2_R, P^3_R of R_G that connect x and y and such that for each i, $1 \leq i \leq 3$, $\kappa_G(P^i_R) = \kappa_G(P^i)$. Resuming the previous equalities, we get $\kappa_G(P^i_R) = \kappa_G(L^i)$, $1 \leq i \leq 3$. Notice that (P^1_R, P^2_R, P^3_R) is a non-trivial Θ-structure in G. In what remains we will show that it is also a vibration of (L^1, L^2, L^3). Notice that $\kappa_G(P^1_R \cup P^2_R) = \kappa_G(L^1 \cup L^2)$ and applying Lemma 3.3, we have that $P^1_R \cup P^2_R \sim^* L^1 \cup L^2$ and this, in turn, implies that $P^1_R \sim^* L^1$ and $P^2_R \sim^* L^2$. Notice now that $P^2_R \cup L^3$ is a noose of G. Recall that $\kappa_G(P^3_R) = \kappa_G(L^3)$ which implies that $\kappa_G(L^2 \cup P^3_R) = \kappa_G(L^2 \cup L^3)$. From Lemma 3.3, we have that $L^2 \cup P^3_R \sim^* L^2 \cup L^3$ and this, in turn, implies that $P^3_R \sim^* L^3$. Therefore, (P^1_R, P^2_R, P^3_R) is a vibration of (L^1, L^2, L^3).

E. A Topological Property of Θ-Structures

The following Lemma is a necessary ingredient for the proofs of the next subsection.

Lemma 3.10. *Let $S = (L_1, L_2, L_3)$ and $S' = (L'_1, L_2, L_3)$ be two non-trivial Θ-structures of some Σ-plane graph G where $S \sim S'$. Then, for one, say D^*, of the closed discs bounded by $L_2 \cup L_3$, holds that $D^* \cap \mathbf{dif}(S, S') \subseteq L_2 \cap L_3$ (recall $L_2 \cap L_3$ is a set of two distinct point).*

Proof. Let $\{x, y\} = L_2 \cap L_3$. Let also L and L' be the length-1 lines comprising the length-2 noose $(S \cup S') - (S \cap S') = L \cup L'$, assuming that $L \subseteq L_1$ and $L' \subseteq L'_1$. In the case analysis that follows, we will define a disc D^* bounded by $L_2 \cup L_3$ and we will show that $L \cup L' \subseteq \overline{\Sigma - D^*}$.

Case 1. $|L_1|$, $|L'_2| \geq 2$. Then, we can choose a vertex $v \in (L \cup L') \cap V(G)$ that is different that x and y. Therefore $v \notin L_2 \cup L_3$ and we can define D^* as the closed disc bounded by $L_2 \cup L_3$ that does not contain v. Notice that $L_1 \cup L'_1$ contains at most one point in common with $L_2 \cup L_3 = \mathbf{bd}(D^*) = \mathbf{bd}(\overline{\Sigma - D^*})$. We need the following topological fact.

Fact 1. Let Δ be a closed disc on a sphere Σ and let N be a simple closed curve where $N \cap \mathbf{bd}(\Delta)$ is either empty or is just a point x. Then $(\Delta - \mathbf{bd}(\Delta)) \cap N \neq \emptyset$ implies $N \subseteq \Delta$.

As $(\Sigma - D^*) \cap (L \cup L') \neq \emptyset$, we apply the fact for $L \cup L'$ and $\overline{\Sigma - D^*}$, obtaining $L \cup L' \subseteq \overline{\Sigma - D^*}$.

Case 2. $|L_1|, |L'_1| = 1$. Notice that, then, $|L_2|, |L_3| \geq 2$. Notice that $L_1 - \{x, y\}$ cannot have common points with the noose $L_2 \cup L_3$. Therefore, it will be a subset of some of the closed discs bounded by $L_2 \cup L_3$. Notice also that the same holds for L'_1. Observe now that $L_1 - \{x, y\}$, $L'_1 - \{x, y\}$ cannot be subsets of different discs bounded by the noose $L_2 \cup L_3$ because then each of the discs bounded by the noose $L_1 \cup L'_1$ should contain a vertex of G. Let D^* be the disc containing none of $L_1 - \{x, y\}$,$L'_1 - \{x, y\}$. This means that the noose $L_1 \cup L'_1$ is a subset of $\overline{\Sigma - D^*}$. As $L_1 = L$ and $L'_1 = L'$, we have that $L_1 \cup L'_1 \subseteq \overline{\Sigma - D^*}$.

Here is the second topological property we use in our proof.

Fact 2. Let Δ be a closed disc on a sphere Σ and let N be a simple closed curve where $N \subseteq \Delta$. Then some of the closed discs bounded by N will be a subset of Δ.

Let A and A' be the discs bounded by $L_1 \cup L'_1$. By Fact 2, one, say A, of A, A' should be a subset of $\overline{\Sigma - D^*}$. Notice that A should be $\mathbf{dif}(S, S')$, otherwise $A = \overline{\Sigma - \mathbf{dif}(S, S')}$ and as $A \subseteq \overline{\Sigma - D^*}$, we have that $\overline{\Sigma - \mathbf{dif}(S, S')} \subseteq \overline{\Sigma - D^*} \Rightarrow D^* \subseteq \mathbf{dif}(S, S')$. Hence $D^* \cap V(G) \subseteq \mathbf{dif}(S, S') \cap V(G) = \{x, y\}$ a contradiction as $|(D^* \cap V(G)) - \{x, y\}| \geq 1$ (this follows from the fact that S is non-trivial). We conclude that $\mathbf{dif}(S, S') \subseteq \overline{\Sigma - D^*}$, therefore $\mathbf{dif}(S, S') - \mathbf{bd}(\mathbf{dif}(S, S')) \subseteq \Sigma - D^* \Rightarrow (\mathbf{dif}(S, S') - \mathbf{bd}(\mathbf{dif}(S, S')) \cap D^* = \emptyset$. As $\mathbf{bd}(\mathbf{dif}(S, S')) = L_1 \cup L'_1$, we have that $\mathbf{bd}(\mathbf{dif}(S, S')) \cap D^* = (L_1 \cup L'_1) \cap D^* \subseteq \{x, y\}$ and the proof is complete. ∎

F. Vibration Invariants of Θ-structures

We are now ready to prove two properties of Θ-structures that will be critical for the proof of Theorem 3.1. Intuitively, we show that vibrations do not alter "interior-exterior" relation of their bounding disks.

Let N, N' be two nooses of some Σ-plane graph G. Let $N \sim N'$ and let $\mathcal{D} = \{D_1, D_2\}$ and $\mathcal{D}' = \{D'_1, D'_2\}$ be the closed discs bounded by N and N', respectively. We set up a bijection $\sigma_{N,N'} : \mathcal{D} \to \mathcal{D}'$ such that if $D \in \mathcal{D}$, then

$$\sigma_{N,N'}(D) = \begin{cases} \overline{D - \mathbf{dif}(N, N')} & \text{if } \mathbf{dif}(N, N') \subseteq D, \\ D \cup \mathbf{dif}(N, N') & \text{if } \mathbf{dif}(N, N') \not\subseteq D. \end{cases}$$

Also, for notational convenience, we enhance the definition of σ so that $\sigma_{N,N}(D) = D$. It is easy to verify that $\sigma_{N,N'} = \sigma^{-1}_{N',N}$ (for an example, see Fig. 10).

Let N and N' be nooses where $N \sim^* N'$. Then if $N = N_0 \sim N_1 \sim \cdots \sim N_{r-1} \sim N_r = N'$, we define $\sigma^*_{N,N'} = \sigma_{N_0,N_1} \circ \sigma_{N_1,N_2} \circ \cdots \circ \sigma_{N_{r-1},N_r}$. Notice that

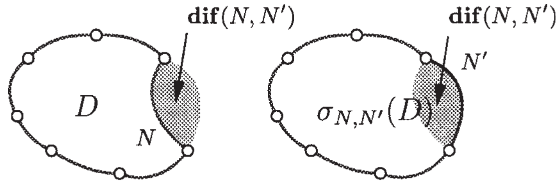

FIGURE 10. An example of the application of the function **dif**.

$\sigma^*_{N,N'}$ is well defined as it does not depend on the way N is transformed to N' (however, we stress that this fact is not used in our proofs). Again it follows that $\sigma^*_{N,N'} = \sigma^{*-1}_{N',N}$.

Intuitively, we define $\sigma^*_{N,N'}$ so that D and $\sigma^*_{N,N'}(D)$ are in the same "interior/exterior location" with respect to the nooses N, N'.

The following lemma is a direct consequence of the fact that $\mathbf{dif}(N, N')$ does not contain vertices that are not met by both N and N'.

Lemma 3.11. *Let N_1, N_2 be nooses of G where $N_1 \sim^* N_2$. If D is some disc bounded by N_1, then $V(G) \cap \sigma^*_{N_1,N_2}(D) = V(G) \cap D$.*

We need the following lemma.

Lemma 3.12. *Let G be a Σ-plane graph and $S = (L_1, L_2, L_3)$ and $S' = (L'_1, L'_2, L'_3)$ be non-trivial Θ-structures in G where $S \sim^* S'$. If D is a closed disc bounded by the noose $L_1 \cup L_2$ and $L_3 \subseteq D$, then $L'_3 \subseteq \sigma^*_{L_1 \cup L_2, L'_1 \cup L'_2}(D)$.*

Proof. It is sufficient to prove the statement of the lemma only for the case $L'_3 \subseteq \sigma_{L_1 \cup L_2, L'_1 \cup L'_2}(D)$. (Using this case as an induction assumption, one can prove the lemma by making use of induction on the number of variations required in order to transform S to S'.)

We set $\{x, y\} = L_1 \cap L_2 \cap L_3$. We also set $\Delta = \mathbf{dif}(S, S')$ and notice that a variation affects only one of the lines in S. Therefore, we can distinguish the following cases.

Case 1. $L_2 \cup L_3 = L'_2 \cup L'_3$. Then $\Delta = \mathbf{dif}(L_1 \cup L_2, L'_1 \cup L'_2)$.

Subcase 1a. If $\Delta \not\subseteq D$, then $\sigma_{L_1 \cup L_2, L'_1 \cup L'_2}(D) = D \cup \Delta$. Therefore, $L'_3 = L_3 \subseteq D \subseteq D \cup \Delta = \sigma_{L_1 \cup L_2, L'_1 \cup L'_2}(D)$.

Subcase 1b. If $\Delta \subseteq D$, apply Lemma 3.10 on S and S' and let $D_{2,3}$ be the closed disc bounded by $L_2 \cup L_3$ where $D_{2,3} \cap \Delta \subseteq \{x, y\}$. As $(L_1 - \{x, y\}) \cap \Delta \neq \emptyset$, it implies that $L_1 - \{x, y\} \subseteq \Sigma - D_{2,3}$. This means that $D_{2,3} \subseteq D$. We now have $D_{2,3} - \{x, y\} \subseteq D_{2,3} - (D_{2,3} \cap \Delta) = D_{2,3} - \Delta \subseteq D - \Delta$. Therefore, $L_3 \subseteq D_{2,3} = \overline{D_{2,3} - \{x, y\}} \subseteq \overline{D - \Delta} = \sigma_{L_1 \cup L_2, L'_1 \cup L'_2}(D)$.

Case 2. $L_1 \cup L_3 = L'_1 \cup L'_3$. This case is symmetric to the Case 1.

Case 3. $L_1 \cup L_2 = L'_1 \cup L'_2$. Again we apply Lemma 3.10 on S and S', and let $D_{1,2}$ be the disc bounded by $L_1 \cup L_2$ where $D_{1,2} \cap \Delta \subseteq \{x, y\}$. As $(L_3 - \{x, y\}) \cap$

$\Delta \neq \emptyset$, we imply that $L_3 - \{x, y\} \subseteq \Sigma - D_{1,2}$. Applying the same argument for L'_3, we get $L'_3 - \{x, y\} \subseteq \Sigma - D_{1,2}$. Therefore, L_3 and L'_3 are both included in the same disc bounded by $L_1 \cup L_2$. As $L_3 \subseteq D$, we conclude $L'_3 \subseteq D = \sigma_{L_1 \cup L_2, L'_1 \cup L'_2}(D)$. ■

Lemma 3.13. *Let G be a Σ-plane graph and $S = (L_1, L_2, L_3)$ and $S' = (L'_1, L'_2, L'_3)$ be non-trivial Θ-structures in G where $S \sim^* S'$. If $D_{1,2}$ is a closed disc bounded by the noose $L_1 \cup L_2$ and $D_{1,3}$ is a closed disc bounded by the noose $L_1 \cup L_3$ such that $D_{1,3} \subseteq D_{1,2}$, then $\sigma^*_{L_3 \cup L_3, L'_1 \cup L'_3}(D_{1,3}) \subseteq \sigma^*_{L_1 \cup L_2, L'_1 \cup L'_2}(D_{1,2})$.*

Proof. As in the previous lemma, it is sufficient to prove only the case $S \sim S'$. (And then use the induction on the number of variations required in order to transform S to S'.)

We set $\{x, y\} = L_1 \cap L_2 \cap L_3$. We also set $\Delta = \mathbf{dif}(S, S')$ and notice that a variation affects only one of the lines in S. Therefore, we can distinguish the following cases.

Case 1. $L_2 \cup L_3 = L'_2 \cup L'_3$. Notice that $\Delta = \mathbf{dif}(L_1 \cup L_3, L'_1 \cup L'_3)$.

Subcase 1a. If $\Delta \not\subseteq D_{1,2}$, then, from, $D_{1,3} \subseteq D_{1,2}$ we also have that $\Delta \not\subseteq D_{1,3}$. Therefore, $\sigma_{L_1 \cup L_2, L'_1 \cup L'_2}(D_{1,2}) = D_{1,2} \cup \Delta$, $\sigma_{L_1 \cup L_3, L'_1 \cup L'_3}(D_{1,3}) = D_{1,3} \cup \Delta$ and the required relation follows as $D_{1,3} \subseteq D_{1,2}$.

Subcase 1b. If $\Delta \subseteq D_{1,2}$ we apply Lemma 3.10 on S and S' and let $D_{2,3}$ be the disc bounded by $L_2 \cup L_3$ where $D_{2,3} \cap \Delta \subseteq \{x, y\}$. As $(L_1 - \{x, y\}) \cap \Delta \neq \emptyset$, we imply that $L_1 - \{x, y\} \subseteq \Sigma - D_{2,3}$. This means that $D_{2,3} \subseteq D_{1,2}$. Combining this with the fact that $D_{1,3} \subseteq D_{1,2}$, we have that $D_{1,2} = D_{1,3} \cup D_{2,3}$. So, we can assume that $D_{1,2} - D_{2,3} \subseteq D_{1,3}$. Notice that $\Delta - \{x, y\} \subseteq \Delta - (D_{2,3} \cap \Delta) = \Delta - D_{2,3} \subseteq D_{1,2} - D_{2,3} \subseteq D_{1,3}$. As also $\{x, y\} \subseteq D_{1,3}$, we have that $\Delta \subseteq D_{1,3}$ and therefore $\sigma_{L_1 \cup L_3, L'_1 \cup L'_3}(D_{1,3}) = \overline{D_{1,3} - \Delta}$. Moreover, $\sigma_{L_1 \cup L_2, L'_1 \cup L'_2}(D_{1,2}) = \overline{D_{1,2} - \Delta}$ and the result follows as $\overline{D_{1,3} - \Delta} \subseteq \overline{D_{1,2} - \Delta}$.

Case 2. $L_1 \cup L_2 = L'_1 \cup L'_2$. Notice that $\Delta = \mathbf{dif}(L_1 \cup L_3, L'_1 \cup L'_3)$.

Observe that in this case, the variation does not affect the noose $L_1 \cup L_2$. Therefore, $\sigma_{L_1 \cup L_2, L'_1 \cup L'_2}(D_{1,2}) = D_{1,2}$. In both subcases that follow, our target will be to prove that $D_{1,2} \supseteq \sigma_{L_1 \cup L_3, L'_1 \cup L'_3}(D_{1,3})$.

Subcase 2a. If $\Delta \not\subseteq D_{1,3}$, we apply Lemma 3.10 on S and S' and let D^* be a disc bounded by $L_1 \cup L_2$ where $D^* \cap \Delta \subseteq \{x, y\}$. As $(L_3 - \{x, y\}) \cap \Delta \neq \emptyset$, we imply that $L_3 - \{x, y\} \subseteq \Sigma - D^*$. As $L_3 \subseteq D_{1,2}$, we get that $D^* = \Sigma - D_{1,2}$. Combining this with $D^* \cap \Delta \subseteq \{x, y\}$ we take $\Delta \subseteq D_{1,2}$. Therefore $\sigma_{L_1 \cup L_3, L'_1 \cup L'_3}(D_{1,3}) = D_{1,3} \cup \Delta \subseteq D_{1,2} \cup \Delta \subseteq D_{1,2}$.

Subcase 2b. If $\Delta \subseteq D_{1,3}$, then $\sigma_{L_1 \cup L_3, L'_1 \cup L'_3}(D_{1,3}) = \overline{D_{1,3} - \Delta} \subseteq D_{1,3} \subseteq D_{1,2}$.

Case 3. $L_1 \cup L_3 = L'_1 \cup L'_3$. Notice that $\Delta = \mathbf{dif}(L_1 \cup L_2, L'_1 \cup L'_2)$.

Observe that in this case, the variation does not affect the noose $L_1 \cup L_3$. Therefore, $\sigma_{L_1 \cup L_3, L'_1 \cup L'_3}(D_{1,3}) = D_{1,3}$. In both subcases that follow, our target will be to prove that $D_{1,3} \subseteq \sigma_{L_1 \cup L_2, L'_1 \cup L'_2}(D_{1,2})$.

Subcase 3a. If $\Delta \nsubseteq D_{1,2}$, then $\sigma_{L_1 \cup L_2, L_1' \cup L_2'}(D_{1,2}) = D_{1,2} \cup \Delta \supseteq D_{1,2} \supseteq D_{1,3}$.

Subcase 3b. If $\Delta \subseteq D_{1,2}$, we apply Lemma 3.10 on S and S' and let D^* be a disc bounded by $L_1 \cup L_3$ where $D^* \cap \Delta \subseteq \{x,y\}$. As $(L_2 - \{x,y\}) \cap \Delta \neq \emptyset$, we imply that $L_2 - \{x,y\} \subseteq \Sigma - D^*$. This means that $D^* = D_{1,3}$. We now have $\overline{D_{1,3} - \{x,y\}} \subseteq D_{1,3} - \overline{(D_{1,3} \cap \Delta)} = D_{1,3} - \Delta \subseteq D_{1,2} - \Delta$. Therefore, $\sigma_{L_1 \cup L_2, L_1' \cup L_2'}(D_{1,2}) = \overline{D_{1,2} - \Delta} \supseteq \overline{D_{1,3} - \{x,y\}} = D_{1,3}$.

G. Proof of Theorem 3.1

Lemma 3.14. *Let G be a triangulated Σ-plane graph without multiple edges and let* **ins** *be a uniform slope of order $k+1$ in R_G for $k \geq 2$. Then, for any face r of R_G,* **ins**(**bd**$(r)) = \bar{r}$.

Proof. As **ins** is uniform, we have that there exists a cycle C' of length $\leq 2k$ such that $r \subseteq$ **ins**(C'). This means that **bd**$(r) \subseteq$ **ins**(C') and from axiom [**S1**] we have that **ins**(**bd**$(r)) \subseteq$ **ins**(C'). Therefore, **ins**(**bd**$(r)) = \bar{r}$. ■

We are now ready to prove the main technical result of this paper.

***Proof of Theorem* 3.1.** Let **ins** be a uniform slope of order $k+1$ in R_G. We define the function **big** as follows. Let N be a noose of G with size $\leq k$. In the trivial case where $|N| \leq 1$, we define **big**(N) as the closed disk bounded by N and containing all the vertices of G. If $|N| \geq 2$ we observe that, as G is triangulated, Lemma 3.7 implies that N is the vibration of some of the cycles, say C of R_G. Observe that C has length $\leq 2k$. We then set **big**$(N) = \sigma^*_{C,N}(\overline{\Sigma - \mathbf{ins}(C)})$. Intuitively, we define **big** so that what is "interiors" according to **ins** becomes "exteriors" for **big** and vice versa.

We claim that the function **big** satisfies the majority axioms on G.

Proof of **[M1].** Let $S = (L_1, L_2, L_3)$ be a Θ-structure of size $\leq k$ where $L_3 \subseteq$ **big**$(L_1 \cup L_2)$. We will prove that **big**$(L_1 \cup L_3) \subseteq$ **big**$(L_1 \cup L_2)$ or **big**$(L_2 \cup L_3) \subseteq$ **big**$(L_1 \cup L_2)$. For this, we distinguish two cases.

Special case. $S = (L_1, L_2, L_3)$ is trivial. Notice that $L_i, i = 1,2,3$ has the same vertices, say x, y of G as endpoints. Also, from Lemma 3.2, $e = \{x,y\}$ is an edge of G. We will first prove the following claim.

Claim. *If $|L_i \cup L_j| = 2$ for some $i,j, 1 \leq i < j \leq 3$, then one, say Δ, of the closed discs bounded by $L_i \cup L_j$ contains all the vertices of G and* **big**$(L_i \cup L_j) = \Delta$ *(recall that $L_i \cup L_j$ is a noose and $|L_i \cup L_j|$ is the number of vertices it meets).*

Proof of Claim. The proof is based on the observation that **ins** maps face $\mathbf{r}_{\{x,y\}}$ of R_G to its "inside" disk. As $\bar{\mathbf{r}}_{x,y}$ is a vibration of $L_i \cup L_j$ then **big** will map it to its "ouside" disk that is Δ. We will now proceed with the formal proof.

The fact that G is triangulated and without multiple edges implies that G is 3-connected. Therefore, one of the closed discs, we denote it Δ, bounded by $L_i \cup L_j$ contains all the vertices of G. It remains to prove that $\mathbf{big}(L_i \cup L_j) = \Delta$.

By Lemma 3.7, the noose $L_i \cup L_j$, is a vibration of some cycle C of R_G. As $|L_i \cup L_j| = 2$, the only cycle of R_G with this property is the boundary of $\mathbf{r}_{\{x,y\}}$. By Lemma 3.14, $\mathbf{ins}(C) = \mathbf{ins}(\mathbf{bd}(\mathbf{r}_{\{x,y\}})) = \overline{\mathbf{r}}_{\{x,y\}}$. From the definition of $\mathbf{big}$, we have that $\mathbf{big}(L_i \cup L_j) = \sigma^*_{C,L_i \cup L_j}(\overline{\Sigma - \overline{\mathbf{r}}_{\{x,y\}}})$. Notice that $\overline{\Sigma - \overline{\mathbf{r}}_{\{x,y\}}} \cap V(G) = V(G)$, and Lemma 3.11 yields that $\sigma^*_{C,L_i \cup L_j}(\overline{\Sigma - \overline{\mathbf{r}}_{\{x,y\}}}) \cap V(G) = V(G)$, therefore $\mathbf{big}(L_i \cup L_j)$ should be equal to Δ and the claim holds.

We now distinguish the following subcases of the special case (recall the length of a line L_i is equal to the number of vertices it meets minus one).

Subcase 1. $|L_i| = 1, i = 1, 2, 3$. Applying the claim above, we have that for $i,j, 1 \leq i < j \leq 3$, $\mathbf{big}(L_i \cup L_j)$ is the closed disc bounded by $L_i \cup L_j$ and containing all the vertices of G. $L_3 \subseteq \mathbf{big}(L_1 \cup L_2)$ implies that either $L_2 - \{x, y\} \subseteq \Sigma - \mathbf{big}(L_1 \cup L_3)$ or $L_1 - \{x, y\} \subseteq \Sigma - \mathbf{big}(L_2 \cup L_3)$. Then either $\mathbf{big}(L_1 \cup L_3) \subseteq \mathbf{big}(L_1 \cup L_2)$, or $\mathbf{big}(L_2 \cup L_3) \subseteq \mathbf{big}(L_1 \cup L_2)$.

Subcase 2. $|L_i| = 1, i = 1, 2$ and $|L_3| = 2$. From Lemma 3.3, we have that $L_1 \cup L_3 \sim^* L_2 \cup L_3$. From the claim above, $\mathbf{big}(L_1 \cup L_2)$ is the closed disc bounded by $L_1 \cup L_2$ and containing all the vertices of G. Therefore, $\overline{\Sigma - \mathbf{big}(L_1 \cup L_2)} = \mathbf{dif}(L_1 \cup L_3, L_2 \cup L_3)$. We now assume that $\mathbf{big}(L_2 \cup L_3) \not\subseteq \mathbf{big}(L_1 \cup L_2)$. This can be rewritten as $\overline{\Sigma - \mathbf{big}(L_1 \cup L_2)} \not\subseteq \overline{\Sigma - \mathbf{big}(L_2 \cup L_3)}$, which implies that $\mathbf{dif}(L_1 \cup L_3, L_2 \cup L_3) \not\subseteq \overline{\Sigma - \mathbf{big}(L_2 \cup L_3)}$ and thus $\mathbf{dif}(L_1 \cup L_3, L_2 \cup L_3) \subseteq \mathbf{big}(L_2 \cup L_3)$. We now have

$$\begin{aligned}
\mathbf{big}(L_1 \cup L_3) &= \sigma_{L_2 \cup L_3, L_1 \cup L_3}(\mathbf{big}(L_2 \cup L_3)) \\
&= \overline{\mathbf{big}(L_2 \cup L_3) - \mathbf{dif}(L_1 \cup L_3, L_2 \cup L_3)} \\
&\subseteq \overline{\Sigma - \mathbf{dif}(L_1 \cup L_3, L_2 \cup L_3)} \\
&= \mathbf{big}(L_1 \cup L_2).
\end{aligned}$$

Subcase 3. $|L_1| = 2$ and $|L_i| = 1, i = 2, 3$. Observe that $L_3 \subseteq \mathbf{big}(L_1 \cup L_2)$ implies that $\mathbf{dif}(L_1 \cup L_2, L_1 \cup L_3) \subseteq \mathbf{big}(L_1 \cup L_2)$. Therefore,

$$\begin{aligned}
\mathbf{big}(L_1 \cup L_3) &= \sigma_{L_1 \cup L_2, L_1 \cup L_3}(\mathbf{big}(L_1 \cup L_2)) \\
&= \overline{\mathbf{big}(L_1 \cup L_2) - \mathbf{dif}(L_1 \cup L_2, L_1 \cup L_3)} \\
&\subseteq \mathbf{big}(L_1 \cup L_2).
\end{aligned}$$

Subcase 4. $|L_1| = 1$, $|L_2| = 2$, and $|L_3| = 1$. This case is symmetric to Case 3.

General Case. $S = (L_1, L_2, L_3)$ is non-trivial. Then, from Lemma 3.9, there exist a non-trivial Θ-structure (P_R^1, P_R^2, P_R^3) of G that is a vibration of S where

P_R^1, P_R^2, and P_R^3 are all paths of R_G. Lemma 3.12 implies that $P_3 \subseteq \mathbf{big}(P_1 \cup P_2)$. As $\mathbf{big}(P_1 \cup P_2)$ is a cycle of R_G, the definition of **big** implies that

$$P_3 \not\subseteq \mathbf{ins}(P_1 \cup P_2). \tag{3}$$

Suppose now that $\mathbf{big}(P_1 \cup P_3) \not\subseteq \mathbf{big}(P_1 \cup P_2)$ and $\mathbf{big}(P_2 \cup P_3) \not\subseteq \mathbf{big}(P_1 \cup P_2)$ and we will show that this assumption leads to a contradiction. As $P_i \cup P_j, 1 \leq i < j \leq 3$, are cycles of R_G, the definition of **big** implies that

$$\mathbf{ins}(P_1 \cup P_2) \not\subseteq \mathbf{ins}(P_2 \cup P_3) \text{ and} \tag{4}$$

$$\mathbf{ins}(P_1 \cup P_2) \not\subseteq \mathbf{ins}(P_1 \cup P_3). \tag{5}$$

From (3), (4), and (5) we have that $\mathbf{ins}(P_1 \cup P_2) \cup \mathbf{ins}(P_1 \cup P_3) \cup \mathbf{ins}(P_2 \cup P_3) = \Sigma$ and this is a contradiction to **[S2]**. Therefore, we get that

$$\mathbf{big}(P_1 \cup P_3) \subseteq \mathbf{big}(P_1 \cup P_2) \quad \text{or} \quad \mathbf{big}(P_2 \cup P_3) \subseteq \mathbf{big}(P_1 \cup P_2). \tag{6}$$

Applying now Lemma 3.13 on each of the relations of (6), we conclude that either $\mathbf{big}(L_1 \cup L_3) \subseteq \mathbf{big}(L_1 \cup L_2)$ or $\mathbf{big}(L_2 \cup L_3) \subseteq \mathbf{big}(L_1 \cup L_2)$.

***Proof of* [M2].** Let N be a noose in G where $|N| = 2$ (in the case where $|N| \leq 1$, [**M2**] follows from the bi-connectivity of G). By Lemma 3.7, there exists a cycle C of R_G such that $N \sim^* C$. By Lemma 3.2, there exists an edge $e = \{x, y\}$ such that $(x, y) = \kappa_G(N)$. Clearly, $C = \mathbf{bd}(\bar{\mathbf{r}}_e)$. By Lemma 3.14, $\mathbf{ins}(C) = \bar{\mathbf{r}}_e$ and thus, $\overline{\Sigma - \mathbf{ins}(C)} \cap V(G) = V(G)$. By Lemma 3.11, $\mathbf{big}(N) \cap V(G) = \sigma^*_{C,N}(\overline{\Sigma - \mathbf{ins}(C)}) \cap V(G) = V(G)$ and [**M2**] follows. ■

A consequence of Theorem 3.1 is the following.

Theorem 3.15. *For any planar graph G,* $\mathbf{bw}(G) \leq \sqrt{4.5|V(G)|} \leq 2.122\sqrt{|V(G)|}$.

Proof. We assume that G has no multiple edges (notice that the duplication of an edge does not increase the branch-width of a graph with branch-width ≥ 2). It is easy to see that G has a triangulation H without multiple edges. It is enough to prove the bound of the theorem for H. By Theorem 2.3, H does not have any majority of order $\geq (3/\sqrt{2})\sqrt{|V(G)|}$. By Theorem 3.1, R_H has no slope of order $\geq (3/\sqrt{2})\sqrt{|V(G)|} + 1$. The result now follows from Theorem 2.2. ■

Theorem 3.15 combined with Euler's formula and the Robertson and Seymour result on the branch-width of dual graphs implies the following.

Theorem 3.16. *For any planar graph G,* $\mathbf{bw}(G) \leq \frac{3}{2}\sqrt{|E(G)|} + 2$.

Proof. By the direct consequence of Robertson and Seymour's min–max Theorem (4.3) in [18] relating tangles and branch-width and Theorem (6.6) in

[20] establishing relations between tangles of dual graphs, we have that for any planar graph G of branch-width ≥ 2, the branch-width of G is equal to the branch-width of its dual. Thus

$$2 \cdot \mathbf{bw}(G) = \mathbf{bw}(G^*) + \mathbf{bw}(G).$$

Let G be a plane graph with n vertices, m edges, and f faces, and let G^* be the dual of G. By Euler's formula, $n - m + f = 2$.

By Theorem 3.15,

$$\mathbf{bw}(G) + \mathbf{bw}(G^*) \leq \sqrt{4.5n} + \sqrt{4.5f}.$$

Thus,

$$2 \cdot \mathbf{bw}(G) \leq \sqrt{4.5n} + \sqrt{4.5f}.$$

By Euler's formula,

$$\sqrt{4.5n} + \sqrt{4.5f} = \sqrt{4.5}(\sqrt{n + f + 2\sqrt{nf}}) \leq \sqrt{4.5}(\sqrt{2m + 4})$$

and we conclude that

$$\mathbf{bw}(G) \leq \frac{\sqrt{4.5}(\sqrt{2m+4})}{2} \leq \frac{3}{2}\sqrt{m} + 2. \qquad \blacksquare$$

Finally, Theorem 2.1 implies the following (notice that $9/(2\sqrt{2}) < 3.182$).

Theorem 3.17. *For any planar graph G,* $\mathbf{tw}(G) \leq 3.182\sqrt{|V(G)|}$.

4. ALGORITHMIC CONSEQUENCES

In this section, we discuss some applications of our results for different problems on planar graphs.

One of the fields for taking advantage of the current bounds on treewidth and branch-width of planar graphs is the design of parameterized algorithms. The last ten years were the evidence of rapid development of a new branch of computational complexity: Parameterized Complexity. (See the book of Downey and Fellows [11].) Roughly speaking, a parameterized problem with parameter k is *fixed parameter tractable* if it admits a solving algorithm with running time $f(k)|I|^{\beta}$. (Here f is a function depending only on k, $|I|$ is the length of the non-parameterized part of the input, and β is a constant.) Typically, $f(k) = c^k$ is an exponential function for some constant k. During the last two years, much attention was paid to the construction of parameterized algorithms with running time where $f(k) = c^{\sqrt{k}}$ for different problems on planar graphs. The first paper on the subject was the paper by Alber et al. [1] describing an algorithm with running time $O(4^{6\sqrt{34k}}n)$ (which is approximately $O(2^{70\sqrt{k}}n)$) for the Planar Dominating Set problem. Different fixed parameter algorithms for solving problems on planar and related graphs are discussed in [4,14]. Surprisingly, the obtained upper bounds on branch-width and tree-width provide much simpler algorithms with better proven worst case time analysis.

Let $\mathcal{L}$ be a parameterized problem, i.e., $\mathcal{L}$ consists of pairs (I, k) where k is the *parameter* of the problem. *Reduction to linear problem kernel* is the replacement of problem inputs (I, k) by a reduced problem with inputs (I', k') (linear kernel) with constants c_1, c_2 such that

$$k' \leq c_1 k,\ |I'| \leq c_2 k',\ \text{and}\ (I, k) \in \mathcal{L} \Leftrightarrow (I', k') \in \mathcal{L}.$$

(We refer to Downey and Fellows [11] for discussions on fixed parameter tractability and the ways of constructing kernels.)

Observation 1. Let $\mathcal{L}$ be a parameterized problem (G, k) (here G is a graph) such that

- There is a linear problem kernel G' computable in time $T_{\text{kernel}}(|V(G)|, k)$ with constants c_1, c_2 and such that an optimal branch decomposition of the kernel is computable in time $T_{bw}(|V(G')|)$.
- On graphs of branch-width $\leq \ell$ and ground set of size n the problem $\mathcal{L}$ can be solved in $O(2^{c_3 \ell} n)$, where c_3 is a constant.
- $\mathbf{bw}(G') \leq c_4\sqrt{k}$, where c_4 is a constant.

Then $\mathcal{L}$ can be solved in time $O(2^{c_3 c_4 \sqrt{k}} k + T_{bw}(|V(G')|) + T_{\text{kernel}}(|V(G)|, k))$.

Proof. The algorithm works as follows. First we compute a linear kernel in time $T_{\text{kernel}}(|V(G)|, k)$. Then we construct a branch decomposition of the kernel G' in $T_{bw}(|V(G')|)$ steps. The size of the kernel is at most $c_1 c_2 k = O(k)$. The branch-width of the kernel is at most $c_4\sqrt{k}$ and it takes $O(2^{c_3 c_4 \sqrt{k}} k + T_{bw}(|V(G')|) + T_{\text{kernel}}(|V(G)|, k))$ to solve the problem. ■

Let us give some examples, where Observation 1 provides *proven* better bounds for different parameterized problems.

Vertex cover. A *vertex cover* C of a graph is a set of vertices such that every edge of G has at least one endpoint in C. The Planar Vertex Cover problem is the task to compute, given a planar graph G and a positive integer k, a vertex cover of size k or to report that no such a set exists.

A linear problem kernel of size $2k$ (with constants $c_1 = 1$ and $c_2 = 2$) for the Vertex Cover problem (not necessary planar) was obtained by Chen et al. [9]. This result is based on the theoretical results of Nemhauser and Trotter [17] and Buss and Goldsmith [8]. The running time of the algorithm constructing a kernel of a graph on n vertices is $O(kn + k^3)$. So in this case, $T_{\text{kernel}}(|I|, k) = O(kn + k^3)$.

It is well known that the Vertex Cover problem on graphs on n vertices and with bounded tree-width $\leq \ell$ can be solved in $O(2^{\ell}n)$ time. The dynamic programming algorithm for the Vertex Cover on graphs with bounded tree-width can be easy translated to the dynamic programming algorithm for graphs with bounded branch-width with running time $O(2^{3/2\ell}m)$, where m is the number of edges in a graph, and we omit it here. For planar graphs, $2^{3/2\ell}m = O(2^{3/2\ell}n)$, thus $c_3 \leq 3/2$.

From the constructions used in the reduction algorithm of Chen et al. [9], it follows that if G is a planar graph, then the kernel graph is also planar. To compute an optimal branch decomposition of a planar graph, one can use the algorithm due to Seymour and Thomas (Sections 7 and 9 in [22]). This algorithm can be implemented in $O(k^4)$ steps. And what is important for practical applications, there is no *large hidden constants* in the running time of this algorithm.

The kernel graph I' has at most $2k$ vertices. Then by Theorem 3.15, $c_4 \leq \sqrt{4.5}\sqrt{2} = 3$. Thus by making use of Observation 1, we conclude that Planar Vertex Cover can be solved in $O(k^4 + 2^{4.5\sqrt{k}}k + kn)$. To our knowledge, this is the best, so far, time bound for this problem.

Dominating set. A *k-dominating set D* of a graph G is a set of k vertices such that every vertex outside D is adjacent to a vertex of D. The Planar Dominating Set problem is the task to compute, given a planar graph G and a positive integer k, a k-dominating set or to report that no such a set exists.

Alber, Fellows, and Niedermeier [2] show that the Planar Dominating Set problem admits a linear problem kernel. (The size of the kernel is $335k$.) This reduction can be performed in $O(n^3)$ time.

Dominating set problem on graphs of branch-width $\leq \ell$ can be solved in $O(2^{3\log_4 3\cdot\ell}m)$ steps [12]. Thus $c_3 \leq 3\log_4 3$.

What about the constant c_4 for the Planar Dominating Set problem? It is proved in [12] that for every planar graph G with dominating set k, the branch-width of G is at most $3\sqrt{4.5}\sqrt{k}$, i.e., $c_4 \leq 3\sqrt{4.5}$. Then by Observation 1, Planar Dominating set can be solved in $O(2^{15.13\sqrt{k}}k + n^3 + k^4)$, which improves any other time bound given before for this problem.

Exact Algorithms. It is well known that by making use of the well-known approach of Lipton and Tarjan [16], based on the celebrated planar separator theorem [15], one can obtain algorithms with time complexity $c^{O(\sqrt{n})}$ for many problems on planar graphs. However, the constants "hidden" in $O(\sqrt{n})$ can be crucial for practical implementations. During the last few years, a lot of work has been done to compute and to improve the "hidden" constants [3,4].

The branch-width based approach can be applied to planar graph problems as well: If a problem can be solved in $O(c^{\ell}n)$ on graphs of branch-width $\leq \ell$ for some constant c, we have that on planar graphs this problem can be solved in time $O(n^4 + c^{\sqrt{4.5n}}n)$. (One needs to construct a branch-decomposition of size $\leq \sqrt{4.5n}$ and apply dynamic programming.) Combining this simple idea with well-known dynamic programming techniques for graphs for bounded tree-width (branch-width), one can obtain sub-exponential solutions to many problems on planar graphs. For example, this approach can be used to obtain the fastest known (so far) algorithms on planar graphs for such problems like Independent Set (the running time of the algorithm is $O(2^{3.182\sqrt{n}}n + n^4))$ and Dominating Set (with running time $O(2^{5.043\sqrt{n}}n + n^4))$. This machinery not only improves the time bounds but also provides an unified approach for many exact algorithms emerging from the planar separator theorem of Lipton and Tarjan [15,16]. (See [13] for further details.)

5. DISCUSSION AND OPEN PROBLEMS

In this section, we present three open problems emerging from our main result and the methodology of our proof.

Improving the Constant 2.122. According to Theorem 3.15, any planar graph on n vertices has branch-width $\leq 2.122\sqrt{n}$. The constant 2.122 follows from the constant of Theorem 2.3 proven by Alon, Seymour, and Thomas in [6]. Any improvement of the constant of Theorem 2.3 implies also an improvement of our bound.

Given a graph G, a function $w : V(G) \rightarrow \mathbb{R}$, and a set $S \subseteq V(G)$, we call S $(2/3)$-*separator of* G if $V(G) - S$ can be partitioned into two sets A_1, A_2 where no edge of $E(G)$ has one endpoint in A_1 and the other in A_2 and such that $w(A_i) \leq 2/3w(V(G))$. If we strengthen the definition of a $(2/3)$-separator by asking that $w(A_i) + 1/2w(S) \leq 2/3w(V(G))$, we define the notion of a *strong* $(2/3)$-*separator of* G. If G is Σ-plane and there exists a noose N bounding the open discs D, D' such that $D \cap V(G) = A_1$, $D' \cap V(G) = A_2$, and $S = N \cap V(G)$, then we call S (strong) *cyclic* $(2/3)$-separator of G.

In [6], Alon, Robertson, and Thomas proved the following.

Theorem 5.1. *Let G be a Σ-plane graph with n vertices, let $w : V(G) \rightarrow \mathbb{R}$ be a function, and let $k \geq 0$ be an integer. If every majority of G has order $\leq k$, then G has a strong $(2/3)$-separator of G of size $\leq k$.*

Theorems 5.1 and 2.3 were proved in [6] in order to imply the following.

Theorem 5.2. *Let G be a Σ-plane graph with n vertices and let $w : V(G) \rightarrow \mathbb{R}$ be a function. Then G has strong cyclic $(2/3)$-separator of size $\leq 2.122\sqrt{n}$.*

Curiously, any proof of Theorem 5.2, for a better constant c, implies the reduction of the constant of Theorem 3.15 from 2.122 to $\max\{2, c\}$. Indeed, this is correct because of Theorems 2.2 and 3.15 and the following interesting result (Statement (3.9) of [6]).

Theorem 5.3. *Let G be a Σ-plane graph with n vertices, let $w : V(G) \rightarrow \mathbb{R}$ be a function, and let k be an integer where $k \geq 2\sqrt{n} - 1$. If G contains a strong $(2/3)$-cyclic separator of size $\leq k$, then every majority of G has order $\leq k$.*

In [10], Djidjev and Venkatesan proved that every Σ-plane graph on n vertices contains a *cyclic* $2/3$-*separator* of size $2\sqrt{n} + O(1)$. It is an interesting challenge to strengthen this result so that it guarantees the existence of a *strong* cyclic $(2/3)$-separator, as required by Theorem 5.2. This would make it possible to reduce to 2 the constant 2.122 of our main result (and to improve the time bounds of our algorithms).

Creating Slopes from Majorities. It is an interesting question whether the inverse of Theorem 3.1 holds for general graphs. In this direction, one should show that majorities can be "transformed" to slopes. As any cycle C of R_G is also a noose of G, we can directly define $\mathbf{ins}(C) = \overline{\Sigma - \mathbf{big}(C)}$, following the idea in the proof of Theorem 3.1 (notice that in this direction, the idea does not need the "vibration" machinery). Moreover it is possible to prove that the axiom [**M2**] for **big** implies the uniformity of **ins** and axiom [**M1**] for **big** implies axiom [**S2**] for **ins**. However, it is not easy to prove that axiom [**S1**] also holds for **ins**, and this is the main obstacle for any possible "translation" of majorities to slopes.

Constructive Upper Bounds. While Theorem 3.15 gives an upper bound to the branch-width of any planar graph, it does not provide any way to *construct* the corresponding branch decomposition. The "non-constructiveness" of our proof emerges from the fact that it makes strong use of the results in [6], [18], and [21] that are not (at least directly) "translatable" to a polynomial time algorithm. However, the algorithmic results of [21] make it possible to construct, for any n-vertex planar graph, a branch decomposition of width $\leq 2.122\sqrt{n}$ in time $O(n^4)$ and such a branch decomposition can be easily transformed to a tree decomposition of width $\leq 3.128\sqrt{n}$ using the results of [19]. It is an open problem, whether Theorems 3.15 and 3.17 can admit simpler proofs implying faster algorithms for the construction of the corresponding decompositions. Robin Thomas (in private communication) mentioned that by adapting the arguments from Alon, Seymour, and Thomas paper [6], one can obtain similar bounds on branch-width/tree-width. Perhaps this can bring us to faster algorithms.

ACKNOWLEDGMENTS

We are grateful to Hans L. Bodlaender, Ton Kloks, and Robin Thomas for answering to our questions.

REFERENCES

[1] J. Alber, H. L. Bodlaender, H. Fernau, T. Kloks, and R. Niedermeier, Fixed parameter algorithms for dominating set and related problems on planar graphs, Algorithmica 33 (2002), 461–493.

[2] J. Alber, M. R. Fellows, and R. Niedermeier, Polynomial-time data reduction for dominating set, J ACM 51 (2004), 363–384.

[3] J. Alber, H. Fernau, and R. Niedermeier, Graph separators: A parameterized view, J Comput System Sci 67 (2003), 808–832.

[4] J. Alber, H. Fernau, and R. Niedermeier, Parameterized complexity: Exponential speed-up for planar graph problems, J Algorithms 52 (2004), 26–56.

[5] N. Alon, P. Seymour, and R. Thomas, A separator theorem for nonplanar graphs, J Am Math Soc 3 (1990), 801–808.

[6] N. Alon, P. Seymour, and R. Thomas, Planar separators, SIAM J Discrete Math 7 (1994), 184–193.

[7] H. L. Bodlaender, A partial *k*-arboretum of graphs with bounded treewidth, Theor Comp Sci 209 (1998), 1–45.

[8] J. F. Buss and J. Goldsmith, Nondeterminism within P, SIAM J Comput 22 (1993), 560–572.

[9] J. Chen, I. A. Kanj, and W. Jia, Vertex cover: Further observations and further improvements, J Algorith 41 (2001), 280–301.

[10] H. N. Djidjev and S. M. Venkatesan, Reduced constants for simple cycle graph separation, Acta Informatica 34 (1997), 231–243.

[11] R. G. Downey and M. R. Fellows, Parameterized complexity, Springer-Verlag, New York, 1999.

[12] F. V. Fomin and D. M. Thilikos, Dominating sets in planar graphs: Branch-width and exponential speed-up, in Proceedings of the 14th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2003), 2003, pp. 168–177.

[13] F. V. Fomin and D. M. Thilikos, A simple and fast approach for solving problems on planar graphs, in Proceedings of the 21st International Symposium on Theoretical Aspects of Computer Science (STACS 2004), Vol. 2996 of LNCS, Springer, Berlin, 2004, pp. 56–67.

[14] I. Kanj and L. Perković, Improved parameterized algorithms for planar dominating set, in Proceedings of the 27th International Symposium on Mathematical Foundations of Computer Science (MFCS 2002), Vol. 2420 of LNCS, Springer, Berlin, 2002, pp. 399–410.

[15] R. J. Lipton and R. E. Tarjan, A separator theorem for planar graphs, SIAM J Appl Math 36 (1979), 177–189.

[16] R. J. Lipton and R. E. Tarjan, Applications of a planar separator theorem, SIAM J Comput 9 (1980), 615–627.

[17] G. L. Nemhauser and L. E. Trotter, Jr., Vertex packings: Structural properties and algorithms, Math Program 8 (1975), 232–248.

[18] N. Robertson and P. D. Seymour, Graph minors. X. Obstructions to tree-decomposition, J Combin Theory Ser B 52 (1991), 153–190.

[19] N. Robertson and P. D. Seymour, Graph minors. XI. Circuits on a surface, J Combin Theory Ser B 60 (1994), 72–106.

[20] N. Robertson and P. D. Seymour, Graph minors. XI. Circuits on a surface, J Combin Theory Ser B 60 (1994), 72–106.

[21] N. Robertson, P. D. Seymour, and R. Thomas, Quickly excluding a planar graph, J Combin Theory Ser B 62 (1994), 323–348.

[22] P. D. Seymour and R. Thomas, Call routing and the ratcatcher, Combinatorica 14 (1994), 217–241.

[23] R. Thomas, *Tree-decompositions of graphs*. http://www.math.gatech.edu/˜thomas/SLIDE/slide2.ps, p. 32.