Variants of Plane Diameter Completion*†

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Abstract

The Plane Diameter Completion problem asks, given a plane graph G and a positive integer d, if it is a spanning subgraph of a plane graph H that has diameter at most d. We examine two variants of this problem where the input comes with another parameter k. In the first variant, called BPDC, k upper bounds the total number of edges to be added and in the second, called BFPDC, k upper bounds the number of additional edges per face. We prove that both problems are NP-complete, the first even for 3-connected graphs of face-degree at most 4 and the second even when $k = 1$ on 3-connected graphs of face-degree at most 5. In this paper we give parameterized algorithms for both problems that run in $O(n^3) + 2^{2^{O((kd)^2 \log d)}} \cdot n$ steps.

1 Introduction

In 1987, Chung [3, Problem 5] introduced the following problem: find the optimum way to add q edges to a given graph G so that the resulting graph has minimum diameter. (Notice that in all problems defined in this paper we can directly assume that G is a simple graph as loops do not contribute to the diameter of a graph and the same holds if we take simple edges instead of multiple ones.) This problem was proved to be NP-hard if the aim is to obtain a graph of diameter at most 3 [19], and later the NP-hardness was shown even for the Diameter-2 Completion problem [14]. It is also know that Diameter-2 Completion is W[2]-hard when parameterized by q [9].

For planar graphs, Dejter and Fellows introduced in [5] the Planar Diameter Completion problem that asks whether it is possible to obtain a planar graph of

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diameter at most d from a given planar graph by edge additions. It is not known whether PLANAR DIAMETER COMPLETION admits a polynomial time algorithm, but Dejter and Fellows showed that, when parameterized by d, PLANAR DIAMETER COMPLETION is fixed parameter tractable [5]. The proof is based on the fact that the YES-instances of the problem are closed under taking minors. Because of the Robertson and Seymour theorem [18] and the algorithm in [16], this implies that, for each d, the set of graphs G for which (G, d) is a YES-instance can be characterized by a *finite* set of forbidden minors. This fact, along with the minor-checking algorithm in [17] implies that there exists an $O(f(d) \cdot n^3)$-step algorithm (i.e. an FPT-algorithm) deciding whether a plane graph G has a plane completion of diameter at most d. Using the parameterized complexity, this means that PLANAR DIAMETER COMPLETION is FPT, when parameterized by d. To make this result constructive, one requires the set of forbidden minors for each d, which is unknown. To find a constructive FPT-algorithm for this parameterized problem remains a major open problem in parameterized algorithm design.

Our results. We denote by $\mathbb{S}_0$ the 3-dimensional sphere. By a *plane* graph G we mean a simple planar graph G with the vertex set $V(G)$ and the edge set $E(G)$ drawn in $\mathbb{S}_0$ such that no two edges of this embedding intersect. A plane graph H is a *a plane completion* (or, simply *completion*) of another plane graph G if H is a spanning subgraph of G. A *q-edge completion* of a plane graph G is a completion H of G where $|E(H)| - |E(G)| \leq q$. A *k-face completion* of a plane graph G is a completion H of G where at most k edges are added in each face of G.

In this paper we consider the variants of the PLANE DIAMETER COMPLETION problem:

> PLANE DIAMETER COMPLETION (PDC)
> *Input*: a plane graph G and $d \in \mathbb{N}_{\geq 1}$.
> *Output*: is there a completion of G with diameter at most d?

Notice that the important difference between PDC and the aforementioned problems is that we consider plane graphs, i.e., the aim is to reduce the diameter of a given embedding of a planar graph preserving the embedding. In particular we are interested in the following variants:

> BOUNDED BUDGET PDC (BPDC)
> *Input*: a plane graph G and $q \in \mathbb{N}, d \in \mathbb{N}_{\geq 1}$
> *Question*: is there a completion H of G of diameter at most d that is also a q-edge completion?

> BOUNDED BUDGET/FACE PDC (BFPDC)
> *Input*: a plane graph G and $k \in \mathbb{N}, d \in \mathbb{N}_{\geq 1}$.
> *Question*: is there a completion H of G of diameter at most d that is also a k-face completion?

We examine the complexity of the two above problems. Our hardness results are the following.

Theorem 1. *Both* BPDC *and* BFPDC *are* NP*-complete. Moreover,* BPDC *is* NP*-complete even for 3-connected graphs of face-degree at most 4, and* BFPDC *is* NP*-complete even for $k = 1$ on 3-connected graphs of face-degree at most 5.*

The hardness results are proved in Section 6 using a series of reductions departing from the PLANAR 3-SATISFIABILITY problem that was shown to be NP-hard by Lichtenstein in [15].

The results of Theorem 1 prompt us to examine the parameterized complexity[1] of the above problems. For this, we consider the following general problem:

BOUNDED BUDGET AND BUDGET/FACE BDC (BBFPDC)
Input: a plane graph G, $q \in \mathbb{N} \cup \{\infty\}$, $k \in \mathbb{N}$, and $d \in \mathbb{N}_{\geq 1}$.
Question: is there a completion H of G of diameter at most d that is also a q-edge completion and a k-face completion?

Notice that when $q = \infty$ BBFPDC yields BFPDC and when $q = k$ BBFPDC yields BPDC. Our main result is that BBFPDC is fixed parameter tractable (belongs in the parameterized class FPT) when parameterized by k and d.

Theorem 2. *It is possible to construct an $O(n^3) + 2^{2^{O((kd)\log d)}} \cdot (\alpha(q))^2 \cdot n$-step algorithm for* BBFPDC.

(In the above statement and in the rest of this paper we use the function $\alpha : \mathbb{N} \cup \{\infty\} \to \mathbb{N}$ such that if $q = \infty$, then $\alpha(q) = 1$, otherwise $\alpha(q) = q$.)

The main ideas of the algorithm of Theorem 2 are the following. We first observe that YES-instances of PDC and all its variants have bounded branchwidth (for the definition of branchwidth, see Section 2). The typical approach in this case is to derive an FPT-algorithm by either expressing the problem in Monadic Second Order Logic – MSOL (using Courcelle's theorem [4]) or to design a dynamic programming algorithm for this problem. However, for completion problems, this is not really plausible as this logic can quantify on *existing* edges or vertices of the graph and not on the "non-existing" completion edges. This also indicates that to design a dynamic programming algorithm for such problems is, in general, not an easy task. In this paper we show how to tackle this problem for BBFPDC (and its special cases BPDC and BFPDC). Our approach is to deal with the input G as a part of a more complicated graph with $O(k^2 \cdot n)$ additional edges, namely its *cylindrical enhancement* G' (see Section 3 for the definition). Informally, sufficiently large cylindrical grids are placed inside the faces of G and then internally vertex disjoint paths in these grids can be used to emulate the edges of a solution of the original problem placed inside the corresponding faces. Thus, by the enhancement we reduce BBFPDC to a new problem on G' certified by a suitable 3-partition of the additional edges. Roughly, this partition consists of the 1-weighted edges that should be added in the completion, the 0-weighted edges that should link these edges to the boundary of the face of G where they will be

[1]For more on parameterized complexity, we refer the reader to [8].

inserted, and the ∞-weighted edges that will be the (useless) rest of the additional edges. The new problem asks for such a partition that simulates a bounded diameter completion. The good news is that, as long as the number of edges per face to be added is bounded, which is the case for BBFPDC, the new graph G' has still bounded branchwidth and it is possible, in the new instance, to quantify this 3-partition of the graph G'. However, even under these circumstances, to express the new problem in Monadic Second Order Logic is not easy. For these reasons we decided to follow the more technical approach of designing a dynamic programming algorithm that leads to the (better) complexity bounds of Theorem 2. This algorithm is quite involved due to the technicalities of the translation of the BBFPDC to the new problem. It runs on a sphere-cut decomposition of the plane embedding of G' and its tables encode how a partial solution is behaving inside a closed disk whose boundary meets only (a few of) the edges of G'. We stress that this encoding takes into account the topological embedding and not just the combinatorial structure of G'. Sphere-cut decompositions as well as some necessary combinatorial structures for this encoding are presented in Section 4. The dynamic programming algorithms is presented in Section 5 and is the most technical part of this paper.

2 Definitions and preliminaries

Given a graph G, we denote by $V(G)$ (respectively $E(G)$) the *set of vertices* (respectively *edges*) of G. A graph G' is a *subgraph* of a graph G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$, and we denote this by $G' \subseteq G$. Also, in case $V(G) = V(G')$, we say that H is a *spanning subgraph* of G. If S is a set of vertices or a set of edges of a graph G, the graph $G \setminus S$ is the graph obtained from G after the removal of the elements of S. If S is a set of edges, we define $G[E]$ as the graph whose vertex set consists of the endpoints of the edges of E and whose edge set of E.

Distance and diameter. Let G be a graph and let $\mathsf{w} : E(G) \to \mathbb{N} \cup \{\infty\}$ (w is a *weighting of the edges of* G). Given two vertices $x, x' \in V(G)$ we call (x, x')*-path* every path of G with x and x' as endpoints. We also define $\mathsf{w}\text{-}\mathbf{dist}_G(x, x') = \min\{\mathsf{w}(E(P)) \mid P \text{ is an } (x, x')\text{-path in } G\}$ and $\mathsf{w}\text{-}\mathbf{diam}(G) = \max\{\mathsf{w}\text{-}\mathbf{dist}_G(x, y) \mid x, y \in V(G)\}$ (if G is not connected then $\mathsf{w}\text{-}\mathbf{diam}(G)$ is infinite). When the graph is unweighted then we use $\mathbf{dist}_G$ and $\mathbf{diam}$ instead of $\mathsf{w}\text{-}\mathbf{dist}_G$ and $\mathsf{w}\text{-}\mathbf{diam}$.

Plane graphs. To simplify notations on plane graphs, we consider a plane graph G as the union of the points of $\mathbb{S}_0$ in its embedding corresponding to its vertices and edges. That way, a subgraph H of G can be seen as a graph H where $H \subseteq G$. The *faces* of a plane graph G, are the connected components of the set $\mathbb{S}_0 \setminus G$. A vertex v (an edge e resp.) of a plane graph G is *incident* to a face f and, vice-versa, f is incident to v (resp. e) if v (resp., e) lies on the boundary of f. Two faces f_1, f_2 are *adjacent* if they have a common incident edges. We denote by $F(G)$ the set of all faces of G. The *degree* of a face $f \in F(G)$ is the number of edges incident to f where bridges of G

count double in this number. The *face-degree* of G is the maximum degree of a face in $F(G)$. Given a face f of G, we define $B_G(f)$ as the graph whose set of points is the boundary of f and whose vertices are the vertices incident to f.

A set $\Delta \subseteq \mathbb{S}_0$ is an open disc if it is homeomorphic to $\{(x, y) : x^2 + y^2 < 1\}$. Also, Δ is a *closed disk* of $\mathbb{S}_0$ if it is the closure of some open disk of $\mathbb{S}_0$.

Branch decomposition. Given a graph H with n vertices, a branch decomposition of H is a pair (T, μ), where T is a tree with all internal vertices of degree three and $\mu : L \to E(H)$ is a bijection from the set of leaves of T to the edges of H. For every edge e of T, we define the middle set $\mathbf{mid}(e) \subseteq V(H)$ as follows: if $T \setminus \{e\}$ has two connected components T_1 and T_2, and for $i \in \{1, 2\}$, let $H_i^e = H[\{\mu(f) : f \in L \cap V(T_i)]$, and set $\mathbf{mid}(e) = V(H_1^e) \cap V(H_2^e)$.

The width of (T, μ) is the maximum order of the middle sets over all edges of T, i.e. $\max\{|\mathbf{mid}(e)| : e \in T\}$. The *branchwidth* of H is the minimum width of a branch decomposition of H and is denoted by $\mathbf{bw}(H)$.

Grid-annulus. Let k and r be positive integers where $k \in \mathbb{N}_{\geq 3}, r \in \mathbb{N}_{\geq 3}$. We define the graph $\Gamma_{k,r}$ as the $(k \times r)$-*grid annulus*, which is the Cartesian product of a path of k vertices and a cycle of r vertices. Notice that $\Gamma_{k,r}$ is uniquely embeddable (up to homeomorphism) in the plane and has exactly two non-square faces (i.e., faces incident to 4 edges) f_1 and f_2 that are incident only with vertices of degree 3. We call one of the faces f_1 and f_2 the *interior* of $\Gamma_{k,r}$ and the other the *exterior* of $\Gamma_{k,r}$. We call the vertices incident to the interior (exterior) of $\Gamma_{k,r}$ *base* (*roof*) of $\Gamma_{k,r}$. Given an edge e in the base of $\Gamma_{k,r}$, we define its *ceilings* as the set of edges of $\Gamma_{k,r}$ that contains e and whose dual edges in $\Gamma_{k,r}^*$ form a minimum length path between the duals of the interior and the exterior face of $\Gamma_{k,r}^*$.

We need the following result.

Proposition 1 ([11])**.** *Let G be a planar graph and r, k be integers with $r \geq 3$ and $k \geq 1$. Then G has either a minor isomorphic to $\Gamma_{r,k}$ or a branch decomposition of width at most $r + 2k - 2$.*

An central feature of the PDC problem and its variants is that its YES-instances have bounded branchwidth.

Lemma 1. *There exists a constant c_1 such that if (G, d) is a* YES*-instance of* PDC*, then $\mathbf{bw}(G) \leq c_1 \cdot d$. The same holds for the graphs in the* YES*-instances of* BPDC*,* BFPDC*, and* BBFPDC*.*

Proof. We examine only the case of PDC as a YES-instance of BPDC, BFPDC, and BBFPDC is also a YES-instance of PDC.

Notice first that if G has a completion of diameter at most d and G' is a minor[2] of some G, then also G' has a completion H of diameter at most d. Notice also

[2]A graph G' is a minor of a graph G if it can obtained applying edge contractions to some subgraph of G.

that every completion of the grid annulus $\Gamma_{r+2,r+2}$ has diameter $> r$, therefore, if (G,d) is a YES-instance of PDC, then G cannot contain a $\Gamma_{r+2,r+2}$ as a minor. From Proposition 1, G has branchwidth bounded by a linear function of d and the lemma follows. $\square$

3 The reduction

3.1 Cylindrical enhancements

Cylindrical enhancement of a plane graph. Let G be a plane graph. We next give the definition of the graph $G^{(k)}$ for $k \in \mathbb{N}_{\geq 3}$. Let $f_i \in F(G)$ and let $C_1^i, \ldots, C_{\rho_i}^i$ be the connected components of $B_G(f_i)$. For each C_j^i, we denote by σ_j^i the number of its edges, agreeing that, in this number, bridge edges count twice and that if C_j^i consists of only one vertex, then $\sigma_j^i = 1$. We then add a copy Γ_j^i of $\Gamma_{k,k\cdot\sigma_j^i}$ in the embedding of G such that C_j^i is contained in the interior of Γ_j^i and all $C_1^i, \ldots, C_{j-1}^i, \ldots, C_{j+1}^i \ldots, C_{\rho_i}^i$ are contained in the exterior of Γ_j^i (In Figure 1 the edges of each Γ_j^i are colored red). We then add, for each $v \in C_j^i$, $\kappa(v)\cdot k$ edges (those around the disks $C_1, \ldots, C_4$ in Figure 1) from v to the base of Γ_j^i, where $\kappa(v)$ is the number of connected components in $C_j^i \setminus v$ (in the trivial case where C_j^i consists of only one vertex v, then $\kappa(v) = 1$). We add these edges in a way that the resulting embedding remains plane and no more than a set $V_{v,i,j}$ of k consecutive vertices of the base of C_j^i are connected with the same vertex v of C_j^i; observe that there is only one way to add edges so to fulfill these restrictions. Notice that the set $V_{v,i,j}$ always induces a path $P_{v,i,j}$ in the resulting graph except in the case where C_j^i consists of a single vertex v where $V_{v,i,j}$ induces a cycle. In the later case we pick a maximal path in this cycle and we denote it by $P_{v,i,j}$. In the example of Figure 1 the $P_{v,i,j}$'s are the bold paths of the innermost cycle of each Γ_j^i. We apply this enhancement for each connected component of the boundary of each face of G and we denote the resulting graph by $R_G^{(k)}$.

We call a face f_i of $R_G^{(k)}$ non-trivial if $B_{R_G^{(k)}}(f_i)$ has more than one connected components $C_1^i, \ldots, C_{\rho_i}^i$. Notice that if f_i is non-trivial, each C_j^i is the roof of some previously added grid-annulus. For each such grid-annulus, let J_j be k consecutive vertices of its roof. We add inside f_i a copy of $\Gamma_{k,k\cdot\rho_i}$ such that its base is a subset of f_i and let $\{I_1, \ldots, I_{\rho_i}\}$ be a partition of its roof in ρ_i parts, each consisting of k consecutive base vertices. In the example of Figure 1, the annulus $\Gamma_{k,k\cdot\rho_i}$ is the one with the edges in the middle of the figure and its base is its innermost cycle. For each $j \in \{1, \ldots, r_i\}$ we add k edges (depicted as the "interconnecting" edges in Figure 1) each connecting a vertex of J_j with some vertex of I_j in a way that the resulting embedding remains plane (again, there is a unique way for this to be done). We apply this enhancement for each non-trivial face of $R_G^{(k)}$ and we denote the resulting graph by $G^{(k)}$. Notice that $G^{(k)}$ is not uniquely defined as its definition depends on the choice of the sets J_j. From now on, we always consider an arbitrary choice for $G^{(k)}$ and we call

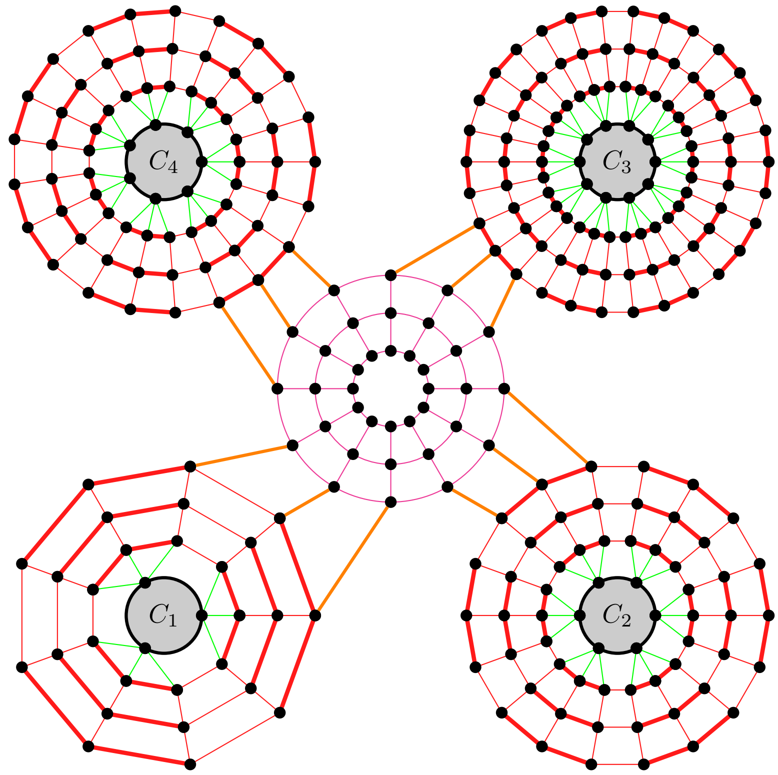

Figure 1: An example of a cylindrical enhancement for $k = 3$ inside a non-trivial face of a graph with 4 connected components (i.e., the boundaries of the disks $C_1, \ldots, C_4$).

$G^{(k)}$ *the k-th cylindrical enhancement of G*. By the construction of $G^{(k)}$, it directly follows that $|V(G^{(k)})| = O(k^2 \cdot n)$. We say that an edge of $G^{(k)}$ is an *expansion edge* if it is an edge of $P_{v,i,j}$ for some i, j, and $v \in V(C_{i,j})$. Also we denote by $\bar{G}^{(k)}$ the graph created by $G^{(k)}$ if we contract all its expansion edges and all their ceilings of the grid-annuli that were added during the construction of $R_G^{(k)}$.

Primal-dual drawings. Let G be a connected plane graph. We denote by $D(G)$ the graph obtained if we draw G together with its dual so that dual edges are intersecting to a single point and then introduce a vertex to each of these intersection points. We recursively define $D^{(k)}(G)$ such that $D^{(0)}(G) = G$ and $D^{(k)}(G) = D^{k-1}(D(G))$ for every $k \geq 1$. The next proposition is a direct consequence of [13, Lemma 4].

Proposition 2. *There exists some constant c such that for every connected plane graph G, it holds that $\mathbf{bw}(D(G)) \leq 2 \cdot \mathbf{bw}(G)$.*

Corollary 1. *For every connected plane graph G and $k \in \mathbb{N}_{\geq 1}$, it holds that $\mathbf{bw}(D^{(k)}(G)) \leq 2^k \cdot \mathbf{bw}(G)$.*

Lemma 2. *If G is a connected plane graph and $k \in \mathbb{N}_{\geq 3}$, then $\bar{G}^{(k)}$ is a minor of $D^{(\lceil \log(k+1) \rceil + 1)}(G)$.*

Proof. Notice first that $\bar{G}^{(3)}$ is a minor of $D^{(3)}(G)$. It is then enough to observe that for every $i \geq 3$, if $\bar{G}^{(i)}$ is a minor of $D^{(i)}(G)$, then $\bar{G}^{(2i+1)}$ is a minor of $D^{(i+1)}(G)$. □

The following lemma indicates that cylindrical enhancements do not considerably increase the branchwidth of a graph.

Lemma 3. *There is a constant c_2 such that if G is an n-vertex plane graph and $k \in \mathbb{N}_{\geq 3}$, then $G^{(k)}$ is 3-connected, $\mathbf{bw}(G^{(k)}) \leq c_2 \cdot k \cdot \mathbf{bw}(G)$.*

Proof. Let H be the graph created from G if we add a vertex v_f to each non trivial face f and for each of the connected components of $B_G(f)$, we arbitrarily pick a vertex and make it adjacent to v_f by a path of $2k$ internal vertices. As the branchwidth of a non-acyclic graph is the maximum branchwidth of its connected components, it follows that $\mathbf{bw}(H) = \mathbf{bw}(G)$. It is also easy to see that $G^{(k)}$ is a minor of $\bar{H}^{(k)}$. From Lemma 2, $\bar{H}^{(k)}$ is a minor of $D^{(r)}(H)$, where $r = \lceil \log(k+1) \rceil + 1$. By Corollary 1, it follows that $\mathbf{bw}(D^{(r)}(H)) \leq 2^r \cdot \mathbf{bw}(H) = O(k \cdot \mathbf{bw}(G))$. □

3.2 Edge colorings of new edges.

Let G and H be two plane graphs such that G is a subgraph of H and let $q \in \mathbb{N} \cup \{\infty\}$, $k \in \mathbb{N}$, and $d \in \mathbb{N}_{\geq 1}$. Given a 3-partition $\mathbf{p} = \{E^0, E^1, E^\infty\}$ of $E(H) \setminus E(G)$, we define the function $\mathsf{w}_\mathbf{p} : E(H) \to \mathbb{N}$ such that

$$\begin{aligned} \mathsf{w}_\mathbf{p} &= \{(e,1) \mid e \in E(G)\} \cup \{(e,0) \mid E \in E^0\} \cup \\ &\quad \{(e,1) \mid e \in E^1\} \cup \{(e, d+1) \mid E \in E^\infty\}. \end{aligned}$$

We say that G has (q,k,d)-*extension* in H if there is a 3-partition $\mathbf{p} = \{E^0, E^1, E^\infty\}$ of $E(H) \setminus E(G)$ such that the following conditions hold

A. There is no path in H with endpoints in $V(G)$ that consists of edges in E^0,

B. every face F of G contains at most k edges of E^1,

C. $\forall x, y \in V(G), \mathsf{w}_\mathbf{p}\text{-}\mathbf{dist}_H(x,y) \leq d$, and

D. $|E^1| \leq q$.

Given a 3-partition $\mathbf{p} = \{E^0, E^1, E^\infty\}$ of $E(H) \setminus E(G)$ we refer to its elements as the *0-edges*, the *1-edges*, and the ∞-*edges* respectively. We also call the edges of G *old-edges*.

Our first step towards our algorithm is to reduce BBFPDC to a problem about (q,k,d)-extensions of G.

Given a plane graph G and an open set Λ of $\mathbb{S}_0$, we define $G\langle\Lambda\rangle$ as the graph whose edge set consists of the edges of G that are subsets of Λ and whose vertex set consists of their endpoints.

Disjoint paths. Let G be a graph. We say that two paths in G are *disjoint* if none of the internal vertices of a path is a vertex of the other. Given a collection $\mathcal{P}$ of pairwise disjoint paths of G, we define $L(\mathcal{P}) = \{\{x, y\} \mid x \text{ and } y \text{ are the endpoints of a path in } \mathcal{P}\}$.

The proofs of the following proposition can be found in [2].

Proposition 3. *Let G be a graph $k \in \mathbb{N}_{\geq 1}$ and let H be a k-face completion of G. For every face $f \in F(G)$, there is a collection $\mathcal{P}$ of k disjoint paths in the graph $G^{(\max\{3,k\})}\langle f \rangle$ such that $E(G\langle f \rangle) = L(\mathcal{P})$.*

Lemma 4. *Let G be a plane graph, with $q \in \mathbb{N} \cup \{\infty\}$, $k \in \mathbb{N}_{\geq 1}$ and $d \in \mathbb{N}_{\geq 1}$. Then (G, q, k, d) is a* YES*-instance of* BBFPDC *if and only if G has a (q, k, d)*-extension *in $G^{(\max\{3,k\})}$.*

Proof. Assume first that (G, q, k, d) is a YES-instance of BBFPDC and let H be a completion H of G of diameter at most d that is also a q-edge completion and a k-face completion. This means that for every $f \in F(G)$, the graph $H_f = H\langle f \rangle$ contains at most k edges and that the graph $H^{\text{new}} = \bigcup_{f \in F(H)} H_f$ contains at most q edges. From Proposition 3, there is a collection $\mathcal{P}_f$ of $y_f = |E(H_f)|$ internally disjoint paths in $G^{(\max\{3,k\})}$. Let E^1 be a set of $y = \sum_{f \in F(G)} y_f$ edges obtained if, for every $f \in F(G)$, we pick one edge from each of the paths in $\mathcal{P}_f$. Let $E^0 = E(\bigcup_{f \in F(G)} \bigcup_{P \in \mathcal{P}_f}) \setminus E^1$ and let $E^\infty = E(H^{\text{new}}) \setminus (E^0 \cup E^1)$. We now observe that $\mathbf{p} = \{E^0, E^1, E^\infty\}$ is a 3-partition of $E(H^{\text{new}}) = E(G^{(\max\{3,k\})}) \setminus E(G)$. By its construction, $\mathbf{p}$ satisfies conditions 1–4 of the definition of a (q, k, d)-extension of G in $G^{(\max\{3,k\})}$ as required.

Let now $\mathbf{p} = \{E^0, E^1, E^\infty\}$ is a 3-partition of $E(H^{\text{new}}) = E(G^{(\max\{3,k\})}) \setminus E(G)$ that is a (q, k, d)-extension of G in $G^{(\max\{3,k\})}$. We construct the graph H by removing from $G^{(\max\{3,k\})}$ all edges in E^∞ and then, in the resulting graph, contract all edges in E^0. It is easy to observe that H is a completion of G that is also an q-edge completion and a k-face completion □

4 Structures for dynamic programming

For our dynamic programming algorithm we need a variant of branchwidth for plane graphs whose middle sets have additional topological properties.

Sphere-cut decomposition. Let H be a plane graph. An arc is a subset O of the plane homeomorphic to a circle and is called a *noose of H* if it meets H only in vertices. We also set $V_O = V(H) \cap O$. An *arc* of a noose O is a connected component of $O \setminus V_O$ while in the trivial case where $V_O = \emptyset$, O does not have arcs. A *sphere-cut decomposition* or *sc-decomposition* of H is a triple (T, μ, π) where (T, μ) is a branch decomposition of H and π is a function mapping each $e \in E(T)$ to cyclic orderings of vertices of H, such that for every $e \in E(T)$ there is a noose O_e of H where the following properties are satisfied.

- O_e meets every face of H at most once,

- H_1^e is contained in one of the closed disks bounded by O_e and H_2^e is contained in the other (H_1^e and H_2^e are as in the definition of branch decomposition).
- $\pi(e)$ is a cyclic ordering of V_{O_e} defined by a clockwise traversal of O_e in the embedding of H.

We denote $X_e = V_{O_e}$ and we always assume that its vertices are clockwise enumerated according to $\pi(e)$. We denote by $\mathbf{A}_e$ the set containing the arcs of O_e. Also, if $\pi(e) = [a_1, \ldots, a_k, a_1]$, then we use the notation $\mathbf{A}_e = \{a_{1,2}, a_{2,3}, \ldots, a_{k-1,k}, a_{k,1}\}$ where the boundary of the arc $a_{i,i+1}$ consists of the vertices a_i and a_{i+1}. We also define $H_e^+ = (V(H), E(H \cup \mathbf{A}_e))$, i.e., H_e^+ is the embedding occurring if we add in H the arcs of O_e as edges. A face of H_e^+ is called *internal* if it is not incident to an arc in $\mathbf{A}_e$, i.e., it is also a face of H. A face of H_e^+ is *marginal* if it is a properly included is some face of H.

For our dynamic programming we require to have in hand an optimal sphere-cut decomposition. This is done combining the main result of [10] and [20, (5.1)] (see also [7]) and is summarized to the following.

Proposition 4. *There exists an algorithm that, with input a 3-connected plane graph G and $w \in \mathbb{N}$, outputs a sphere-cut decomposition of G of width at most w or reports that* $\mathbf{bw}(G) > w$.

Our next step is to define a series of combinatorial structures that are necessary for our dynamic programming. Given two sets A and B we denote by A^B the set of all functions from B to A.

(d, k, q)-configurations. Given a set X and a non-negative integer t, we say that the pair $(\mathcal{X}, \chi)$ is a *t-labeled partition of X* if $\mathcal{X}$ is a collection of pairwise disjoint non-empty subsets of X and χ is a function mapping the integers in $\{1, \ldots, |\mathcal{X}|\}$ to integers in $\{0, \ldots, t\}$. In case $X = \emptyset$, a *t-labeled partition* corresponds to the pair $\{\emptyset, \varnothing\}$ where $\varnothing$ is the "empty" function, i.e. the function whose domain is empty. Let X and A be two finite sets. Given $d, k \in \mathbb{N}$ and $q \in \mathbb{N} \cup \{\infty\}$, we define a *$(d, k, q)$-configuration* of (X, A) as a quintuple $((\mathcal{X}, \chi), (\mathcal{A}, \alpha), (\mathcal{F}, \mathcal{E}), \delta, z)$ where

1. $(\mathcal{X}, \chi)$ is a 1-labeled partition of X,
2. $(\mathcal{A}, \alpha)$ is a k-labeled partition of A,
3. $(\mathcal{F}, \mathcal{E})$ is a graph (possibly with loops) where $\mathcal{F} \subseteq \{0, \ldots, d+1\}^X$,
4. $\delta \in \{0, \ldots, d+1\}^{X^2}$, and
5. if $q \in \mathbb{N}$, then $z \leq q$, otherwise $z = \infty$.

Fusions and restrictions. Let $(\mathcal{X}_1, \chi_1)$ and $(\mathcal{X}_2, \chi_2)$ be two t-labeled partitions of the sets X_1 and X_2 respectively such that $\mathcal{X}_i = \{X_1^i, \ldots, X_{\rho_1}^i\}, i \in \{1, 2\}$. We define $\mathcal{X}_1 \oplus \mathcal{X}_2$ as follows: if $x, x' \in X_1 \cup X_2$ we say that $x \sim x'$ if there is a set in $\mathcal{X}_1 \cup \mathcal{X}_2$ that contains both of them. Let $\sim_T$ be the transitive closure of $\sim$. Then $\mathcal{X}_1 \oplus \mathcal{X}_2$ contains the equivalence classes of $\sim_T$. We now define $\chi_1 \oplus \chi_2$ as follows: let $\mathcal{X}_1 \oplus \mathcal{X}_2 = \{Y_1, \ldots, Y_\rho\}$. Then for each $i \in \{1, \ldots, \rho\}$, we define $\chi_1 \oplus \chi_2(i) = \min\{t, \sum_{X^1_{i'} \subseteq Y_i} \chi_1(i') + \sum_{X^2_{i'} \subseteq Y_i} \chi_2(i')\}$.

The *fusion* of the t-labeled partitions $(\mathcal{X}_1, \chi_1)$ and $(\mathcal{X}_2, \chi_2)$ is the pair $(\mathcal{X}_1 \oplus \mathcal{X}_2, \chi_1 \oplus \chi_2)$ that is a $(t+1)$-labeled partition and is denoted by $(\mathcal{X}_1, \chi_1) \oplus (\mathcal{X}_2, \chi_2)$. Given a t-labeled partition $(\mathcal{X}, \chi)$ of a set X and given a subset X' of X we define the *restriction* of $(\mathcal{X}, \chi)$ to X' as the t-labeled partition $(\mathcal{X}', \chi')$ of X' where $\mathcal{X}' = \{X_i \cap X' \mid X_i \in \mathcal{X}\} \setminus \{\emptyset\}$ and $\chi' = \{(i, \chi(i)) \mid X_i \cap X' \neq \emptyset\}$ and we denote it by $(\mathcal{X}, \chi)|_{X'}$. We also define the intersection of $(\mathcal{X}, \chi)$ with X' as the t-labeled partition $(\mathcal{X}', \chi')$ where $\mathcal{X}' = \{X_i \in \mathcal{X} \mid X_i \cap (X \setminus X') \neq \emptyset\}$ and $\chi' = \{(i, \chi(i)) \mid X_i \cap X'' \neq \emptyset\}$ where $X'' = \cup_{X'_i \in \mathcal{X}'} X_i$ and we denote it by $(\mathcal{X}, \chi) \cap X'$. Notice that $(\mathcal{X}, \chi)|_{X'}$ and $(\mathcal{X}, \chi) \cap X'$ are not always the same.

5 Dynamic programming

The following result is the main algorithmic contribution of this paper.

Lemma 5. *There exists an algorithm that, given (G, H, q, k, d, D, b) as input where G and H are plane graphs such that G is a subgraph of H, H is 3-connected, $q \in \mathbb{N} \cup \{\infty\}$, $k \in \mathbb{N}$, $d \in \mathbb{N}_{\geq 1}$, $b \in \mathbb{N}$, and $D = (T, \mu, \pi)$ is a sphere-cut decomposition of H with width at most b, decides whether G has (q, k, d)*-extension *in H in $(\alpha(q))^2 \cdot 2^{O(b^2 \log d) + 2^{O(b \log d)}} \cdot n$ steps.*

Proof. We use the notation $E^{\text{old}} = E(G)$ and $E^{\text{new}} = E(H) \setminus E(G)$, $V^{\text{old}} = V(G)$ and $V^{\text{new}} = V(H) \setminus V(G)$. We choose an arbitrary edge $e^* \in E(T)$, subdivide it by adding a new vertex v_{new} and update T by adding a new vertex r adjacent to v_{new}. We then root T at this vertex r and we extend μ by setting $\mu(r) = \emptyset$. In T we call *leaf-edges* all its edges that are incident to its leaves except from the edge $e_r = \{r, v_{\text{new}}\}$. An edge of T that is not a leaf-edge is called *internal*. We denote by $L(T)$ the set of the leaf-edges of T and we denote by $I(T)$ the internal edges of T. We also call e_r *root-edge*. For each $e \in E(T)$, let T_e be the tree of the forest $T \setminus \{e\}$ that does not contain r as a leaf and let E_e be the edges that are images, via μ, of the leaves of T that are also leaves of T_e. We denote $H_e = H[E_e]$ and $V_e = V(H_e)$ and observe that $H_{e_r} = H$. For each edge $e \in I(T)$, we define its children as the two edges that both belong in the connected component of $T \setminus e$ that does not contain the root r and that share a common endpoint with e. Also, for each edge $e \in E(T)$, we define Δ_e as the closed disk bounded by O_e such that $G \cap \Delta_e = H_e$. Finally, for each edge $e \in E(T)$, we set $X_e = \mathbf{mid}(e)$, $V_e^{\text{new}} = V_e \cap V^{\text{new}}$, $V_e^{\text{old}} = V_e \cap V^{\text{old}}$, $E_e^{\text{new}} = E_e \cap E^{\text{new}}$, and $E_e^{\text{old}} = E_e \cap E^{\text{old}}$.

Distance signatures and dependency graphs. Let $\mathbf{p} = \{E_e^0, E_e^1, E_e^\infty\}$ be a 3-partition of E_e^{new}. For each vertex $v \in V_e$, we define the $(X_e, \mathbf{p})$-*distance vector* of v as the function $\phi_v : X_e \to \{0, \ldots, d+1\}$ such that if $x \in X_e$ then $\phi_v(x) = \min\{\mathsf{w}_{\mathbf{p}}\text{-}\mathbf{dist}_{G_e}(v, x), d+1\}$. We define the $(e, \mathbf{p})$-*dependency graph* $\mathcal{G}_{e,\mathbf{p}} = (\mathcal{F}_{e,\mathbf{p}}, \mathcal{E}_{e,\mathbf{p}})$ (that may contain loops) where $\mathcal{F}_{e,\mathbf{p}} = \{\phi_v \mid v \in V_e\}$ and such that two (not necessarily distinct) vertices ϕ and ϕ' of $\mathcal{F}_{e,\mathbf{p}}$ are connected by an edge in $\mathcal{E}_{e,\mathbf{p}}$ if and only if there exist $v, v' \in V_e$ such that $\phi = \phi_v$, $\phi' = \phi_{v'}$ and $\mathsf{w}_{\mathbf{p}}\text{-}\mathbf{dist}_{H_e}(v, v') > d$. Notice that the set $\Phi_e = \{\mathcal{G}_{e,\mathbf{p}} \mid \mathbf{p}$ is a 3-partition of $E_e^{\text{new}}\}$ has at most $2^{(d+2)^{|X_e|}}$ elements because $\{\mathcal{F}_{e,\mathbf{p}} \mid \mathbf{p}$ is a 3-partition of $E_e^{\text{new}}\} \subseteq \{0, \ldots, d+1\}^{X_e}$ and, to each $\mathcal{F}_{e,\mathbf{p}}$, assign a unique edge set $\mathcal{E}_{e,\mathbf{p}}$. Intuitively, each $\mathcal{F}_{e,\mathbf{p}}$ corresponds to a partition of the elements of V_e such that vertices in the same part have the same $(X_e, \mathbf{p})$-distance signature. Moreover the existence of an edge in the $(e, \mathbf{p})$-dependency graph between two such parts implies that they contain vertices, one from each part, whose $\mathsf{w}_{\mathbf{p}}$-distance in H_e is bigger than d.

The tables. Our aim is to give a dynamic programming algorithm running on the sc-decomposition T. For this, we describe, for each $e \in E(T)$, a table $\mathfrak{T}(e)$ containing information on partial solutions of the problem for the graph G_e in a way that the table of an edge $e \in E(T)$ can be computed using the tables of the two children of e, the size of each table does not depend on G and the final answer can be derived by the table of the root-edge e_r.

We define the function $\mathfrak{T}$ mapping each $e \in E(T)$ to a collection $\mathfrak{T}(e)$ of (d, k, q)-configurations of $(X_e, \mathbf{A}_e)$. In particular, $Q = ((\mathcal{X}, \chi), (\mathcal{A}, \alpha), (\mathcal{F}, \mathcal{E}), \delta, z) \in \mathfrak{T}(e)$ iff there exists a 3-partition $\mathbf{p} = \{E_e^0, E_e^1, E_e^\infty\}$ of E_e^{new} such that the following hold:

1. $C_1, \ldots, C_h$ are the connected components of $(V(H_e), E_e^0)$, then
   - $\mathcal{X} = \{V(C_1) \cap X_e, \ldots, V(C_h) \cap X_e\}$ and
   - $\forall_{i \in \{1,\ldots,h\}}$ $\chi(i) = 1$ if C_i contains some vertex of V_e^{old}, otherwise $\chi(i) = 0$.

   (The pair $(\mathcal{X}, \chi)$ encodes the connected components of the 0-edges that contain vertices of X_e and for each of them registers the number (0 or 1) of the vertices in V_e^{old} in them. This information is important to control Condition A.)

2. $\mathcal{A}$ is a partition of $\mathbf{A}_e$ such that two arcs $A, A' \in \mathbf{A}_e$ belong in the same set, say A_i of $\mathcal{A}$ if and only if they are incident to the same marginal face f_i of H_e^+. Moreover, for each $i \in \{1, \ldots, |\mathcal{A}|\}$, $\alpha(i)$ is equal to the number of edges in E_e^1 that are inside f_i.

   (Here $(\mathcal{A}, \alpha)$ encodes the "partial" faces of the embedding of G_e that are inside Δ_e. To each of them we correspond the number of 1-edges that they contain in H_e. This is useful in order to guarantee that during the algorithm, faces that stop being marginal do not contain more than k 1-edges, as required by Condition B.)

3. $(\mathcal{F}, \mathcal{E})$ is the $(e, \mathbf{p})$-dependency graph, i.e., the graph $\mathcal{G}_{e,\mathbf{p}} = (\mathcal{F}_{e,\mathbf{p}}, \mathcal{E}_{e,\mathbf{p}})$.

   (Recall that $\mathcal{F}$ is the collection of all the different distance vectors of the vertices of V_e. Notice also that there might be pairs of vertices $x, x' \in V_e$ whose $\mathbf{w}_{\mathbf{p}}$-distance in G_e is bigger than d. In order for G to have a completion of diameter d, these two vertices should become connected, at some step of the algorithm, by paths passing *outside* Δ_e. To check this possibility, it is enough to know the distance vectors of x and x' and these are encoded in the set $\mathcal{F}$. Moreover the fact that x and x' are still "far away" inside G_e is certified by the existence of an edge (or a loop) between their distance vectors in $\mathcal{F}$.)

4. For each pair $x, x' \in X_e$, $\delta(x, x') = \min\{\mathsf{w}_{\mathbf{p}}\text{-}\mathbf{dist}_{H_e}(x, x'), d + 1\}$.

   (This information is complementary to the one stored in $\mathcal{F}$ and registers the distances of the vertices in X_e inside H_e. As we will see, $\mathcal{F}$ and δ will be used in order to compute the distance vectors as well as their dependencies during the steps of the algorithm.)

5. There is no path in H_e with endpoints in V_e^{old} that consists of edges in E_e^0.

   (This ensures that Condition A is satisfied for the current graph G_e.)

6. Every internal face of G_e^+ contains at most k edges in E_e^1.

   (This ensures that Condition B holds for all the internal faces of G_e.)

7. $\forall v, v' \in V_e$, either $\mathsf{w}_{\mathbf{p}}\text{-}\mathbf{dist}_{H_e}(v, v') \le d$ or there are two vertices $x, x' \in X_e$ such that $\phi_v(x) + \phi_{v'}(x') \le d$.

   (Here we demand that if two vertices x_1, x_2 of V_e are "far away" (have $\mathbf{w_p}$-distance $> d$) inside H_e then they have some chance to come "close" (obtain $\mathbf{w_p}$-distance $\le d$) in the final graph, so that Condition C is satisfied. This fact is already stored by an edge in $\mathcal{E}$ between the two distance vectors of x and x' and the possibility that x_1 and x_2 may come close at some step of the algorithm, in what concerns the graph G_e, depends only on these distance vectors and not on the vertices x_1 and x_2 themselves.)

8. There are at most z edges of E_e^1 inside the internal faces of G_e^+ (clearly, this last condition becomes void when $q = \infty$).

   (This information helps us control Condition D during the algorithm.)

Notice that in case $X_e = \emptyset$ the only graph that can correspond to the 6th step is the graph $(\{\varnothing\}, \emptyset)$ which, from now on will be denoted by $G_\varnothing$.

Bounding the set of characteristics. Our next step is to bound $\mathfrak{T}(e)$ for each $e \in E(T)$. Notice first that $|X_e| = |\mathbf{A}_e| \le b$. This means that there are $2^{O(b \log b)}$ instantiations of $(\mathcal{X}, \chi)$ and $2^{O(k + b \log b)}$ instantiations of $(\mathcal{A}, \alpha)$. As we previously noticed, the different instantiations of $(\mathcal{F}, \mathcal{E})$ are $|\Phi_e| = 2^{2^{O(b \log d)}}$. Moreover, there are $2^{O(b^2 \log d)}$ instantiations of δ and $\alpha(q)$ instantiations of z. We conclude that there exists a function f such that for each $e \in V(T)$, $|\mathfrak{T}(e)| \le f(k, q, b, d)$. Moreover, $f(k, q, b, d) = \alpha(q) \cdot 2^{O(b^2 \log d) + 2^{O(b \log d)}}$.

The characteristic function on the root edge. Observe that $E_{\rm new}$ is (k, d, q, w)-edge colorable in H if and only if $\mathfrak{T}(e_r) \neq \emptyset$, i.e., $((\emptyset, \varnothing), (\emptyset, \varnothing), G_{\varnothing}, \varnothing, z) \in \mathfrak{T}(e_r)$ for some $z \leq q$. Indeed, if this happens, conditions 1–4 become void while conditions 5, 6, 7, and 8 imply that $H = H_e$ satisfies the conditions A, B, C, and D respectively in the definition of the (k, d, q, w)-edge colorability of $E^{\rm new}$.

The computation of the tables. We will now show how to compute $\mathfrak{T}(e)$ for each $e \in E(T)$.

We now give the definition of $\mathfrak{T}(e)$ in the case where e is a leaf of T is the following: Given a $q \in \mathbb{N} \cup \{\infty\}$, we define $A(q) = \{\infty\}$ if $q = \infty$, otherwise $A(q) = \{z \mid z \leq q\}$.

Suppose now that e_l is a leaf-edge of T where $\pi(e_l) = [a_1, a_2, a_1]$ and $\mathbf{A}_{e_l} = \{a_{1,2}, a_{2,1}\}$.

1. If $\{a_1, a_2\} \in E_e^{\rm old}$, then

$$\begin{aligned}\mathfrak{T}(e_l) \;=\; \{ \big(& (\{\{a_1\}, \{a_2\}\}, \{(1,1), (2,1)\}), \\ & (\{\{a_{1,2}\}, \{a_{2,1}\}\}, \{(1,0), (2,0)\}), \\ & \big(\big\{\{(a_1, 0), (a_2, \mathsf{w}(\{a_1, a_2\}))\}, \{(a_1, \mathsf{w}(\{a_1, a_2\})), (a_2, 0)\}\big\}, \emptyset\big), \\ & \{((a_1, a_2), \mathsf{w}(\{a_1, a_2\}))\}, z\big) \mid z \in A(q)\},\end{aligned}$$

2. if $\{a_1, a_2\} \in E_e^{\rm new}$ and $\{a_1, a_2\} \subseteq V_e^{\rm old}$, then $\mathfrak{T}(e_l) = \mathcal{Q}^1 \cup \mathcal{Q}^{\infty}$ where

$$\begin{aligned}\mathcal{Q}^1 \;=\; \{ \big(& (\{\{a_1\}, \{a_2\}\}, \{(1,1), (2,1)\}) \\ & (\{\{a_{1,2}, a_{2,1}\}\}, \{(1,1)\}) \\ & (\big\{\{(a_1, 0), (a_2, 1)\}, \{(a_1, 1), (a_2, 0)\}\big\}, \emptyset) \\ & \{((a_1, a_2), s)\}, z\big) \mid z \in A(q) - \{0\}\}\end{aligned}$$

$$\begin{aligned}\mathcal{Q}^{\infty} \;=\; \{ \big(& (\{\{a_1\}, \{a_2\}\}, \{(1,1), (2,1)\}) \\ & (\{\{a_{1,2}, a_{2,1}\}\}, \{(1,0)\}) \\ & (\big\{\{(a_1, 0), (a_2, d+1)\}, \{(a_1, d+1), (a_2, 0)\}\big\}, K) \\ & \{((a_1, a_2), d+1)\}, z\big) \mid z \in A(q)\}\end{aligned}$$

(the set K above contains a single edge that is not a loop), and if $\{a_1, a_2\} \in E_e^{\rm new}$ and $\{a_1, a_2\} \not\subseteq V_e^{\rm old}$, then $\mathfrak{T}(e_l) = \mathcal{Q}^1 \cup \mathcal{Q}^{\infty} \cup \mathcal{Q}^0$ where

$$\begin{aligned}\mathcal{Q}^0 \;=\; \{ \big(& (\{\{a_1, a_2\}\}, \{(1, 1 - \langle \{a_1, a_2\} \subseteq V_e^{\rm new} \rangle)\}) \\ & (\{\{a_{1,2}, a_{2,1}\}\}, \{(1,0)\}) \\ & (\{\{(a_1, 0), (a_2, 0)\}\}, \emptyset) \\ & \{((a_1, a_2), 0)\}, z\big) \mid z \in A(q)\}.\end{aligned}$$

Assume now that e is a non-leaf edge of T with children e_l and e_r, the collection $\mathfrak{T}(e)$ is given by $\textbf{join}(\mathfrak{T}(e_1), \mathfrak{T}(e_2))$ where **join** is a procedure that is depicted below.

Notice that $\mathbf{A}_e$ is the symmetric difference of $\mathbf{A}_{e_l}$ and $\mathbf{A}_{e_r}$ and X_e consists of the endpoints of the arcs in $\mathcal{A}_e$. We also set $X_e^F = (X_{e_l} \cup X_{e_r}) \setminus X_e$.

Procedure **join**
Input: two collections $\mathcal{C}_{e_l}$ and $\mathcal{C}_{e_r}$ of (d,k,q)-configurations of $(X_{e_l}, \mathbf{A}_{e_l})$ and $(X_{e_r}, \mathbf{A}_{e_r})$.
Output: a collection $\mathcal{C}_r$ of (d,k,q)-configurations of $(X_e, \mathbf{A}_e)$
(1) set $\mathcal{C}_e = \emptyset$
(2) for every pair $(Q_{e_l}, Q_{e_r}) \in \mathcal{C}_{e_l} \times \mathcal{C}_{e_r}$, if $\mathbf{merge}(Q_{e_l}, Q_{e_r}) \neq \mathsf{void}$,
then let $\mathcal{C}_e \leftarrow \mathcal{C}_e \cup \{\mathbf{merge}(Q_{e_l}, Q_{e_r})\}$.
(3) return $\mathcal{C}_e$

It remains to describe the routine **merge**. For this, assume that it receives as inputs the (d,k,q)-configurations $Q_l = ((\mathcal{X}_l, \chi_l), (\mathcal{A}_l, \alpha_l), (\mathcal{F}_l, \mathcal{E}_l), \delta_l, z_l)$ and $Q_r = ((\mathcal{X}_r, \chi_r), (\mathcal{A}_r, \alpha_r), (\mathcal{F}_r, \mathcal{E}_r), \delta_r, z_r)$ of $(X_{e_l}, \mathbf{A}_{e_l})$ and $(X_{e_r}, \mathbf{A}_{e_r})$ respectively. Procedure $\mathbf{merge}(Q_{e_l}, Q_{e_r})$ returns a (d,k,q)-configuration $((\mathcal{X}, \chi), (\mathcal{A}, \alpha), (\mathcal{F}, \mathcal{E}), \delta, z)$ of $(X_e, \mathbf{A}_e)$ constructed as follows:

1. If $z_r + z_r > q$, then return void, otherwise $z = z_l + z_r$

(This controls the number of 1-edges that are now contained in Δ_e)

2. Let $(\mathcal{X}', \chi') = (\mathcal{X}_l, \chi_l) \oplus (\mathcal{X}_r, \chi_r)$ and if $\chi'^{-1}(2) \neq \emptyset$ then return void.

(This compute the "fusion" of the connected components of $(V(H_{e_l}, E^0_{e_l}))$ and $(V(H_{e_r}, E^0_{e_r}))$ with vertices in V_{e_l} and V_{e_r} and makes sure that none of the created components contains 2 or more 0-vertices.)

3. Let $(\mathcal{X}, \chi) = (\mathcal{X}'_l, \chi'_l)|_{V_e}$

(This computes the fusion $(\mathcal{X}'_l, \chi'_l)$ is restricted on the boundary O_e of Δ_e.)

4. Let $(\mathcal{A}', \alpha') = (\mathcal{A}_l, \alpha_l) \oplus (\mathcal{A}_r, \alpha_r)$ and if $\alpha'^{-1}(k+1) \neq \emptyset$ then return void.

5. Let $(\mathcal{A}, \alpha) = (\mathcal{A}_l, \alpha_l) \oplus (\mathcal{A}_r, \alpha_r)|_{\mathbf{A}_e}$.

6. Compute the function $\gamma : (\mathcal{F}_{e_l} \cup \mathcal{F}_{e_r} \cup X_e) \times (\mathcal{F}_{e_l} \cup \mathcal{F}_{e_r} \cup X_e) \to \{0, \ldots, d+1\}$, whose description is given latter.

7. Take the disjoint union of the graphs $(\mathcal{F}_l, \mathcal{E}_l)$ and $(\mathcal{F}_r, \mathcal{E}_r)$ and remove from it every edge $\{\phi_1, \phi_2\}$ for which $\gamma(\phi_1, \phi_2) \leq d$. Let $\mathcal{G}^+ = (\mathcal{F}^+, \mathcal{E}^+)$ be the obtained graph.

8. If for some edge $\{\phi_1, \phi_2\} \in \mathcal{E}^+$ it holds that for every $x_1, x_2 \in V_e$, $\gamma(\phi_1, x_1) + \gamma(\phi_2, x_2) > d$, then return void.

9. Consider the function $\lambda : \mathcal{F}_l \cup \mathcal{F}_r \to \{1, \ldots, d\}^{X_e}$ such that $\lambda(\phi) = \{(x, \gamma(\phi, x)) \mid x \in X_e\}$.

10. For every $\phi' \in \lambda(\mathcal{F}_l \cup \mathcal{F}_r)$, do the following for every set $\mathsf{F} = \lambda^{-1}(\phi')$: identify in $\mathcal{G}^+$ all vertices in F and if at least one pair of them is adjacent in $\mathcal{G}^+$, then add an loop on the vertex created after this identification. Let $\mathcal{G} = (\mathcal{F}, \mathcal{E})$ be the resulting graph (notice that $\mathcal{F} = \lambda(\mathcal{F}_l \cup \mathcal{F}_r)$).

11. $\delta = \{((x, x'), \gamma(x, x')) \mid x, x' \in V_e\}$.

The definition of function γ. We present here the definition of the function γ used in the above description of the tables of the dynamic programming procedure.

Given a non-empty set X and $q \in \{0, 1\}$ we define

$$\begin{aligned}\mathsf{ord}^q(X) \quad &= \quad \{\pi \mid \exists X' \subseteq X : X' \neq \emptyset \wedge \ |X'| \bmod 2 = q \\ &\qquad\qquad \wedge\ \pi \text{ is an ordering of } X'\}\end{aligned}$$

Given γ_l and γ_r, we define $\gamma : (\mathcal{F}_{e_l} \cup \mathcal{F}_{e_r} \cup X_e) \times (\mathcal{F}_{e_l} \cup \mathcal{F}_{e_r} \cup X_e) \to \{0, \ldots, d+1\}$ by distinguishing the following cases:

1. If $(x \in X_e \setminus X_{e_r} \wedge \phi \in \mathcal{F}_{e_l})$ or $(x \in X_e \setminus X_{e_l} \wedge \phi \in \mathcal{F}_{e_r})$, then

$$\begin{aligned}\gamma(\phi, x) \quad &= \quad \min\big\{\phi(x), \min\{\phi(p_1) + \sum_{[\![1,\rho-1]\!]} \delta_{\mathbf{s}(i)}(p_i, p_{i+1}) + \\ &\qquad \delta_{\mathbf{s}(\rho)}(p_\rho, x) \mid [p_1, \ldots, p_\rho] \in \mathsf{ord}^0(X_e^F)\}\big\},\end{aligned}$$

   where $\mathbf{s}(i) =$ "l" if $\langle x \in X_e \setminus X_{e_l}\rangle = (i \bmod 2)$, otherwise $\mathbf{s}(i) =$ "r".

2. If $(x \in X_e \setminus X_{e_l} \wedge \phi \in \mathcal{F}_{e_l})$ or $(x \in X_e \setminus X_{e_r} \wedge \phi \in \mathcal{F}_{e_r})$, then

$$\begin{aligned}\gamma(\phi, x) \quad &= \quad \min\big\{\phi(p_1) + \sum_{[\![1,\rho-1]\!]} \delta_{\mathbf{t}(i)}(p_i, p_{i+1}) + \delta_{\mathbf{t}(\rho)}(p_\rho, x) \\ &\qquad \mid [p_1, \ldots, p_\rho] \in \mathsf{ord}^1(X_e^F)\}\big\},\end{aligned}$$

   where $\mathbf{t}(i) =$ "l" if $\langle x \in X_e \setminus X_{e_l}\rangle \neq (i \bmod 2)$, otherwise $\mathbf{t}(i) =$ "r".

3. If x is one of the (at most two) vertices in $(X_{e_r} \cap X_{e_r}) \setminus X_e^F$ and $\phi \in \mathcal{F}_{e_l} \cup \mathcal{F}_{e_r}$, then

$$\begin{aligned}\gamma(\phi, x) \quad &= \quad \min\big\{\phi(x), \\ &\qquad \min\{\phi(p_1) + \sum_{[\![1,\rho-1]\!]} \delta_{\mathbf{u}(i)}(p_i, p_{i+1}) + \delta_{\mathbf{u}(q)}(p_\rho, x) \\ &\qquad \mid [p_1, \ldots, p_\rho] \in \mathsf{ord}^q(X_e^F)\} \mid q \in \{0, 1\}\big\}\end{aligned}$$

   where $\mathbf{u}(i) =$ "r" if $\langle \phi \in \mathcal{F}_{e_l}\rangle = (i \bmod 2)$, otherwise $\mathbf{u}(i) =$ "l".

4. If $\phi, \phi' \in \mathcal{F}_l \cup \mathcal{F}_r$, then

$$\begin{aligned}\gamma(\phi, \phi') \quad &= \quad \min\big\{\phi(p_1) + \sum_{[\![1,\rho-1]\!]} \delta_{\mathbf{u}(i)}(p_i, p_{i+1}) + \phi'(p_\rho) \\ &\qquad \mid [p_1, \ldots, p_\rho] \in \mathsf{ord}^q(X_e^F)\big\}\end{aligned}$$

In this equality, $q=1$ if ϕ and ϕ' belong in different sets in $\{\mathcal{F}_l, \mathcal{F}_r\}$, otherwise $q=0$. The function $\mathbf{u}$ is the same as in the previous case.

5. If $x_1, x_2 \in X_e \setminus X_{e_r}$ or $x_1, x_2 \in X_e \setminus X_{e_l}$, then

$$\begin{aligned}\delta(x_1,x_2) \quad = \quad & \min\big\{\delta_{\mathbf{y}(0,x_1)}(x_1,x_2), \min\{\delta_{\mathbf{y}(0,x_1)}(x_1,p_1) + \\ & \sum_{i\in[\![1,\rho-1]\!]} \delta_{\mathbf{y}(i,x_1)}(p_i,p_{i+1}) + \\ & \delta_{\mathbf{y}(0,x_2)}(p_\rho,x_2) \mid [p_1,\ldots,p_\rho] \in \mathsf{ord}^0(X_e^F)\}\big\}\end{aligned}$$

In this equality $\mathbf{y}(i,x) =$ "l" if $\langle x \in X_e \setminus X_{e_r}\rangle = \langle i \bmod 2 = 0\rangle$ otherwise $\mathbf{y}(i,x) =$ "r".

6. If x_1, x_2 belong in different sets is $\{X_e \setminus X_{e_r}, X_e \setminus X_{e_l}\}$, then

$$\begin{aligned}\delta(x_1,x_2) \quad = \quad & \min\big\{\delta_{\mathbf{y}(0,x_1)}(x_1,p_1) + \sum_{[\![1,\rho-1]\!]} \delta_{\mathbf{y}(i,x_1)}(p_i,p_{i+1}) + \\ & \delta_{\mathbf{y}(0,x_2)}(p_\rho,x_2) \mid [p_1,\ldots,p_\rho] \in \mathsf{ord}^1(X_e^F)\big\}\end{aligned}$$

The function $\mathbf{y}$ is the same as in the previous case.

7. If exactly one, say x_2, of x_1, x_2 belongs in $X_{e_r} \cap X_{e_r}) \setminus X_e^F$, then

$$\begin{aligned}\delta(x_1,x_2) \quad = \quad & \min\Big\{\delta_{\mathbf{y}(0,x_1)}(x_1,x_2), \\ & \min\big\{\min\{\delta_{\mathbf{y}(0,x_1)}(x_1,p_1) + \sum_{[\![1,\rho-1]\!]} \delta_{\mathbf{y}(i,x_1)}(p_i,p_{i+1}) + \\ & \delta_{\mathbf{y}(0,x_2)}(p_\rho,x_2) \,|\, [p_1,\ldots,p_\rho] \in \mathsf{ord}^q(X_e^F)\} \,|\, q \in \{0,1\}\big\}\Big\}\end{aligned}$$

The function $\mathbf{y}$ is the same as in the two previous cases. In case x_1 belongs in $X_{e_r} \cap X_{e_r}) \setminus X_e^F$, then just swap the positions of x_1 and x_2 in the above equation.

8. If both x_1, x_2 belong in $X_{e_r} \cap X_{e_r}) \setminus X_e^F$, then

$$\begin{aligned}\delta(x_1,x_2) \quad = \quad & \min\big\{\delta_l(x_1,x_2), \delta_r(x_1,x_2), \\ & \min\{\min\{\delta_{\mathbf{z}(0,j)}(x_1,p_1) + \\ & \sum_{[\![1,\rho-1]\!]} \delta_{\mathbf{z}(i,j)}(p_i,p_{i+1}) + \delta_{\mathbf{z}(q,j)}(p_\rho,x_2) \,| \\ & [p_1,\ldots,p_\rho] \in \mathsf{ord}^q(X_e^F)\} \,|\, (q,j) \in \{0,1\}^2\}\big\}\end{aligned}$$

In the previous equality, $\mathbf{z}(i,j) =$ "l" if $(i+j \bmod 2) = 0$, otehrwise $\mathbf{z}(i,x) =$ "r".

Running time analysis. It now remains to prove that procedure **join** runs in $(\alpha(q))^2 \cdot 2^{O(k^2)+2^{O(b\log d)}}$ steps. Recall that there exists a function f such that $|\mathfrak{T}(e)| \leq f(k,q,b,d)$. Therefore **merge** will be called in Step (2) at most $(f(k,q,b,d))^2$ times.

The first computationally non-trivial step of **merge** is Step 5, where function γ is computed. Notice that γ has at most $((d+1)^{|X_{e_l}|}+(d+1)^{|X_{e_r}|}+|X_e|)^2=2^{O(b\cdot\log d)}$ entries and each of their values require running over all permutations of the subsets of X_e^F that are at most $b!=2^{O(b\cdot\log b)}$. These facts imply that the computation of γ takes $2^{O(b\cdot\log b)}$ steps. As Steps 6–10 deal with graphs of $2^{O(b\cdot\log d)}$ vertices, the running time of **join** is the claimed one. □

We are now in position to prove the main algorithmic result of this paper.

Proof of Theorem 2. Given an input $I=(G,q,k,d)$ of BBFPDC, we consider the graph $H=G^{(\max\{3,k\})}$ whose construction takes $O(k^2n)$ steps, because of Lemma 3. Then run the algorithm of Proposition 4 with (H,w) as input, where $w=c_1\cdot c_2\cdot k\cdot d$. If the answer is that $\mathbf{bw}(H)>w$, then, from Proposition 3, $\mathbf{tw}(G)>c_1\cdot d$, therefore, from Lemma 1, we can safely report that I is a NO-instance. If the algorithm of Proposition 4 outputs a sphere-cut decomposition $D=(T,\mu)$ of width at most $w=O(k\cdot d)$ then we call the dynamic programming algorithm of Lemma 5, with input (G,H,q,k,d,D,b). This, from Lemma 4, provides an answer to BBFPDC for the instance I in $(\alpha(q))^2\cdot 2^{O((kd)^2\log d)+2^{O((kd)\log d)}}\cdot n=(\alpha(q))^2\cdot 2^{2^{O((kd)\log d)}}\cdot n$ steps and this completes the proof of the theorem. □

6 NP-hardness proofs

In this section we show that the Bounded Budget Plane Diameter Completion and Bounded Budget/Face Plane Diameter Completion problems are NP-complete.

Here we consider $\mathbb{R}^2$-plane graphs, i.e., graphs embedded in the plane $\mathbb{R}^2$. Each $\mathbb{R}^2$-plane graph has exactly one unbounded face, called the *outer* face, and all other faces are called *inner faces*. Take in mind that every $\mathbb{S}_0$-plane graph has as many embeddings in $\mathbb{R}^2$ as the number of its faces (each correspond on which face of the embedding in $\mathbb{S}_0$ will be chosen to be the outer face in $\mathbb{R}^2$). All our problems can be equivalently restated on $\mathbb{R}^2$-plane graphs. We choose such embeddings because they facilitate the presentation of the result of this section.

We also need some additional terminology. A *walk* in a graph G of is a sequence $P=v_0,e_1,v_1,e_2,\ldots,e_s,v_s$ of vertices and edges of G such that $v_0,\ldots,v_s\in V(G)$, $e_1,\ldots,e_s\in E(G)$, the edges $e_1,\ldots,e_s$ are pairwise distinct, and for $i\in\{1,\ldots,s\}$, $e_i=\{v_{i-1},v_i\}$; v_0,v_s are the *end-vertices* of the walk. A walk is *closed* if its end-vertices are the same. The *length* of a walk P is the number of edges in P. For a walk P with end-vertices u,v, we say that P is a (u,v)-walk. A walk is a *path* if $v_0,\ldots,v_s$ and $e_1,\ldots,e_s$ are pairwise distinct with possible exception $v_0=v_s$, and a *cycle* is a closed path. We write $P=v_0\ldots v_s$ to denote a walk $P=v_0,e_1,\ldots,e_s,v_s$ omitting edges.

Recall that the 3-Satisfiability problem for a given Boolean formula $\phi=C_1\wedge\ldots\wedge C_m$ with clauses $C_1,\ldots,C_m$ with 3 literals each over variables $x_1,\ldots,x_n$, asks whether $x_1, \ldots, x_n$ have an assignment that satisfies ϕ. We write that a literal $x_i \in C_j$ ($\overline{x}_i \in C_j$ resp.) if this interval is in C_j. For an instance ϕ of 3-Satisfiability, we define the graphs G_ϕ and G'_ϕ as follows. The vertex set of G_ϕ is $\{x_1, \ldots, x_n\} \cup \{C_1, \ldots, C_m\}$, and for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, $\{x_i, C_j\} \in E(G_\phi)$ if and only if C_j contains either x_i or $\overline{x}_i$. Respectively, $V(G'_\phi) = \{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\} \cup \{C_1, \ldots, C_m\}$ and $E(G'_\phi) = \{\{x_i, \overline{x}_i\} | 1 \leq i \leq n\} \cup \{\{x_i, C_j\} | x_i \in C_j, 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{\{\overline{x}_i, C_j\} | \overline{x}_i \in C_j, 1 \leq i \leq n, 1 \leq j \leq m\}$.

Let ϕ over variables $x_1, \ldots, x_n$ be an instance of 3-Satisfiability such that G'_ϕ is planar, and let G' be a plane embedding of G'_ϕ. Let also $R_\phi = \{\{x_i, \overline{x}_i\} | 1 \leq i \leq n\} \subseteq E(G')$. We define the bipartite graph $H(G')$ as the graph with the vertex set $R_\phi \cup F(G')$ and the edge set $\{\{e, f\} | e \in R_\phi, f \in F(G')$ such that e is incident to $f\}$.

We consider the following special variant of Satisfiability.

Plane Satisfiability with Connectivity of Variables *Input*: A Boolean formula $\phi = C_1 \wedge \ldots \wedge C_m$ with clauses $C_1, \ldots, C_m$ with at most 3 literals each over variables $x_1, \ldots, x_n$ such that G'_ϕ is planar, and a plane embedding G' of G'_ϕ such that $H(G')$ is connected. *Output*: Is it possible to satisfy ϕ?

We show that this problem is hard.

Lemma 6. Plane Satisfiability with Connectivity of Variables *is* NP-*complete.*

Proof. It is straightforward to see that Plane Satisfiability with Connectivity of Variables is in NP. To show NP-hardness, we reduce Planar 3-Satisfiability, i.e. the 3-Satisfiability problem restricted to instances ϕ such that G_ϕ is planar. This problem was shown to be NP-complete by Lichtenstein in [15].

Let $\phi = C_1 \wedge \ldots \wedge C_m$ over variables $x_1, \ldots, x_n$ be an instance of Planar 3-Satisfiability. For the plane graph G_ϕ, we construct its plane embedding G. It is well known that it can be done in polynomial time, e.g., by the classical algorithm of Hopcroft and Tarjan [12] or by the algorithm of Boyer and Myrvold [1]. We consequently consider variables $x_1, \ldots, x_n$ and modify ϕ and G.

Suppose that a variable x_i occurs in the clauses $C_{j_1}, \ldots, C_{j_{p(i)}}$. Without loss of generality we assume that the edges $\{x_i, C_{j_1}\}, \ldots, \{x_i, C_{j_{p(i)}}\}$ are ordered clockwise in G as shown in Fig. 2 a). We perform the following modifications of ϕ and G.

- Replace x_i by $2p(i)$ new variables $x_{i,1}, \ldots, x_{i,2p(i)}$.
- For $k \in \{1, \ldots, p(i)\}$, replace x_i in C_{j_k} by $x_{i,2k-1}$.
- Construct $2p(i)$ clauses $C_i^1, \ldots, C_i^{2p(i)}$ where $C_i^k = \overline{x}_{i,k-1} \vee x_{i,k}$ for $k \in \{1, \ldots 2p(i)\}$; we assume that $x_{i,0} = x_{i,2p(i)}$.
- Modify the current plane graph as it is shown in Fig. 2.[3]

[3]Here and further we demonstrate constructions of plane embeddings in figures instead of long technical formal descriptions.

Denote the obtained Boolean formula and plane graph by $\hat{\phi}$ and $\hat{G}$ respectively. By the construction, $\hat{G}$ is a plane embedding of $G_{\hat{\phi}}$.

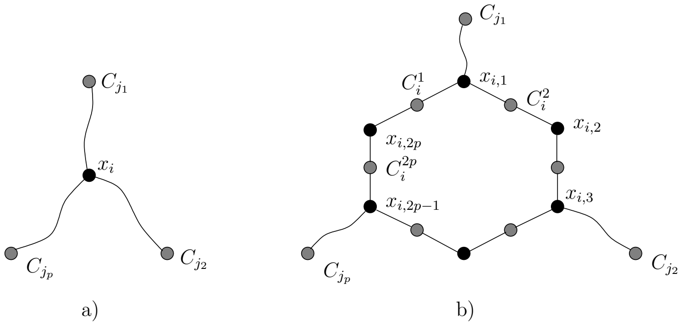

Figure 2: Modification of ϕ and G: a) before the modification and b) after; $p = p(i)$.

We show that ϕ can be satisfied if and only if $\hat{\phi}$ has a satisfying assignment. Suppose that the variables have assigned values such that $\phi = true$. For each $i \in \{1, \ldots, n\}$, we assign the same value as x_i for all the variables $x_{i,1}, \ldots, x_{i,2p(i)}$ that replace x_i in $\hat{\phi}$. It is straightforward to verify that $\hat{\phi} = true$ for this assignment. Assume now that $\hat{\phi} = true$ for some values of the variables. Observe that for each $i \in \{1, \ldots, n\}$, the variables $x_{i,1}, \ldots, x_{i,2p(i)}$ that replace x_i should have the same value to satisfy $C_i^1, \ldots, C_i^{2p(i)}$. It remains to observe that if each x_i has the same value as $x_{i,1}, \ldots, x_{i,2p(i)}$, then $\phi = true$ by the construction of $\hat{\phi}$.

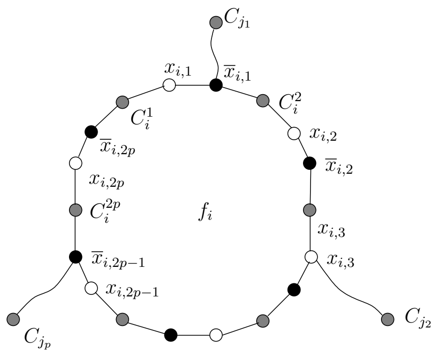

Figure 3: Construction of $\hat{G}'$; it is assumed that C_{j_1} contains $\overline{x}_i$, C_{j_2} contains x_i and $C_{j_{p(i)}}$ contains $\overline{x}_i$, and $p = p(i)$.

Observe that each variable $x_{i,k}$ in $\hat{\phi}$ occurs in at most 3 clauses, and it occurs at least once in positive and at least once with negation. It implies that a plane embedding $\hat{G}'$ of $G'_{\hat{\phi}}$ can be constructed from $\hat{G}$ by "splitting" the variable vertices as shown in Fig. 3. Clearly, $\hat{G}'$ can be constructed in polynomial time.

We claim that $H(\hat{G}')$ is connected. To see it, observe that $\hat{G}'$ is constructed from G by replacing each variable-vertex x_i by the cycle $L_i = C_i^1 x_{i,1} \overline{x}_{i,1} C_i^2 \ldots C_i^1$ (see Fig. 3).

Respectively, this graph has n new faces that are inner faces of these cycles. All other faces correspond to the faces of G. Denote by f_i the inner face of L_i for $i \in \{1, \ldots, n\}$. Notice that $R_{\hat{\phi}}$ contains edges from the cycles L_i. It follows that each vertex of $R_{\hat{\phi}}$ is adjacent to some vertex f_i in $H(\hat{G}')$. Hence, to prove the connectivity of $H(\hat{G}')$, it is sufficient to show that for any two vertices $h_1, h_2 \in F(\hat{G}')$, $H(\hat{G}')$ has a (h_1, h_2)-walk.

Consider the dual G^* of $\hat{G}'$. Recall that $V(G^*) = F(\hat{G}')$ and two vertices of G^* are adjacent if and only if the corresponding faces of $\hat{G}'$ are adjacent. It is straightforward to observe that the dual of any plane graph is always connected. Hence, to show that for any two vertices $h_1, h_2 \in F(\hat{G}')$ of $H(\hat{G}')$, $H(\hat{G}')$ has a (h_1, h_2)-walk, it is sufficient to prove that it holds for any two h_1, h_2 that are adjacent vertices of G^*, i.e., adjacent faces of $\hat{G}'$. Suppose that $h_1 = f_i$ for some $i \in \{1, \ldots, n\}$. Then h_2 is a face corresponding to a face h_2' of G such that the vertex x_i lies on the boundary of h_2'. Then by the construction of $\hat{G}'$, there is an edge $e = \{x_{i,j}, \overline{x}_{i,j}\}$ of $\hat{G}'$ that lies on the boundaries of h_1 and h_2. Because e is a vertex of $H(\hat{G}')$ adjacent to h_1, h_2, there is a (h_1, h_2)-walk in $H(\hat{G}')$. Assume now that h_1, h_2 are faces of $\hat{G}'$ distinct from f_i for $i \in \{1, \ldots, n\}$. Because h_1, h_2 are adjacent in G^*, the faces h_1, h_2 correspond to faces h_1', h_2' of G such that h_1', h_2' has a common vertex x_i on their boundaries. It implies that h_1, h_2 are adjacent to f_i in G^*. We already proved that $H(\hat{G}')$ has (f_i, h_1) and (f_i, h_2)-walks. Therefore, $H(\hat{G}')$ has an (h_1, h_2)-walk.

It completes the proof of connectedness of $H(\hat{G}')$ and the proof of the lemma. □

For the proof of our main result, we need some special gadgets. We introduce them and prove their properties that will be useful further.

Let $r \geq 3$ be a positive integer. We construct the graph $W_r(v_1, \ldots, v_r)$ as follows (see Fig. 4).

- Construct vertices $v_1, \ldots, v_r$ and a vertex u.
- For $i \in \{1, \ldots, r\}$, construct a (v_i, u) path $x_0^i \ldots x_r^i$ of length r, $v_i = x_0^i$, $u = x_r^i$.
- For $j \in \{1, \ldots, r-1\}$, construct a cycle $x_j^1 \ldots x_j^r x_j^1$.
- For $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, r-1\}$, construct an edge $\{x_{j-1}^{i-1}, x_j^i\}$; we assume that $x_j^0 = x_j^r$ for $j \in \{0, \ldots, r\}$.

We say that the vertices of $V(W_r(v_1, \ldots, v_r)) \setminus \{v_1, \ldots, v_r\}$ are the *inner* vertices of the gadget.

Let G be a plane graph with a face f, and let $v_1 \ldots v_r v_1$, $r \geq 3$, be a facial walk for f. We say that G' is obtained from G by *attaching a web* to f if G' is constructed by adding a copy of $W_r(v_1, \ldots, v_r)$, where the vertices $v_1, \ldots, v_r$ of the gadget are identified with the vertices with the same names in the facial walk, and embedding $W_r(v_1, \ldots, v_r)$ if f as is shown in Fig. 5. Notice that some vertices in the facial walk can occur several times.

Lemma 7. *Let G be a plane graph with a face f that has a facial walk of length $r \geq 3$, and let G' be a plane graph obtained from G by attaching a web to f.*

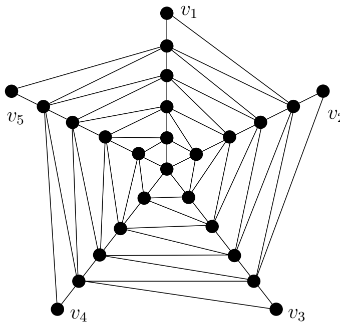

Figure 4: Construction of $W_5(v_1,\dots,v_5)$.

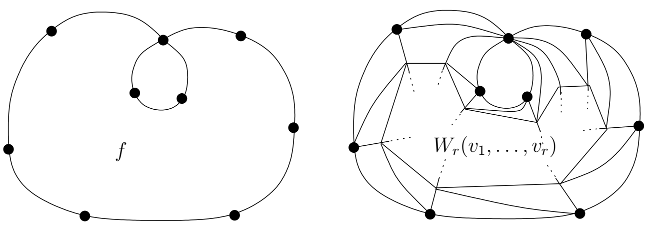

Figure 5: Attachment of a web.

i) For any two vertices $u,v\in V(G)$, $\mathbf{dist}_{G'}(u,v)=\mathbf{dist}_G(u,v)$. Moreover, any shortest (u,v)-path in G' has no inner vertices of $W_r(v_1,\dots,v_r)$ attached to f.

ii) For any vertex $v\in V(W_r(v_1,\dots,v_r))$, there is a vertex $u\in V(G)$ such that $\mathbf{dist}_{G'}(u,v)\le r$.

Proof. Let $v_1\dots v_rv_0$ be a facial walk for f. To prove i), it is sufficient to observe that for all v_i,v_j, the length of any (v_i,v_j)-path in $W_r(v_1,\dots,v_r)$ is greater that the length of a shortest (v_i,v_j)-path in G that lies on the boundary of f. The definition of $W_r(v_1,\dots,v_r)$ immediately implies ii). □

Let h be a positive integer. The graph $M_h(u_1,u_2,u_3)$ is defined as follows (see Fig. 6).

- Construct vertices u_1,u_2,u_3 and v_1,v_2,v_3.
- For $i\in\{1,2,3\}$, construct a (u_i,v_i) path $x_0^i\dots x_r^i$ of length ℓ, $u_i=x_0^i$, $v_i=x_h^i$.
- For $j\in\{1,\dots,h\}$, construct a cycle $x_j^1\dots x_j^rx_j^1$.
- For $j\in\{1,\dots,h\}$, construct edges $\{x_{j-1}^1,x_j^2\}$, $\{x_{j-1}^1,x_j^3\}$ and $\{x_{j-1}^2,x_j^3\}$.

We say that the vertices of $V(M_h(u_1,u_2,u_3))\setminus\{u_1,u_2,u_3\}$ are the *inner* vertices of the gadget. We also say that u_1 is the *root* and v_1 is the *pole* of $M_\ell(u_1,u_2,u_3)$.

Let G be a plane graph, and let $u_1\in V(G)$ be a vertex incident to a face f with a triangle facial walk $u_1u_2u_3u_1$. Let also ℓ be a positive integer. We say that G' is

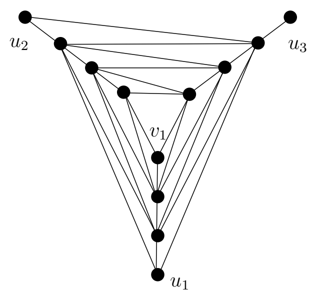

Figure 6: Construction of $M_3(u_1, u_2, u_3)$.

obtained from G by *attaching a mast of height* h rooted in u_1 to f if G' is constructed by adding a copy of $M_h(u_1, u_2, u_3)$, where the vertices u_1, u_2, u_3 of the gadget are identified with the vertices with the same names in the facial walk, and embedding $M_h(u_1, u_2, u_3)$ in f. We need the properties summarized in the following straightforward lemma.

Lemma 8. *Let ℓ be a positive integer. Let G be a plane graph, and let u_1 be a vertex of G incident to a face f with a triangle facial walk $u_1u_2u_3u_1$. Let also G' be a plane graph obtained from G by attaching a mast of height h rooted in u_1 to f.*

i) For any two vertices $u, v \in V(G)$, $\mathbf{dist}_{G'}(u, v) = \mathbf{dist}_G(u, v)$. Moreover, any shortest (u, v)-path in G' has no inner vertices of $M_h(u_1, u_2, u_2)$ attached to f.

ii) For any vertex $v \in V(M_h(u_1, u_2, u_2))$, $\mathbf{dist}_{G'}(u_1, v) \leq h$.

iii) If v is the pole of $M_h(u_1, u_2, u_2)$, then $\mathbf{dist}_{G'}(u_1, v) = h$ and $\mathbf{dist}_{G'}(u_2, v) > h, \mathbf{dist}_{G'}(u_3, v) > h$.

iv) For any inner vertices x, y of $M_h(u_1, u_2, u_2)$, $\mathbf{dist}_{G'}(x, y) \leq h$.

Now we are ready to prove the main result of the section.

Proof of Theorem 1. It is straightforward to see that BPDC and BFPDC are in NP. To show NP-hardness, we reduce Plane Satisfiability with Connectivity of Variables that was shown to be NP-complete in Lemma 6.

First, we consider BPDC.

Let (ϕ, G') be an instance of Plane Satisfiability with Connectivity of Variables, where $\phi = C_1 \wedge \ldots \wedge C_m$ is a Boolean formula with clauses $C_1, \ldots, C_m$ with at most 3 literals each over variables $x_1, \ldots, x_n$ such that G'_ϕ is planar, and G' is a plane embedding of G'_ϕ such that $H(G')$ is connected. Recall that $H(G')$ is the bipartite graph with the bipartition of the vertex set $(R_\phi, F(G'))$, where $R_\phi = \{\{x_i, \overline{x}_i\} | 1 \leq i \leq n\} \subseteq E(G')$, and $F(G')$ is the set of faces of G', and for $e \in R_\phi$ and $f \in F(G')$, $\{e, f\} \in E(H(G'))$ if and only if the edge e is incident to the face f in G'. Notice that $\deg_{H(G')}(e) \leq 2$ for any $e \in R_\phi$.

We select an arbitrary vertex $r \in F(G')$ of $H(G')$. Using the connectedness of $H(G')$, we find in polynomial time a tree T of shortest (r, e)-paths for $e \in R_\phi$ by the breadth-first search. We assume that T is rooted in r and it defines the parent-child relation on T. Let $L \subseteq R_\phi$ be the set of leaves of T, and let $s = \max\{\mathbf{dist}_T(r, e) | e \in L\}$.

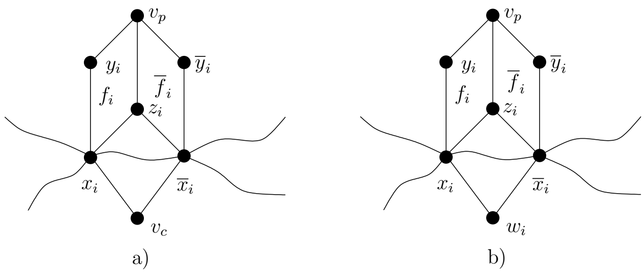

Figure 7: Construction of gadgets for $\{x_i, \overline{x}_i\}$.

We construct the plane graph $\hat{G}$ as follows.

i) Construct a copy of G'.

ii) For each vertex $f \in V(T)$ such that $f \in F(G')$, crate a vertex v_f embedded in the face f.

iii) For each $e = \{x_i, \overline{x_i}\} \in R_\phi \setminus L$, denote by p its parent and by c its child in T, construct vertices $y_i, \overline{y}_i, z_i$ and edges $\{x_i, y_i\}, \{y_i, v_p\}, \{x_i, v_c\}, \{x_i, z_i\}$, $\{\overline{x}_i, \overline{y}_i\}, \{\overline{y}_i, v_p\}, \{\overline{x}_i, v_c\}, \{\overline{x}_i, z_i\}, \{z_i, v_p\}$ and embed them as is shown in Fig. 7 a). Denote by f_i the inner face of the cycle $x_i y_i v_p z_i x_i$ and by $\overline{f}_i$ the inner face of the cycle $\overline{x}_i \overline{y}_i v_p z_i \overline{x}_i$.

iv) For each $e = \{x_i, \overline{x_i}\} \in L$, denote by p its parent in T, construct vertices $y_i, \overline{y}_i, z_i, w_i$ and edges $\{x_i, y_i\}, \{y_i, v_p\}, \{x_i, w_i\}, \{x_i, z_i\}, \{\overline{x}_i, \overline{y}_i\}, \{\overline{y}_i, v_p\}, \{\overline{x}_i, w_i\}$, $\{\overline{x}_i, z_i\}, \{z_i, v_p\}$ and embed them as is shown in Fig 7 b). Denote by f_i the inner face of the cycle $x_i y_i v_p z_i x_i$ and by $\overline{f}_i$ the inner face of the cycle $\overline{x}_i \overline{y}_i v_p z_i \overline{x}_i$.

v) For each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, if $\{x_i, C_j\} \in E(G')$ ($\{\overline{x}_i, C_j\} \in E(G')$ resp.), replace this edge by a (x_i, C_j)-path (by $(\overline{x}_i, C_j)$-path resp.) of length $2s - \mathbf{dist}_T(r, \{x_i, \overline{x}_i\})$.

We denote the constructed at this stage graph by $\hat{G}_1$. Observe that $\hat{G}_1$ is connected. Hence, each face has a facial walk. Denote by ℓ the length of a longest facial walk in $\hat{G}_1$. Now we proceed with construction of $\hat{G}$.

vi) For each face $f \in F(\hat{G}_1)$ distinct from the faces $f_i, \overline{f}_i$ for $i \in \{1, \ldots, n\}$, attach a web to f.

Denote the constructed at this stage graph by $\hat{G}_2$. Notice that $\hat{G}_2$ is 3-connected due to attached webs.

vii) For $j \in \{1, \ldots, m\}$, select a face f of the obtained graph such that C_j is incident to f and attach a mast of height $\ell + 2s$ rooted in C_j to f (notice that the boundary of f is a triangle because of attached webs).

viii) For each $e = \{x_i, \overline{x_i}\} \in L$, attach a mast of height $\ell + 4s - 1 - \mathbf{dist}_T(r, e)$ rooted in w_i to the face with the facial walk $w_i x_i \overline{x}_i w_i$.

ix) For the vertex v_r, select a face f with a triangle boundary such that v_r is incident to f (such a face always exists due to attached webs) and attach a mast of height $\ell + 8s$ rooted in v_r to f.

Notice that the obtained graph $\hat{G}$ is 3-connected because $\hat{G}_2$ is 3-connected and attachments of masts cannot destroy 3-connectivity. Also only the faces $f_i, \overline{f}_i$ for $i \in \{1, \ldots, n\}$ have degree 4, and all other faces have degree 3.

To complete the construction of an instance of BPDC, we set $q = n$ and $d = 2\ell + 12s$.

We show that (ϕ, G') is a yes-instance of Plane Satisfiability with Connectivity of Variables if and only if $(\hat{G}, q, d)$ is a yes-instance of BPDC.

Suppose that (ϕ, G') is a yes-instance of Plane Satisfiability with Connectivity of Variables. Assume that the variables $x_1, \ldots, x_n$ have values such that $\phi = true$. For $i \in \{1, \ldots, n\}$, if $x_i = true$, then we add an edge $\{x_i, v_p\}$ for the parent p of $\{x_i, \overline{x}_i\}$ in T and embed this edge in f_i. Respectively, we add an edge $\{\overline{x}_i, v_p\}$ and embed this edge in $\overline{f}_i$ if $x_i = false$. Denote the obtained graph by $\hat{G}'$. We show that $\mathbf{diam}(\hat{G}') \leq d$.

By the construction of $\hat{G}_1$, for any vertex $v \in V(\hat{G}_1)$, $\mathbf{dist}_{\hat{G}_1}(v_r, v) \leq 3s$. By Lemma 7, any vertex $v \in V(\hat{G}_2)$ is at distance at most ℓ from a vertex of $\hat{G}_1$ in $\hat{G}_2$. Hence, for any vertex $v \in V(\hat{G}_2)$, $\mathbf{dist}_{\hat{G}_2}(v_r, v) \leq \ell + 3s$. Observe also that for any $e = \{x_i, \overline{x_i}\} \in L$, $\mathbf{dist}_{\hat{G}'}(v_r, w_i) = \mathbf{dist}_T(r, e) + 1$. To show that for any $u, v \in V(\hat{G}')$, $\mathbf{dist}_{\hat{G}'}(u, v) \leq d$, we consider five cases.

Case 1. $u, v \in V(\hat{G}_2)$. Because $\mathbf{dist}_{\hat{G}_2}(v_r, u) \leq \ell + 3s$ and $\mathbf{dist}_{\hat{G}_2}(v_r, v) \leq \ell + 3s$, $\mathbf{dist}_{\hat{G}'}(u, v) \leq \mathbf{dist}_{\hat{G}_2}(u, v) \leq 2\ell + 6s \leq d$.

Case 2. u, v are vertices of the same mast attached to a face of $\hat{G}_2$. By Lemma 8, $\mathbf{dist}_{\hat{G}'}(u, v)$ is at most the height of the mast, and we have that $\mathbf{dist}_{\hat{G}'}(u, v) \leq \ell + 8s \leq d$.

Case 3. $u \in V(\hat{G}_2)$ and v is a vertex of a mast attached to a face of $\hat{G}_2$. By Lemma 8, $\mathbf{dist}_{\hat{G}'}(u, v_r) \leq \ell + 8s$ if the mast is rooted in v_r. Suppose that this mast is rooted in some other vertex z, i.e., $z = w_i$ or $z = C_j$ for some $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$. Then $\mathbf{dist}_{\hat{G}'}(u, v_r) \leq \ell + 4s - 1 + \mathbf{dist}_{\hat{G}_1}(z, r) \leq \ell + 8s$. Because $\mathbf{dist}_{\hat{G}'}(v_r, v) \leq \mathbf{dist}_{\hat{G}_2}(v_r, v) \leq \ell + 3s$, $\mathbf{dist}_{\hat{G}'}(u, v) \leq 2\ell + 11s \leq d$.

Case 4. u, v are vertices of distinct masts attached to faces of $\hat{G}_2$ that are rooted in $z, z' \neq v_r$ respectively. If $z = w_i$ for some $i \in \{1, \ldots, n\}$, then $\mathbf{dist}_{\hat{G}'}(u, v_r) \leq \ell + 4s - 1 - \mathbf{dist}_T(r, e) + \mathbf{dist}_{\hat{G}'}(v_r, w_i) \leq (\ell + 4s - 1 - \mathbf{dist}_T(r, e)) + (\mathbf{dist}_T(r, e) + 1) \leq \ell + 4s$ where $e = \{x_i, \overline{x}_i\}$. If $z = C_j$ for some $j \in \{1, \ldots, m\}$, then $\mathbf{dist}_{\hat{G}'}(u, v_r) \leq \ell + 2s + \mathbf{dist}\hat{G}_1(C_j, v_r) \leq \ell + 5s$. Clearly, the same bounds hold for $\mathbf{dist}_{\hat{G}'}(v, v_r)$. We have that $\mathbf{dist}_{\hat{G}'}(u, v) \leq \mathbf{dist}_{\hat{G}'}(u, v_r) + \mathbf{dist}_{\hat{G}'}(v_r, v) \leq 2\ell + 10s \leq d$.

It remains to consider the last case.

Case 5. u, v are vertices of masts attached to faces of $\hat{G}_2$ such that u is in the mast rooted in v_r and v is in a mast rooted in $z \neq v_r$. Suppose that $z = w_i$ for some $i \in \{1, \ldots, n\}$. Then $e = \{x_i, \overline{x}_i\} \in L$. We have that $\mathbf{dist}_{\hat{G}'}(u, v) \leq \mathbf{dist}_{\hat{G}'}(u, v_r) +$

$\mathbf{dist}_{\hat{G}'}(v_r,w_i)+\mathbf{dist}_{\hat{G}'}(w_i,v) \leq (\ell+8s)+(\mathbf{dist}_T(r,e)+1)+(\ell+4s-1-\mathbf{dist}_T(r,e)) \leq 2\ell+12s \leq d$. Assume that $z=C_j$ for $j\in\{1,\ldots,m\}$. Then the clause C_j in ϕ contains a literal that has the value *true*. Let x_i be such a literal (the case when C_j contains some $\overline{x}_i = true$ is symmetric). Notice that if $x_i = true$, then for the vertex $x_i \in V(\hat{G}')$, $\mathbf{dist}_{\hat{G}'}(x_i,v_r) = \mathbf{dist}_T(e,r)$ for $e=\{x_i,\overline{x}_i\}$ by the construction of $\hat{G}$ and the selection of the added edges. Then, $\mathbf{dist}_{\hat{G}'}(u,v) \leq \mathbf{dist}_{\hat{G}'}(u,v_r)+\mathbf{dist}_{\hat{G}'}(v_r,x_i)+\mathbf{dist}_{\hat{G}'}(x_i,C_j)+\mathbf{dist}_{\hat{G}'}(C_j,v) \leq (\ell+8s)+\mathbf{dist}_T(r,e)+(2s-\mathbf{dist}_T(r,e))+(\ell+2s) \leq 2\ell+12s \leq d$.

Suppose now that $(\hat{G},q,d)$ is a yes-instance of BPDC. Let A be a set of at most q edges such that the graph $\hat{G}'$ obtained from $\hat{G}$ by the addition of A has diameter at most d. Because only the faces $f_i,\overline{f}_i$ for $i\in\{1,\ldots,n\}$ have degree 4 and all other faces have degree 3, each edge of A has its end-vertices in the boundary of some f_i or $\overline{f}_i$ and is embedded in this face. Using this observation, denote by $\hat{G}_1'$ and $\hat{G}_2'$ the graphs obtained from $\hat{G}_1$ and $\hat{G}_2$ respectively by the addition of A. Let v_r' be the pole of the mast rooted in v_r. Because $\mathbf{diam}(\hat{G}') \leq d$, for any $u \in V(\hat{G}')$, $\mathbf{dist}_{\hat{G}'}(v_r',u) \leq d$ and, in particular, it holds for poles of other masts.

Consider masts rooted in w_i for $e=\{x_i,\overline{x}_i\} \in L$. For a mast rooted in w_i, denote by w_i' its pole. By Lemma 8, $\mathbf{dist}_{\hat{G}'}(v_r',w_i') = \mathbf{dist}_{\hat{G}'}(v_r',v_r)+\mathbf{dist}_{\hat{G}'}(v_r,w_i)+\mathbf{dist}_{\hat{G}'}(w_i,w_i') = (\ell+8s)+\mathbf{dist}_{\hat{G}_2'}(v_r,w_i)+(\ell+4s-1-\mathbf{dist}_T(r,e))$, and by Lemma 7, $\mathbf{dist}_{\hat{G}_2'}(v_r,w_i) = \mathbf{dist}_{\hat{G}_1'}(v_r,w_i)$. We conclude that $\mathbf{dist}_{\hat{G}_1'}(v_r,w_i) \leq \mathbf{dist}_T(r,e)+1$. Because $\mathbf{dist}_T(r,e)+1 \leq s+1$, a shortest (v_r,w_i)-path in $\hat{G}_1'$ does not contain the vertices C_j for $j\in\{1,\ldots,m\}$. We obtain that for every edge $e'=\{x_h,\overline{x}_h\}$ that lies on the unique (r,e)-path in T, $\{x_i,v_p\}\in A$ or $\{\overline{x}_i,v_p\}\in A$ where p is the parent of e' in T. This holds for each leaf of T. Because $R_\phi \subseteq V(T)$ and $k=n$, we have that for each $h\in\{1,\ldots,n\}$, either $\{x_i,v_p\}\in A$ or $\{\overline{x}_i,v_p\}\in A$ where p is the parent of $\{x_h,\overline{x}_h\}$ in T. For $h\in\{1,\ldots,n\}$, we let the variable $x_h = true$ if $\{\overline{x}_i,v_p\}\in A$ and $x_h = false$ otherwise. We show that this assignment satisfies ϕ.

Consider a clause C_j for $j\in\{1,\ldots,m\}$. To simplify notations, assume that C_j contains literals x_{i_1},x_{i_2},x_{i_3} (the cases when C_j contains two literals and/or some literals are negations of variables are considered in the same way). Let C_j' be the pole of the mast rooted in the vertex C_j. We have that $\mathbf{dist}_{\hat{G}'}(v_r',C_j') \leq d$. By Lemma 8,$\mathbf{dist}_{\hat{G}'}(v_r',C_j') = \mathbf{dist}_{\hat{G}'}(v_r',v_r)+\mathbf{dist}_{\hat{G}'}(v_r,C_j)+\mathbf{dist}_{\hat{G}'}(C_j,C_j') = (\ell+8s)+\mathbf{dist}_{\hat{G}_2'}(v_r,C_j)+(\ell+2s)$, and by Lemma 7, $\mathbf{dist}_{\hat{G}_2'}(v_r,C_j) = \mathbf{dist}_{\hat{G}_1'}(v_r,C_j)$. Therefore, $\mathbf{dist}_{\hat{G}_1'}(v_r,C_j) \leq 2s$. Let $e_h = \{x_{i_h},\overline{x}_{i_h}\}$ for $h\in\{1,2,3\}$. By the construction of $\hat{G}'$, $\mathbf{dist}_{\hat{G}_1'}(v_r,C_j) = \min\{\mathbf{dist}_{\hat{G}_1'}(v_r,x_{i_h})+(2s-\mathbf{dist}_T(r,e_h)) | 1 \leq h \leq 3\}$. Let $\mathbf{dist}_{\hat{G}_1'}(v_r,C_j) = \mathbf{dist}_{\hat{G}_1'}(v_r,x_{i_h})+(2s-\mathbf{dist}_T(r,e_h))$ for $h\in\{1,2,3\}$. It follows that $\mathbf{dist}_{\hat{G}_1'}(v_r,x_{i_h}) \leq \mathbf{dist}_T(r,e_h)$, and this immediately implies that $\{v_p,x_{i_h}\}\in A$ where p is the parent of e_h in T. By the definition, $x_{i_h} = true$ and, therefore, $C_j = true$. This holds for each C_j for $j\in\{1,\ldots,m\}$, and we conclude that $\phi = true$.

To complete the proof of the NP-hardness of BPDC, it remains to observe that $\hat{G}$ can be constructed in polynomial time.

To show NP-hardness of BFPDC, we use similar arguments.

Let (ϕ, G') be an instance of PLANE SATISFIABILITY WITH CONNECTIVITY OF VARIABLES, where $\phi = C_1 \wedge \ldots \wedge C_m$ is a Boolean formula with clauses $C_1, \ldots, C_m$ with at most 3 literals each over variables $x_1, \ldots, x_n$ such that G'_ϕ is planar, and G' is a plane embedding of G'_ϕ such that $H(G')$ is connected. As before, we pick an arbitrary vertex $r \in F(G')$ of $H(G')$ and find a tree T rooted in r of shortest (r,e)-paths for $e \in R_\phi$ with the set of leaves $L \subseteq R_\phi$. Let $s = \max\{\mathbf{dist}_T(r,e) | e \in L\}$.

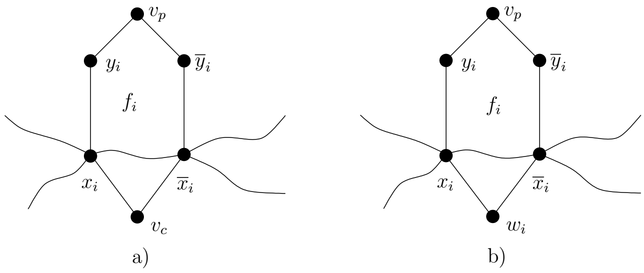

Figure 8: Construction of gadgets for $\{x_i, \overline{x}_i\}$.

We construct the plane graph $\tilde{G}$ similarly to the construction of $\hat{G}$ above. The only difference is that Steps iii) and iv) are replaced by the following steps iii*) and iv*).

iii*) For each $e = \{x_i, \overline{x_i}\} \in R_\phi \setminus L$, denote by p its parent and by c its child in T, construct vertices $y_i, \overline{y}_i$ and edges $\{x_i, y_i\}, \{y_i, v_p\}, \{x_i, v_c\}$, $\{\overline{x}_i, \overline{y}_i\}, \{\overline{y}_i, v_p\}, \{\overline{x}_i, v_c\}$ and embed them as is shown in Fig. 8 a). Denote by f_i the inner face of the cycle $x_i y_i v_p \overline{y}_i \overline{x}_i x_i$.

iv*) For each $e = \{x_i, \overline{x_i}\} \in L$, denote by p its parent in T, construct vertices $y_i, \overline{y}_i, w_i$ and edges $\{x_i, y_i\}, \{y_i, v_p\}, \{x_i, w_i\}$, $\{\overline{x}_i, \overline{y}_i\}, \{\overline{y}_i, v_p\}, \{\overline{x}_i, w_i\}$ and embed them as is shown in Fig 8 b). Denote by f_i the inner face of the cycle $x_i y_i v_p \overline{y}_i \overline{x}_i x_i$.

Observe that $\tilde{G}$ can be obtained from $\hat{G}$ by the deletion of the vertices $z_1, \ldots, z_n$, and for any $u, v \in V(\tilde{G})$, $\mathbf{dist}_{\tilde{G}}(u,v) = \mathbf{dist}_{\hat{G}}(u,v)$. Notice that the obtained graph $\tilde{G}$ is 3-connected, the faces $f_1, \ldots, f_n$ have degree 5, and all other faces have degree 3. To complete the construction of an instance of BFPDC, we set $k = 1$ and $d = 2\ell + 12s$.

We show that (ϕ, G') is a yes-instance of PLANE SATISFIABILITY WITH CONNECTIVITY OF VARIABLES if and only if $(\tilde{G}, k, d)$ is a yes-instance of BFPDC.

Suppose that (ϕ, G') is a yes-instance of PLANE SATISFIABILITY WITH CONNECTIVITY OF VARIABLES. Assume that the variables $x_1, \ldots, x_n$ have values such that $\phi = true$. For $i \in \{1, \ldots, n\}$, if $x_i = true$, then we add an edge $\{x_i, v_p\}$ for the parent p of $\{x_i, \overline{x}_i\}$ in T and embed this edge in f_i. Respectively, we add an edge $\{\overline{x}_i, v_p\}$ and embed this edge in f_i if $x_i = false$. Denote the obtained graph by $\tilde{G}'$. By exactly the same arguments as for the proof of the inequality $\mathbf{diam}(\hat{G}') \leq d$, we have that $\mathbf{diam}(\tilde{G}') \leq d$.

Suppose now that $(\hat{G}, k, d)$ is a yes-instance of BFPDC. Let A be a set of edges such that the graph $\tilde{G}'$ obtained from $\tilde{G}$ by the addition of A has diameter at most

d. Because only the faces $f_1, \dots, f_n$ have degree 5 and all other faces have degree 3, each edge of A has its end-vertices in the boundary of some f_i and is embedded in this face. Because $k = 1$, at most one edge of A is embedded in f_i for $i \in \{1, \dots, n\}$. Let v'_r be the pole of the mast rooted in v_r. Because $\mathbf{diam}(\tilde{G}') \leq d$, for any $u \in V(\tilde{G}')$, $\mathbf{dist}_{\tilde{G}'}(v'_r, u) \leq d$ and, in particular, it holds for poles of other masts. Consider masts rooted in w_i for $e = \{x_i, \overline{x}_i\} \in L$. For a mast rooted in w_i, denote by w'_i its pole. Because $\mathbf{dist}_{\tilde{G}'}(v'_r, w'_i) \leq d$, by the same arguments that were used above in the proof of the NP-hardness of BPDC, we obtain that it implies that for each $h \in \{1, \dots, n\}$, either $\{x_i, v_p\} \in A$ or $\{\overline{x}_i, v_p\} \in A$ where p is the parent of $\{x_h, \overline{x}_h\}$ in T. For $h \in \{1, \dots, n\}$, we let the variable $x_h = true$ if $\{\overline{x}_i, v_p\} \in A$ and $x_h = false$ otherwise. To prove that this assignment satisfies ϕ, we again use the same arguments as above: it follows from the fact that for each clause C_j, $\mathbf{dist}_{\tilde{G}'}(v'_r, C'_j) \leq d$ where C'_j is the pole of the mast rooted in the vertex C_j.

To complete the proof of the NP-hardness of BPDC, it remains to observe that $\tilde{G}$ can be constructed in polynomial time. □

We proved that BPDC is NP-complete for 3-connected planar graphs. By the Whitney's theorem (see, e.g., [6]), any two plane embeddings of a 3-connected plane graphs are equivalent. It gives the following corollary.

Bounded Budget Planar Diameter Completion
Input: A planar graph G, non-negative integers k and d.
Output: Is it possible to obtain a planar graph G' of diameter at most d from G by adding at most k edges?

Corollary 2. Bounded Budget Planar Diameter Completion *is* NP*-complete for 3-connected planar graphs.*

7 Discussion

We remark that our algorithm still works for the classic PDC problem when the face-degree of the input graph is bounded. For this we define the following problem:

Bounded Face BDC (FPDC)
Input: a plane graph G with face-degree at most $k \in \mathbb{N}_{\geq 3}$, and $d \in \mathbb{N}$
Question: is it possible to add edges in G such that the resulting embedding remains plane and has diameter at most d?

We directly have the following corollary of Theorem 2.

Theorem 3. *It is possible to construct an* $O(n^3) + 2^{2^{O((kd)\log d)}} \cdot n$*-step algorithm for* FPDC.

To construct an FPT-algorithm for PDC when parameterized by d remains an insisting open problem. The reason why our approach does not apply (at least directly) for PDC is that, as long as a completion may add an arbitrary number of edges in each face, we cannot guarantee that our dynamic programming algorithm will be applied on a graph of bounded branchwidth. We believe that our approach and, in particular, the machinery of our dynamic programming algorithm, might be useful for further investigations on this problem.

All the problems in this paper are defined on plane graphs. However, one may also consider the "non-embedded" counterparts of the problems PDC and BPDC by asking that their input is a planar combinatorial graphs (without a particular embedding). Similarly, such a counterpart can also be defined for the case of BFPDC if we ask whether the completion has an embedding with at most k new edges per face. Again, all these parameterized problems are known to be (non-constructively) in FPT, because of the results in [18, 16]. However, our approach fails to design the corresponding algorithms as it strongly requires an embedding of the input graph. For this reason we believe that even the non-embedded versions of BPDC and BFPDC are as challenging as the general Planar Diameter Completion problem.

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