A Polynomial-time Algorithm for Outerplanar Diameter Improvement*

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Abstract

The OUTERPLANAR DIAMETER IMPROVEMENT problem asks, given a graph G and an integer D, whether it is possible to add edges to G in a way that the resulting graph is outerplanar and has diameter at most D. We provide a dynamic programming algorithm that solves this problem in polynomial time. OUTERPLANAR DIAMETER IMPROVEMENT demonstrates several structural analogues to the celebrated and challenging PLANAR DIAMETER IMPROVEMENT problem, where the resulting graph should, instead, be planar. The complexity status of this latter problem is open.

Keywords: diameter improvement; outerplanar graphs; completion problems; polynomial-time algorithms; dynamic programming.

1 Introduction

In general, a *graph completion problem* asks whether it is possible to add edges to some given input graph so that the resulting graph satisfies some target property. There are two different ways of defining the optimization measure for such problems. The first, and most common, is the number of edges to be added, while the second is the value of some graph invariant on the resulting graph. Problems of the first type are HAMILTONIAN COMPLETION [16], INTERVAL GRAPH COMPLETION [19], PROPER INTERVAL GRAPH COMPLETION [17, 23], CHORDAL GRAPH COMPLETION [23, 27],

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and STRONGLY CHORDAL GRAPH COMPLETION [23], where the property is being Hamiltonian, interval, proper interval, chordal, and strongly chordal, respectively.

We focus our attention to the second category of problems where, for some given parameterized graph property $\mathcal{P}_k$, the problem asks, given a graph G and an integer k, whether it is possible to add edges to G such that the resulting graph belongs to $\mathcal{P}_k$. Usually $\mathcal{P}_k$ is a parameterized graph class whose graphs are typically required (for every k) to satisfy some sparsity condition. There are few problems of this type in the bibliography. Such a completion problem is PLANAR DISJOINT PATHS COMPLETION that asks, given a plane graph and a collection of k pairs of terminals, whether it is possible to add edges such that the resulting graph remains plane and contains k vertex-disjoint paths between the pairs of terminals. This problem is easily seen to be NP-complete. Indeed, observe that the special case of PLANAR DISJOINT PATHS COMPLETION, in which the input graphs are triangulated, is equivalent to special case of the PLANAR DISJOINT PATHS problem, in which the input graphs are triangulated. This is because a triangulated plane graph is maximal, i.e., no edge can be added. It remains to see that the latter problem remains NP-complete via the following transformation from PLANAR DISJOINT PATHS without restriction: given a graph G and a collection $\mathcal{F}$ of pairs of terminals, we obtain a new graph G' by adding a vertex v at every face of G which is not a triangle and adding edges between v and each vertex on the boundary of the face. The new collection of pairs of terminals is obtained by adding $\{v, v\}$ to $\mathcal{F}$ for every newly added vertex v. On the other hand, PLANAR DISJOINT PATHS COMPLETION has been studied from the point of view of parameterized complexity [1]. In particular, when all edges should be added in the same face, it can be solved in $f(k) \cdot n^2$ steps [1], i.e., it is fixed parameter tractable (FPT in short; for details about fixed parameter tractability, we refer to the monographs [7, 12, 14, 24]).

Perhaps the most challenging problem of the second category is the PLANAR DIAMETER IMPROVEMENT problem (PDI in short), which was first mentioned by Dejter and Fellows [9] (and made an explicit open problem in [12]). Here we are given a planar (or plane) graph and we ask for the minimum integer D such that it is possible to add edges so that the resulting graph is a planar graph with diameter at most D (according to the general formalism, for each D, the parameterized property $\mathcal{P}_D$ contains all planar graphs with diameter at most D). The computational complexity of PLANAR DIAMETER IMPROVEMENT is open, as it is not even known whether it is an NP-complete problem, even in the case where the embedding is part of the input. Interestingly, PLANAR DIAMETER IMPROVEMENT is known to be FPT: it is easy to verify that, for every D, its YES-instances are closed under taking minors. To see this, if a graph G can be completed into a planar graph G' of diameter D, then G' is also a valid completion of any subgraph $H \subseteq G$. Similarly, by merging two adjacent vertices uv in both G and G', one sees that the diameter is also closed under edge contraction. This fact, according to the meta-algorithmic consequence of the Graph Minors series of Robertson and Seymour [25, 26], implies that PLANAR DIAMETER IMPROVEMENT is FPT. Unfortunately, this implication only proves the *existence* of such an algorithm for each D, while it does not give any way to construct it. Whether this problem is uniformly FPT[1] remains as one of the most intriguing open questions in parameterized algorithm design. To our knowledge, when it comes to explicit algorithms, it is not even clear how to get an $O(n^{f(D)})$-algorithm for this problem (in parameterized complexity terminology, such an algorithm is called an XP-algorithm).

Notice that, in both aforementioned problems of the second type, the planarity of the graphs in $\mathcal{P}_D$ is an important restriction, as it is essential for generating a

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[1] As opposed to having a possibly different algorithm for each D, a problem is *uniformly* FPT if the algorithm solving the problem is the same for each D.

non-trivial problem; otherwise, one could immediately turn a graph into a clique that trivially belongs to $\mathcal{P}_1$. For practical purposes, such problems are relevant where instead of generating few additional links, we mostly care about maintaining the network topology. The algorithmic and graph-theoretic study on diameter improvement problems has focused both on the case of minimizing the number (or weight) of added edges [2, 4, 5, 11, 13, 20], as well as on the case of minimizing the diameter [4, 15]. In contrast, the network topology, such as acyclicity or planarity, as a constraint to be preserved has received little attention in the context of complementing a graph; see for example [13]. See also [21, 22] for other completion problems in outerplanar graphs, where the objective is to add edges in order to achieve a prescribed connectivity.

In this paper we study the OUTERPLANAR DIAMETER IMPROVEMENT problem, or OPDI in short. An instance of OPDI consists of an outerplanar graph $G = (V, E)$ and a positive integer D, and we are asked to add a set F of missing edges to G so that the resulting graph $G' = (V, E \cup F)$ has diameter at most D, while G' remains outerplanar. Note that we are allowed to add arbitrarily many edges as long as the new graph is outerplanar. Given a graph $G = (V, E)$, we call $G' = (V, E \cup F)$ a *completion* of G.

It appears that the combinatorics of OPDI demonstrate some interesting parallelisms with the notorious PDI problem. We denote by $\mathbf{opdi}(G)$ (resp. $\mathbf{pdi}(G)$) the minimum diameter of an outerplanar (resp. planar) completion of G. It is not difficult to see that the treewidth of a planar graph with bounded $\mathbf{pdi}(G)$ is bounded, while the pathwidth of an outerplanar graph with bounded $\mathbf{opdi}(G)$ is also bounded. Indeed, to see the former relation, it is known that if the treewidth of a planar graph G is at least $5D$ for some integer D, then G contains the $(D \times D)$-grid, say Γ_D, as a minor [18], and it is easy to see that $\mathbf{pdi}(\Gamma_D) = \Omega(D)$. Combined with the fact that $\mathbf{pdi}$ is closed under taking minors, this implies that $\mathbf{pdi}(G) = \Omega(D)$. This establishes that the treewidth of G is $O(\mathbf{pdi}(G))$.

To see the second relation, let G be an outerplanar graph whose pathwidth is at least $3^{D+1} - 1$ for some integer D. Using the fact that for every forest F, every graph with pathwidth at least $|V(F)| - 1$ has a minor isomorphic to F [3], it follows that G contains a full ternary tree of depth D, say T_D, as a minor. (Namely, T_D is the rooted tree in which every non-leaf vertex has three children, and every leaf is at distance exactly D from the root.) It can be proved by induction on D that $\mathbf{opdi}(T_D) = \Omega(D)$. Combined with the fact that $\mathbf{opdi}$ is also closed under taking minors, this implies that $\mathbf{opdi}(G) = \Omega(D)$. This establishes that the pathwidth of G is $2^{O(\mathbf{opdi}(G))}$. In that sense, the OPDI can be seen as the "linear counterpart" of PDI. We stress that the same "small pathwidth" behavior of OPDI holds even if, instead of outerplanar graphs, we consider any class of graphs with bounded outerplanarity.

Note also that both $\mathbf{pdi}(G)$ and $\mathbf{opdi}(G)$ are trivially 2-approximable in the particular case where the embedding is given. To see this, let G' be a triangulation of a plane (resp. outerplane) embedding of G where, in every face of G, all edges added to it have a common endpoint. Then, for each edge uv in each shortest path in an optimal completion of G, a u-v-path of length at most two exists in G'. Thus, for both graph invariants, the diameter of G' does not exceed twice the optimal value.

Our results. In this work, we show that OUTERPLANAR DIAMETER IMPROVEMENT is polynomial-time solvable. Our algorithm, described in Section 2, is based on dynamic programming and works in full generality, even when the input graph may be disconnected. Also, our algorithm does *not* assume that the input comes with some specific embedding (in the case of an embedded input, the problem becomes considerably easier to solve).

2 Description of the algorithm

The aim of this section is to describe a polynomial-time dynamic program that, given an outerplanar graph G and an integer D, decides whether G admits an outerplanar completion with diameter at most D, denoted *diameter-D outerplanar completion* for simplicity. Note that such an algorithm easily yields a polynomial-time algorithm to compute, given an outerplanar graph G, the smallest integer D such that G admits a diameter-D outerplanar completion.

Before describing the algorithm, we show some properties of outerplanar completions. In particular, Subsection 2.1 handles the case where the input outerplanar graph has cut vertices. Its objective is to prove that we can apply a *reduction rule* to such a graph which is safe for the OPDI problem. In Subsection 2.2 we deal with 2-vertex separators, and in Subsection 2.3 we present a polynomial-time algorithm for *connected* input graphs. Finally, we present the algorithm for disconnected input graphs in Subsection 2.4.

Some notation. We use standard graph-theoretic notation, see for instance [10]. It is well known that a graph is outerplanar if and only if it excludes K_4 and $K_{2,3}$ as a minor. An outerplanar graph is *triangulated* if all its inner faces (in an outerplanar embedding) are triangles. An outerplanar graph is *maximal* if it is 2-connected and triangulated. Note that, when solving the OPDI problem, we may always assume that the completed graph G' is maximal.

2.1 Reducing the input graph when there are cut vertices

Given a graph G, let the *eccentricity* of a vertex u be $\mathrm{ecc}(u, G) = \max_{v \in V(G)} \mathrm{dist}_G(u, v)$. Given an outerplanar graph G, a vertex $u \in V(G)$, and an integer D, let us define $\mathrm{ecc}_D^*(u, G)$ as $\min_H \mathrm{ecc}(u, H)$, where the minimum is taken over all the diameter-D outerplanar completions H of G. If all the outerplanar completions have diameter more than D, we set this value to ∞. Unless said otherwise, we assume henceforth that D is a fixed given integer, so we may just write $\mathrm{ecc}^*(u, G)$ instead of $\mathrm{ecc}_D^*(u, G)$. (The value of D will change only in the description of the algorithm at the end of Subsection 2.3, and in that case we will make the notation explicit).

As admitting an outerplanar completion with bounded eccentricity is a minor-closed property, let us observe the following:

Lemma 1 *For any connected outerplanar graph G, any vertex $v \in V(G)$, and any connected subgraph H of G with $v \in V(H)$, we have that $\mathrm{ecc}^*(v, H) \leq \mathrm{ecc}^*(v, G)$.*

Proof: Let G' be an outerplanar completion of G achieving $\mathrm{ecc}^*(v, G)$, that is, such that $\mathrm{ecc}(v, G') = \mathrm{ecc}^*(v, G)$. Contracting, one at a time, the edges of G' that have exactly one endpoint in $V(H)$ one finally obtains an outerplanar completion H' of H (as outerplanar graphs are minor-closed). As contracting an edge does not elongate any shortest path, we have that $\mathrm{dist}_{H'}(v, u) \leq \mathrm{dist}_{G'}(v, u)$ for any vertex $u \in V(H)$, and in particular the diameter of H' is at most the diameter of G', so $\mathrm{ecc}^*(v, H) < \infty$. We thus have that $\mathrm{ecc}^*(v, H) \leq \mathrm{ecc}(v, H') \leq \mathrm{ecc}(v, G') = \mathrm{ecc}^*(v, G)$. $\qquad\square$

Consider a connected graph G with at least two vertices and a vertex $v \in V(G)$. Let $C_1, \ldots, C_t$ be the vertex sets of the connected components of $G \setminus \{v\}$ (note that in the case where v is not a cut vertex of G, it holds that $t = 1$). For $1 \leq i \leq t$, we call the vertex set $B_i = C_i \cup \{v\}$ a *branch* of G at v. To shorten notations, we abbreviate $B_i \cup \ldots \cup B_j =: B_{i \ldots j}$, for any $1 \leq i < j \leq t$. Also, when referring to the eccentricity, we

simply write B_i to denote the subgraph of G that is induced by B_i (i.e., $G[B_i]$). Thus, for example, when considering the value $\text{ecc}^*(v, B_{1...i})$, it will refer to the minimum eccentricity with respect to v that a diameter-D outerplanar completion of the graph $G[B_{1...i}]$ can have.

The following lemma, which is crucial in order to obtain a polynomial-time algorithm, implies that if G has a cut vertex v with many branches, it is safe to remove most of them.

Lemma 2 *Consider an outerplanar graph G with a cut vertex v that belongs to at least 7 branches. Denote these branches $B_1, \ldots, B_t$, with $t \geq 7$, in such a way that $\text{ecc}^*(v, B_1) \geq \text{ecc}^*(v, B_2) \geq \ldots \geq \text{ecc}^*(v, B_t)$. The graph G has an outerplanar completion with diameter at most D if and only if $\text{ecc}^*(v, B_{1...6}) + \text{ecc}^*(v, B_7) \leq D$.*

Proof: "$\Leftarrow$": If $\text{ecc}^*(v, B_{1...6}) + \text{ecc}^*(v, B_7) \leq D$, gluing on v the outerplanar completions of $G[B_{1...6}], G[B_7], \ldots, G[B_t]$, respectively achieving $\text{ecc}^*(v, B_{1...6}), \text{ecc}^*(v, B_7), \ldots, \text{ecc}^*(v, B_t)$, one obtains a diameter-D outerplanar completion G' of G. Indeed,

- The graph obtained is outerplanar and contains G.

- Two vertices x, y of $G[B_{1...6}]$ (resp. of $G[B_i]$ for $7 \leq i \leq t$) are at distance at most D from each other, as $\text{ecc}^*(v, B_{1...6}) < \infty$ (resp. as $\text{ecc}^*(v, B_i) < \infty$).

- Any vertex x of $G[B_{1...6}]$ and y of $G[B_i]$, with $7 \leq i \leq t$, are respectively at distance at most $\text{ecc}^*(v, B_{1...6})$ and $\text{ecc}^*(v, B_i) \leq \text{ecc}^*(v, B_7)$ from v. They are thus at distance at most $\text{ecc}^*(v, B_{1...6}) + \text{ecc}^*(v, B_7) \leq D$ from each other.

- Any vertex x of $G[B_i]$ and y of $G[B_j]$, with $7 \leq i < j \leq t$, are respectively at distance at most $\text{ecc}^*(v, B_i) \leq \text{ecc}^*(v, B_1) \leq \text{ecc}^*(v, B_{1...6})$ (By Lemma 1) and $\text{ecc}^*(v, B_j) \leq \text{ecc}^*(v, B_7)$ from v. They are thus at distance at most D from each other.

"$\Rightarrow$": In the following, we consider towards a contradiction an outerplanar graph G admitting a diameter-D outerplanar completion, but such that

$$\text{ecc}^*(v, B_{1...6}) + \text{ecc}^*(v, B_7) > D. \tag{1}$$

Among the triangulated diameter-D outerplanar completions of G, let G' be one that maximizes the number of branches at v. Let $t' > 0$ be the number of branches at v in G', and denote these branches $B'_1, \ldots, B'_{t'}$, in such a way that $\text{ecc}^*(v, G') = \text{ecc}^*(v, B'_1) \geq \text{ecc}^*(v, B'_2) \geq \ldots \geq \text{ecc}^*(v, B'_{t'})$. Let $S_{i'} := \{i \mid B_i \subseteq B'_{i'}\}$ for all $1 \leq i' \leq t'$ (note that $\{S_1, \ldots, S_{t'}\}$ is a partition of $\{1, \ldots, t\}$). Furthermore, among all $B'_{i'}$ maximizing $\text{ecc}^*(v, B'_{i'})$, we choose B'_1 such that $\min S_1$ is minimal. Then, since G' has diameter at most D and shortest paths among distinct branches of G' contain v, it is clear that

$$\underset{1 \leq i' < j' \leq t'}{\forall} \text{ecc}^*(v, B'_{i'}) + \text{ecc}^*(v, B'_{j'}) \leq D. \tag{2}$$

The branches $B'_{i'}$ with $|S_{i'}| = 1$ are called *atomic*.

Claim 1 *Let $B'_{i'}$ be a non-atomic branch and let $S' \subsetneq S_{i'}$. Then, $\text{ecc}^*(v, \bigcup_{i \in S'} B_i) + \text{ecc}^*(v, \bigcup_{i \in S_{i'} \setminus S'} B_i) > D$.*

Proof: Let $\mathcal{B} := \bigcup_{i \in S'} B_i$ and $\bar{\mathcal{B}} := B'_{i'} \setminus \mathcal{B}$. If the claim is false, then $\text{ecc}^*(v, \mathcal{B}) + \text{ecc}^*(v, \bar{\mathcal{B}}) \leq D$. Furthermore, for all $j' \neq i'$,

$$\text{ecc}^*(v, \mathcal{B}) + \text{ecc}^*(v, B'_{j'}) \overset{\text{Lemma 1}}{\leq} \text{ecc}^*(v, B'_{i'}) + \text{ecc}^*(v, B'_{j'}) \overset{(2)}{\leq} D$$

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and, likewise, $\text{ecc}^*(v, \bar{\mathcal{B}}) + \text{ecc}^*(v, B'_{j'}) \leq D$. Thus, the result of replacing $G'[B'_{i'}]$ with the disjoint union of an outerplanar completion achieving $\text{ecc}^*(v, \mathcal{B})$ and an outerplanar completion achieving $\text{ecc}^*(v, \bar{\mathcal{B}})$ yields a diameter-D outerplanar completion containing more branches than G', contradicting our choice of G'. $\square$

In the following, we abbreviate $|S_1| =: s$.

Claim 2 $S_1 = \{j \mid 1 \leq j \leq s\}$.

Proof: Towards a contradiction, assume that there is some $i \notin S_1$ with $i + 1 \in S_1$. Let $i' > 1$ be such that $B_i \subseteq B'_{i'}$. Note that B'_1 is not atomic, as otherwise $\text{ecc}^*(v, B'_1) = \text{ecc}^*(v, B_{i+1}) \leq \text{ecc}^*(v, B_i) \leq \text{ecc}^*(v, B'_{i'})$, contradicting the numbering of the B'_j's. Then,

$$
\begin{aligned}
\text{ecc}^*(v, B'_1 \setminus (B_{i+1} \setminus v))) + \text{ecc}^*(v, B_{i+1}) &\overset{\text{Lemma 1}}{\leq} \text{ecc}^*(v, B'_1) + \text{ecc}^*(v, B_{i+1}) \\
&\leq \text{ecc}^*(v, B'_1) + \text{ecc}^*(v, B_i) \\
&\leq \text{ecc}^*(v, B'_1) + \text{ecc}^*(v, B'_{i'}) \overset{(2)}{\leq} D,
\end{aligned}
$$

contradicting Claim 1. Note that, as $S_1 \neq \emptyset$, the claim follows. $\square$

Claim 3 For all i, we have $\text{ecc}^*(v, B_{1\ldots i}) + \text{ecc}^*(v, B_{i+1}) > D$ if and only if $i < s$.

Proof: "$\Leftarrow$": Towards a contradiction, assume there is some $i < s$ such that $\text{ecc}^*(v, B_{1\ldots i}) + \text{ecc}^*(v, B_{i+1}) \leq D$. Then the graph obtained from the diameter-D outerplanar completions of $B_{1\ldots i}$ and B_j for all $j > i$, respectively achieving $\text{ecc}^*(v, B_{1\ldots i})$ and $\text{ecc}^*(v, B_j)$, would be a diameter-D outerplanar completion of G with more branches than G', a contradiction.

"$\Rightarrow$": Assume towards a contradiction that there is some $i \geq s$ such that $\text{ecc}^*(v, B_{1\ldots i}) + \text{ecc}^*(v, B_{i+1}) > D$. By (2) and Lemma 1, we have $D \geq \text{ecc}^*(v, B_{1\ldots s}) + \text{ecc}^*(v, B_{i+1})$ and, hence $\text{ecc}^*(v, B_{1\ldots i}) > \text{ecc}^*(v, B_{1\ldots s})$. But this contradicts Lemma 1, as $\text{ecc}^*(v, B_{1\ldots s}) = \text{ecc}(v, G') \geq \text{ecc}^*(v, G)$. $\square$

By (1), Claim 3 implies that $s \geq 7$.

Claim 4 Let $S' \subseteq \{1, \ldots, t\}$ and let $\mathcal{B} := \bigcup_{i \in S'} B_i$. Then, there is a vertex in $\mathcal{B}$ that is, in G', at distance at least $\text{ecc}^*(v, \mathcal{B})$ to any vertex of $V(G) \setminus (\mathcal{B} \setminus v)$.

Proof: Towards a contradiction, assume that for any vertex $u \in \mathcal{B}$ there exists a vertex $w \in V(G) \setminus (\mathcal{B} \setminus v))$ such that $\text{dist}_{G'}(u, w) < \text{ecc}^*(v, \mathcal{B})$. From G', contracting all vertices of $V(G) \setminus \mathcal{B}$ onto v yields a graph H with a path between u and v of length strictly smaller than $\text{ecc}^*(v, \mathcal{B})$. As this argument holds for any vertex $u \in \mathcal{B}$, it implies that $\text{ecc}(v, H) < \text{ecc}^*(v, \mathcal{B})$. Since H is an outerplanar completion of $G[\mathcal{B}]$, this contradicts the definition of ecc^*. $\square$

Two sub-branches B_i and B_j of B'_1 are *linked* if G' has an edge intersecting both $B_i \setminus \{v\}$ and $B_j \setminus \{v\}$.

Claim 5 Let $1 \leq i < j \leq s$ and let $\text{ecc}^*(v, B_{1\ldots i}) + \text{ecc}^*(v, B_j) > D$. Then $\text{ecc}^*(v, B_{1\ldots i}) + \text{ecc}^*(v, B_j) = D + 1$, and B_j is linked to one of $B_1, \ldots, B_i$.

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Proof: By Claim 4, there is a vertex $x \in B_j$ that is, in G', at distance at least $\mathrm{ecc}^*(v, B_j)$ to any vertex in $B_{1...i}$. Likewise, there is a vertex $y \in B_{1...i}$ that is, in G', at distance at least $\mathrm{ecc}^*(v, B_{1...i})$ to any vertex in B_j. Let P be any shortest path of G' between x and y (hence P has length at most D). By construction, the maximal subpath of P in $B_j \setminus v$ containing x has length at least $\mathrm{ecc}^*(v, B_j) - 1$ and the maximal subpath of P in $B_{1...i} \setminus v$ containing y has length at least $\mathrm{ecc}^*(v, B_{1...i}) - 1$. Since these subpaths are vertex disjoint the remaining part of P has length $d_P \geq 1$. Hence $D \geq \mathrm{ecc}^*(v, B_j) + \mathrm{ecc}^*(v, B_{1...i}) + d_P - 2$. As $\mathrm{ecc}^*(v, B_{1...i}) + \mathrm{ecc}^*(v, B_j) > D$, we have that $d_P = 1$, and thus there is a single edge in P linking B_j and $B_{1...i}$. This also yields to $\mathrm{ecc}^*(v, B_j) + \mathrm{ecc}^*(v, B_{1...i}) = D + 1$. □

Claim 3 and Claim 5 clearly imply that for any $1 \leq i < s$, B_{i+1} is linked to one of $B_1, \ldots, B_i$. A consequence of the next claim will be that B_{i+1} is linked to exactly one of these branches.

In the following, consider the graph L on the vertex set $\{1, \ldots, t\}$ such that ij is an edge of L if and only if B_i is linked to B_j in G'. For all $1 \leq k \leq t$, let L_k be the subgraph of L that is induced by $\{1, \ldots, k\}$.

Claim 6 *For each $1 \leq k \leq s$, the graph L_k is a path.*

Proof: Let $1 \leq k \leq s$. Then,

1. L_k is connected since otherwise, v would be a cut vertex in $G'[B_1']$, contradicting the definition of branch.

2. L_k has maximum degree 2: towards a contradiction, assume that some branch B_i is linked to three branches B_{j_1}, B_{j_2}, and B_{j_3}. As each of $B_i \setminus v$, $B_{j_1} \setminus v$, $B_{j_2} \setminus v$, and $B_{j_3} \setminus v$ induces a connected graph in G', these four sets together with v induce a $K_{2,3}$-minor in G', contradicting its outerplanarity.

3. L_k is not a cycle since otherwise, as each $B_i \setminus v$ induces a connected graph in G', these sets together with v would induce a K_4-minor in G' (since $s \geq 3$), contradicting its outerplanarity. □

Hence, for any $1 \leq i \leq s$, the graph $G'[B_{1...i} \setminus v]$ is connected.

Claim 7 *For any $3 \leq i < s$, $\mathrm{ecc}^*(v, B_{1...i}) > \mathrm{ecc}^*(v, B_{1...i-2})$.*

Proof: The monotonicity property given by Lemma 1 implies that $\mathrm{ecc}^*(v, B_{1...i}) \geq \mathrm{ecc}^*(v, B_{1...i-1}) \geq \mathrm{ecc}^*(v, B_{1...i-2})$. Towards a contradiction, suppose that $\mathrm{ecc}^*(v, B_{1...i}) = \mathrm{ecc}^*(v, B_{1...i-1}) = \mathrm{ecc}^*(v, B_{1...i-2}) =: c$. Then, Claim 3 implies that $c + \mathrm{ecc}^*(v, B_j) > D$ for all $j \in \{i-1, i, i+1\}$. Thus, by Claim 5, each of B_{i-1}, B_i, and B_{i+1} is linked to one of $B_1, \ldots, B_{i-2}$. As each of $B_{1...i-2} \setminus v$, $B_{i-1} \setminus v$, $B_i \setminus v$, and $B_{i+1} \setminus v$ induces a connected graph in G', these sets together with vertex v induce a $K_{2,3}$-minor, contradicting the outerplanarity of G'. □

In the following let q be any integer such that $3 \leq q \leq s$ and B_q is not linked to B_1. Let $p < q$ be such that B_p and B_q are linked. Note that p is unique since otherwise, L_q would not be a path, contradicting Claim 6.

Consider a shortest cycle containing v, a vertex $u \in B_p$ and some vertex of B_q. Since G' is triangulated, this cycle is a triangle. Thus, u is a a neighbor of v (in G') with $u \in B_p$ and u is adjacent to some vertex in $B_q \setminus v$ (see Figure 1 for an illustration).

Since, by Claim 6, all paths in G' between a vertex in B_1 and a vertex in B_q contain u or v, it is clear that $\{v, u\}$ separates $B_1 \setminus v$ and $B_q \setminus v$. Let (X, Y) be a separation of
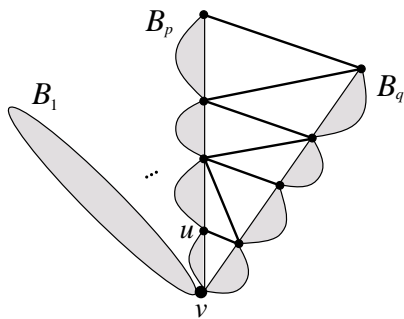
Figure 1: Structure of $G'[B_{1...q}]$.

G' (that is, two sets $X, Y \subseteq V(G')$ such that $X \cup Y = V(G')$ and such that there are no edges between $X \setminus Y$ and $Y \setminus X$) such that $X \cap Y = \{v, u\}$, $B_{1...q-1} \setminus B_p \subsetneq X$ and $B_q \subseteq Y$ (such a separation exists by Claim 6).

Claim 8 $\mathrm{ecc}^*(v, B_{1...q}) = \mathrm{ecc}^*(v, B_{1...q-1})$.

Proof: By Lemma 1, it suffices to show $\mathrm{ecc}^*(v, B_{1...q}) \leq \mathrm{ecc}^*(v, B_{1...q-1})$. To this end, let H be the outerplanar completion of $B_{1...q}$ obtained from G' by contracting every branch B_i, with $i > q$, onto v. Since H is a minor of G', H is a diameter-D outerplanar completion of $B_{1...q}$. We show $\mathrm{ecc}(v, H) \leq \mathrm{ecc}^*(v, B_{1...q-1})$.

Consider any vertex $x \in X$, and let $y \in B_q \subseteq Y$ be a vertex that is at distance at least $\mathrm{ecc}^*(v, B_q)$ to both v and u (such a vertex y exists by Claim 4). As all shortest paths between x and y (of length at most D) contain v or u, the vertex x is at distance at most $D - \mathrm{ecc}^*(v, B_q)$ to v or u. As v and u are adjacent, the vertex x is at distance at most $D + 1 - \mathrm{ecc}^*(v, B_q)$ ($= \mathrm{ecc}^*(v, B_{1...q-1})$) by Claim 5, which is applicable since, by Claim 3, $\mathrm{ecc}^*(v, B_{1...q-1}) + \mathrm{ecc}^*(v, B_q) > D$) to v. Since x is chosen arbitrarily in X, every vertex in X is at distance at most $\mathrm{ecc}^*(v, B_{1...q-1})$ to v in H.

Consider now any vertex $y \in Y \cap V(H)$, and let $x \in B_1 \subsetneq X$ be a vertex that is at distance at least $\mathrm{ecc}^*(v, B_1)$ to both v and u (such a vertex x exists by Claim 4). As a shortest path between x and y (of length at most D) goes through v or u, the vertex y is thus at distance at most $D - \mathrm{ecc}^*(v, B_1)$ to v or u. As v and u are adjacent, the vertex y is at distance at most $D + 1 - \mathrm{ecc}^*(v, B_1)$ ($= \mathrm{ecc}^*(v, B_2)$ by Claim 5, which is applicable since, by Claim 3, $\mathrm{ecc}^*(v, B_1) + \mathrm{ecc}^*(v, B_2) > D$) to v. As $\mathrm{ecc}^*(v, B_2) \leq \mathrm{ecc}^*(v, B_{1...q-1})$ by Lemma 1, every vertex $y \in Y \cap V(H)$ is at distance at most $\mathrm{ecc}^*(v, B_{1...q-1})$ to v in H. $\qquad \square$

We now claim that there exist two consecutive such values q between 3 and 6. Indeed, note first that B_1 is linked to at most two other branches, as otherwise these branches together with v and B_1 would induce a $K_{2,3}$-minor. Note also that by Claim 3 and Claim 5, B_2 is linked to B_1, so it follows that B_1 is linked to at most one branch B_j with $j \geq 3$. Therefore, for $3 \leq q \leq 6$, there are at least two consecutive values of q such that B_q is not linked to B_1. Once we have these two consecutive values, say $i - 1$ and i, we have by Claim 8 that $\mathrm{ecc}^*(v, B_{1...i-2}) = \mathrm{ecc}^*(v, B_{1...i})$, for some $i \leq 6$, contradicting Claim 7. This concludes the proof of the lemma. $\qquad \square$

Our algorithm will compute the eccentricity of a given "root" vertex r in a diameter-D outerplanar completion G' of G in which this eccentricity is minimal, that is, $\mathrm{ecc}^*(r, G)$. Then, however, the branch containing the root (B_0 in Algorithm 1, Subsection 2.3) should not be removed. Therefore, although Lemma 2 already implies that G

has a diameter-D outerplanar completion if and only if $G[B_{1...7}]$ does, we instead use the following corollary to identify removable branches.

Corollary 1 *Let G be an outerplanar graph with a cut vertex v that belongs to at least 8 branches. Denote these branches $B_1, \ldots, B_t$, with $t \geq 8$, in such a way that $\mathrm{ecc}^*(v, B_1) \geq \mathrm{ecc}^*(v, B_2) \geq \ldots \geq \mathrm{ecc}^*(v, B_t)$. For each $8 \leq i \leq t$, the graph $G_i = \bigcup_{j \in \{1, \ldots, 7, i\}} B_j$ has a diameter-D outerplanar completion if and only if G does.*

Proof: Recall that the property of having an outerplanar completion with bounded diameter is minor closed. Thus G_i being a minor of G, we have that if G admits a diameter-D outerplanar completion, then so does G_i.

On the other hand, if G_i admits a diameter-D outerplanar completion, by Lemma 2 applied to G_i we have that $\mathrm{ecc}^*(v, B_{1...6}) + \mathrm{ecc}^*(v, B_7) \leq D$. Thus gluing on v the outerplanar completions of $G[B_{1...6}], G[B_7], \ldots, G[B_t]$, respectively achieving $\mathrm{ecc}^*(v, B_{1...6}), \mathrm{ecc}^*(v, B_7), \ldots, \mathrm{ecc}^*(v, B_t)$, one obtains a diameter-D outerplanar completion of G. □

2.2 Dealing with 2-vertex separators

In this subsection, we extend the definition of eccentricity to the pairs (u, v) such that $uv \in E(G)$. Namely, $\mathrm{ecc}(u, v, G)$ is defined as the set of pairs obtained by taking the maximal elements of the set $\{(\mathrm{dist}_G(u, w), \mathrm{dist}_G(v, w)) \mid w \in V(G)\}$. The pairs are ordered such that $(d_1, d_2) \leq (d_1', d_2')$ if and only if $d_1 \leq d_1'$ and $d_2 \leq d_2'$. As u and v are adjacent, note that $\mathrm{dist}_G(u, w)$ and $\mathrm{dist}_G(v, w)$ differ by at most one. Hence, $\mathrm{ecc}(u, v, G)$ equal to one of $\{(d, d)\}$, $\{(d, d+1)\}$, $\{(d+1, d)\}$, and $\{(d, d+1), (d+1, d)\}$, for some positive integer d. As before, we abbreviate $\mathrm{ecc}(u, v, G[X])$ by $\mathrm{ecc}(u, v, X)$. Given a graph G and a subset $S \subseteq V(G)$, we denote by $\partial(S)$ the set of vertices in S that have at least one neighbor in $V(G) \setminus S$.

Lemma 3 *Consider a connected graph G with $V(G) =: X$ and a triangle uvw and two sets $X_u, X_v \subseteq X$ such that $X_u \cup X_v = X$, $X_u \cap X_v = \{w\}$, $\partial(X_u) \subseteq \{u, w\}$, and $\partial(X_v) \subseteq \{v, w\}$. Then $\mathrm{ecc}(u, v, G)$ equals the maximal elements of the set*

$$\{(d_u, \min\{d_u + 1, d_w + 1\}) \mid (d_u, d_w) \in \mathrm{ecc}(u, w, X_u)\} \quad \cup$$
$$\{(\min\{d_w + 1, d_v + 1\}, d_v) \mid (d_w, d_v) \in \mathrm{ecc}(w, v, X_v)\}.$$

Proof: It is clear from the fact that a shortest path from $X_u \setminus \{u\}$ to u does not go through $X_v \setminus \{w\}$ (as it should go through $w \in N(u)$), from the fact that a shortest path from X_u to v goes through $\{u, w\} \subseteq N(v)$, and from the fact that any subpath of a shortest path is a shortest path (for some pair of vertices). □

Given a connected outerplanar graph G, for any two vertices $u, v \in V(G)$ and any vertex set $X \subseteq V(G)$ with $u, v \in X$ such that $\partial(X) \subseteq \{u, v\}$, let us define $\mathrm{ecc}_D^*(u, v, X)$ as the minimal elements of the set

$$\left\{ \mathrm{ecc}(u, v, H) \;\middle|\; \begin{array}{l} H \text{ is a diameter-}D \text{ outerplanar completion of } G[X] \text{ such} \\ \text{that } uv \in E(H) \text{ and such that } uv \text{ lies on the outer face.} \end{array} \right\}$$

If this set is empty, we set $\mathrm{ecc}_D^*(u, v, X)$ to (∞, ∞). Similarly to Subsection 2.1, we may drop the subindex D from the notation $\mathrm{ecc}_D^*(u, v, X)$. Here, $\mathrm{ecc}(u, v, H) \leq \mathrm{ecc}(u, v, H')$ if and only if for any $(d_1, d_2) \in \mathrm{ecc}(u, v, H)$ there exists a $(d_1', d_2') \in \mathrm{ecc}(u, v, H')$ such that $(d_1, d_2) \leq (d_1', d_2')$. According to the possible forms of $\mathrm{ecc}(u, v, H)$, we have that $\mathrm{ecc}^*(u, v, X)$ is of one of the following five forms:

- $\{\{(d, d)\}\}$,

- $\{\{(d, d+1)\}\}$,

- $\{\{(d+1, d)\}\}$,

- $\{\{(d, d+1), (d+1, d)\}\}$, or

- $\{\{(d, d+1)\}, \{(d+1, d)\}\}$,

for some positive integer d.

Considering $\mathrm{ecc}^*(u, X)$ for some u and X, note that u has at least one incident edge uv on the outer face in an outerplanar completion achieving $\mathrm{ecc}^*(u, X)$. Thus, we can observe the following.

Observation 1 $\mathrm{ecc}^*(u, X) = \min_{v \in X} \min_{S \in \mathrm{ecc}^*(u,v,X)} \max_{(d_u, d_v) \in S} d_u$.

2.3 The algorithm for connected outerplanar graphs

We now proceed to describe a polynomial-time algorithm that solves OUTERPLANAR DIAMETER IMPROVEMENT when the input outerplanar graph is assumed to be connected. In Subsection 2.4 we will deal with the disconnected case. In a graph, a *block* is either a 2-connected component or a bridge. Before proceeding to the formal description of the algorithm, let us provide a high-level sketch.

Algorithm 1 described below receives a *connected* outerplanar graph G, an arbitrary non-cut vertex r of G, called the *root* (such a vertex is easily seen to exist in any graph), and a positive integer D. In order to decide whether G admits a diameter-D outerplanar completion, we will compute in polynomial time the value of $\mathrm{ecc}_D^*(r, G)$, which, by definition, is distinct from ∞ if and only if G admits a diameter-D outerplanar completion.

In order to compute $\mathrm{ecc}_D^*(r, G)$, the algorithm proceeds as follows. In the first step (lines 1-9), we consider an arbitrary block B of G containing r (line 1), and in order to reduce the input graph G, we consider all cut vertices v of G in B. For each such cut vertex v, we order its corresponding branches according to their eccentricity w.r.t. v (line 8), and by Corollary 1 it is safe to keep just a constant number of them, namely 8 (line 9). For computing the eccentricity of the branches not containing the root (lines 5-7), the algorithm calls itself recursively, by considering the branch as input graph, and vertex v as the new root.

In the second step of the algorithm (lines 10-17), we try all 2-vertex separators u, v in the eventual completed graph G' (note that G' cannot be 3-connected, as otherwise it would contain a $K_{2,3}$-minor), together with a set X consisting of a subset of the connected components of $G' \setminus \{u, v\}$, not containing the root r. For each such triple (u, v, X), our objective is to compute the value of $\mathrm{ecc}_D^*(u, v, X)$. For doing so, after initializing its value (lines 11-12), we consider all possible triples w, X_u, X_v chosen as in Lemma 3 after adding the triangle uvw to $G[X]$ (line 13), for which we already know the values of $\mathrm{ecc}_D^*(u, w, X_u)$ and $\mathrm{ecc}_D^*(w, v, X_v)$, since the sets X are processed by increasing size. Among all choices of one element in $\mathrm{ecc}_D^*(u, w, X_u)$ and another in $\mathrm{ecc}_D^*(w, v, X_v)$ (line 14), only those whose corresponding completion achieves diameter at most D are considered for updating the value of $\mathrm{ecc}_D^*(u, v, X)$ (line 15). For updating $\mathrm{ecc}_D^*(u, v, X)$ (line 17), we first compute $\mathrm{ecc}_D(u, v, X)$ using Lemma 3 (line 16).

Finally, once we have computed all values of $\mathrm{ecc}_D^*(u, v, X)$, we can easily compute the value of $\mathrm{ecc}_D^*(u, X)$ by using Observation 1 (line 18). We can now provide a formal description of the algorithm.

Algorithm 1: `OPDI-Connected`

Input : A connected outerplanar graph G, a root $r \in V(G)$ such that $G \setminus \{r\}$ is connected, and a positive integer D.

Output: $\mathrm{ecc}_D^*(r, G)$.

```
// all over the recursive calls of the algorithm, G is a global
   variable, which gets updated whenever some vertices are removed
   in line 9
```

1 Let B be a block of G containing r

```
// we consider all cut vertices of B and we reduce G
```

2 **foreach** *cut vertex* $v \in V(B)$ **do**

3   Let $C_0, \ldots, C_t$ be the connected components of $G \setminus \{v\}$, where $r \in C_0$

4   Let $B_0 \leftarrow C_0$ `// the branch containing the root`

5   **for** $i \leftarrow 1$ **to** t **do**

6   Let $B_i \leftarrow G[C_i \cup \{v\}]$ `// the branches around v`

7 $\mathrm{ecc}_i \leftarrow$ `OPDI-Connected`(B_i, v, D).

          `// recursive call to compute` $\mathrm{ecc}_D^*(v, B_i)$

8   Reorder the B_i's so that $\mathrm{ecc}_1 \geq \mathrm{ecc}_2 \geq \ldots \geq \mathrm{ecc}_t$

9   Remove $B_8, \ldots, B_t$ from G `// by Corollary 1`

```
// guess all size-2 separators u,v in the target completion G',
   together with a subset X of the connected components of
   G' \ {u, v}
```

10 **foreach** *triple* (u, v, X) *such that* $r \notin X \setminus \{u, v\}$ *and* $\partial(X) \subseteq \{u, v\}$ **do**

```
// by increasing size of X, and only if the triple (u, v, X) has
   not already been considered before in a previous iteration
```

11 $\mathrm{Tab}_{\mathrm{ECC}}(u, v, X) \leftarrow \{\{(\infty, \infty)\}\}$.   `// it corresponds to` $\mathrm{ecc}_D^*(u, v, X)$

12   **if** $X = \{u, v\}$ **then** $\mathrm{Tab}_{\mathrm{ECC}}(u, v, X) \leftarrow \{\{(0, 1), (1, 0)\}\}$.

13   **else foreach** w, X_u, X_v *satisfying the hypothesis of Lemma 3 in the graph obtained from* $G[X]$ *by adding the triangle* uvw **do**

```
// eccentricities of smaller subgraphs have been already
   computed
```

14   **foreach** $S_u \in \mathrm{Tab}_{\mathrm{ECC}}(u, w, X_u)$ and $S_v \in \mathrm{Tab}_{\mathrm{ECC}}(w, v, X_v)$ **do**

15   **if** *for all* $(d_u, d_w^u) \in S_u$ *and* $(d_v, d_w^v) \in S_v$, *we have* $(d_w^u + d_w^v \leq D)$ *or* $(d_u + 1 + d_v \leq D)$ **then**

```
// if the diameter of the considered completion of X
   is ≤ D, we compute eccD(u, v, X) using Lemma 3
```

16 $\mathrm{Ecc} \leftarrow \max \left\{ \begin{array}{l} \{(d_u, \min\{d_u + 1, d_w + 1\}) \mid (d_u, d_w) \in S_u\} \cup \\ \{(\min\{d_v + 1, d_w + 1\}, d_v) \mid (d_w, d_v) \in S_v\}\} \end{array} \right\}$

```
// update eccD*(u, v, X)
```

17 $\mathrm{Tab}_{\mathrm{ECC}}(u, v, X) \leftarrow \min\{\mathrm{Tab}_{\mathrm{ECC}}(u, v, X) \cup \mathrm{Ecc}\}$

```
// finally, we compute eccD*(r, G) using Observation 1
```

18 **return** $\min_{v \in V(G) \setminus \{r\}} \min_{S \in \mathrm{Tab}_{\mathrm{ECC}}(r, v, V(G))} \max_{(d_u, d_v) \in S} d_u$.

The correctness of Algorithm 1 follows from the results proved in Subsections 2.1 and 2.2, and the following lemma, which guarantees that the value of $\mathrm{ecc}_D^*(u, v, X)$ can indeed be computed as done in lines 13-17.

Lemma 4 *There exists an outerplanar completion H of $G[X]$ with the edge uv on the outerboundary if and only if there is $w \in X$ and two sets X_u, X_v such that:*

11

(a) $X_u \cup X_v = X$, $X_u \cap X_v = \{w\}$,

(b) $\partial_G(X_u) \subseteq \{u, w\}$ and $\partial_G(X_v) \subseteq \{v, w\}$, and

(c) there exists an outerplanar completion H_u of $G[X_u]$ with the edge uw on the outerboundary, and an outerplanar completion H_v of $G[X_v]$ with the edge vw on the outerboundary.

Proof: Given an outerplanar completion H of $G[X]$ with the edge uv on the outerboundary, let H' be any triangulated outerplanar completion of H with the edge uv on the outerboundary. Now let w be the unique common neighbor of u and v in H', and let X_u (resp. X_v) be the set of vertices whose every path towards v (resp. u) goes through u (resp. v) or w. One easily verifies that w, X_v and X_v satisfy (a) (b) and (c).

Conversely, given a vertex w, and sets X_v and X_v satisfying (a) (b) and (c), let H be the graph obtained by gluing H_u and H_v on a triangle uvw. One easily verifies that such graph H is an outerplanar completion of $G[X]$. $\qquad\square$

It remains to analyze the running time of the algorithm.

Running time analysis of Algorithm 1. Note that at line 6 each B_i is recursively replaced by an equivalent (by Corollary 1) subgraph such that its cut vertices have at most 8 branches attached.

Let us first focus on the second step of the algorithm, that is, on lines 10-17. The algorithm considers in line 10 at most $O(n^2)$ pairs $\{u, v\}$. As each of u and v has at most 7 attached branches avoiding the root, and $G \setminus \{u, v\}$ has at most 2 connected components with vertices adjacent to both u and v (as otherwise G would contain a $K_{2,3}$-minor), there are at most $2^7 \cdot 2^7 \cdot 2^2 = 2^{16}$ possible choices for assigning these branches or components to X or not. In line 13, the algorithm considers $O(n)$ vertices w. Similarly, as w belongs to at most 7 branches not containing u nor v, there are at most 2^7 choices for assigning these branches to X_u or X_v. In lines 14-17, the algorithm uses values that have been already computed in previous iterations, as the sets X are considered by increasing order. Note that each of $\mathrm{ecc}_D^*(u, w, X_u)$ and $\mathrm{ecc}_D^*(w, v, X_v)$ contains at most 2 elements, so at most 4 choices are considered in line 14. Again, at most 4 choices are considered in line 15. Therefore, lines 14-17 are executed in constant time.

As for the first step of the algorithm (lines 1-9), the algorithm calls itself recursively. The number of recursive calls is bounded by the number of blocks of G, as by construction of the algorithm each block is assigned a single root. Therefore, the number of recursive calls is $O(n)$. Once the algorithm calls itself and the corresponding branch has no cut vertex other than the root, the algorithm enters in lines 10-17, whose time complexity has already been accounted above. (Note that each triple (u, v, X) is considered only once, and the value of $\mathrm{ecc}_D^*(u, v, X)$ is stored in the tables.)

Finally, in line 18, the algorithm considers $O(n)$ vertices, and for each of them it chooses among constantly many numbers. Summarizing, we have that the algorithm has overall complexity $O(n^3)$.

It is worth mentioning that Algorithm 1 can also compute the actual completion achieving diameter at most D, if any, within the same time bound. Indeed, it suffices to keep track of which edges have been added to G when considering the guessed triangles uvw (recall that we may assume that the completed graph is triangulated).

Theorem 1 *Algorithm 1 solves* OUTERPLANAR DIAMETER IMPROVEMENT *for connected input graphs in time $O(n^3)$.*

Note that we can compute **opdi**(G) by calling Algorithm 1 with an arbitrary root $r \in V(G)$ (such that $G \setminus \{r\}$ is connected) for increasing values of D, or even binary search on these values.

Corollary 2 *Let G be a connected outerplanar graph. Then, **opdi**(G) can be computed in time $O(n^3 \log n)$.*

2.4 The algorithm for disconnected outerplanar graphs

In this subsection we will focus on the case where the input outerplanar graph is disconnected. The *radius* of a graph is defined as the eccentricity of a "central" vertex, that is, the minimum eccentricity of any of its vertices.

Lemma 5 ([8], Theorem 3) *Let G be a maximal outerplanar graph of diameter D and radius r. Then, $r \leq \lfloor D/2 \rfloor + 1$.*

In the following, we denote the minimum radius of a diameter-D outerplanar completion of a graph or connected component G by $r^*(G)$. If G has no diameter-D outerplanar completion, then let $r^*(G) = \infty$.

Definition 1 *Let G be a connected graph and let D be an integer. Let G' be the graph resulting from G by adding an isolated vertex v. Let G^* be a diameter-D outerplanar completion of G' that minimizes the eccentricity of v. Then, G^* is called* escalated completion *of (G, D) with respect to v and the eccentricity* $\mathrm{ecc}(v, G^*)$*, denoted by $r^+(G)$, is called* escalated eccentricity *of (G, D). Again, if such a G^* does not exist, let $r^+(G) = \infty$.*

We will apply Definition 1 also to connected components of a graph and, if clear from context, we omit D. Note that we can compute $r^+(G)$ by guessing an edge between the isolated vertex v and G and running `OPDI-Connected`, the algorithm for connected graphs. Hence this can be done in $O(n^4)$ time. Also note that $r^*(G) \leq r^+(G) \leq r^*(G) + 1$. Indeed, by contracting an edge uv incident to v in an escalated completion of (G, D) one obtains a diameter-D outerplanar completion of G where the eccentricity of u is bounded by $r^+(G)$, hence $r^*(G) \leq r^+(G)$. Furthermore, connecting a new vertex v to a central vertex of a diameter-D outerplanar completion of minimum radius one can deduce that $r^+(G) \leq r^*(G) + 1$.

Lemma 6 *Given a graph G with a connected component C such that $r^+(C) < D/2$, then G has a diameter-D outerplanar completion if and only if $G \setminus C$ does.*

Proof: In a diameter-D outerplanar completion of $G \setminus C$ there is a vertex v with eccentricity at most $\lfloor D/2 \rfloor + 1$, by Lemma 5. In this completion, adding the completion of $C + v$ achieving $r^+(C) < D/2$, yields a diameter-D outerplanar completion of G. $\square$

Observation 2 *Let C be a connected component of G, let G' be an outerplanar completion of G and let C' be a connected component of $G' \setminus C$. Then, there is a vertex $v \in C$ at distance at least $r^+(C)$ to each vertex of C' in G'.*

Proof: Let G'' be the result of contracting all vertices in $G' \setminus (C \cup C')$ onto vertices in C and contracting C' onto a single vertex u. Then, G'' is a subgraph of an outerplanar completion of the result of adding u as isolated vertex to $G'[C]$. By definition, $\mathrm{ecc}(u, G'') \geq r^+(C)$, implying that there is a vertex $v \in C$ at distance at least $r^+(C)$ to u in G''. Thus, v is at distance at least $r^+(C)$ to each vertex of C'

in G'. □

Observation 2 immediately implies that any cutset separating two connected components C_1 and C_2 of G in G' has distance at least $r^+(C_1)$ and $r^+(C_2)$ to some vertex in C_1 and C_2, respectively. Thus, these two vertices are at distance at least $r^+(C_1) + r^+(C_2)$ in G'.

Corollary 3 *Let C_1 and C_2 be connected components of G such that $r^+(C_1) + r^+(C_2) > D$ and let G' be a diameter-D outerplanar completion of G. Then, C_1 and C_2 are adjacent in G', i.e., G' has an edge with an end in C_1 and an end in C_2.*

Corollary 3 allows us to conclude that all connected components C with $r^+(C) > D/2$ have to be pairwise adjacent in any diameter-D outerplanar completion of G. Thus, there cannot be more than three such components.

Lemma 7 *An outerplanar graph G with more than 3 connected components C such that $r^+(C) > D/2$ has no diameter-D outerplanar completion.*
On the other hand, if G has no connected component C such that $r^+(C) > D/2$, then G necessarily has a diameter-D outerplanar completion.

Proof: The first statement comes from the above comments. The proof of the second statement is similar to the one of Lemma 6. For some component C of G, let v be such that $ecc(v, C) = r^*(C) \le r^+(C) \le D/2$, and complete C in order to achieve this value. Then for the other components C' consider their escalated completion with respect to v. As $r^+(C') \le D/2$ this graph has diameter at most D. □

Hence, assume G has $p = 1, 2$, or 3 connected components C such that $r^+(C) > D/2$. By Corollary 3 these p components are pairwise adjacent in the desired completion. Note that with $O(n^{2p-2})$ tries, we can guess $p - 1$ edges connecting all such components into one larger component. Thus, in the following, we assume that there is only one component C with $r^+(C) > D/2$, denoted $\mathcal{C}_{max}$.

Lemma 8 *Consider an outerplanar graph G with exactly one connected components $\mathcal{C}_{max}$ such that $D/2 < r^+(\mathcal{C}_{max}) < \infty$. If $r^*(\mathcal{C}_{max}) \le D/2$, then G necessarily has a diameter-D outerplanar completion.*

Proof: Same proof as Lemma 7. □

Let us now distinguish two cases according to the parity of D.

Lemma 9 *For odd D, if an outerplanar graph G has at most one component $\mathcal{C}_{max}$ such that $D/2 < r^+(\mathcal{C}_{max}) < \infty$, then G has a diameter-D outerplanar completion.*

Proof: Indeed, by Lemma 6 it is sufficient to consider the component $\mathcal{C}_{max}$ alone. As $r^+(\mathcal{C}_{max}) < \infty$, $\mathcal{C}_{max}$ has a diameter-D outerplanar completion, and so does G. □

The case where D is even is more technical.

Lemma 10 *For even D, Let p and q respectively denote the number of connected components C such that $D/2 < r^+(C) < \infty$ and $r^+(C) = D/2$, of an outerplanar graph G. If $p \ge 2$ and $p + q \ge 5$, then G has no diameter-D outerplanar completion.*

Proof: By Corollary 3, in a diameter-D outerplanar completion G' of G the p components are pairwise adjacent, and any of the q components is adjacent to every of the p ones. For $p = 2$, as $q \ge 3$, this would induce a $K_{2,3}$-minor in G', a contradition. For the other cases, this would induce a K_4-minor in G', a contradiction. □

Lemma 11 *For even D, if an outerplanar graph G has one component, denoted $\mathcal{C}_{max}$, such that $D/2 < r^*(\mathcal{C}_{max}) < \infty$, and at least 4 other components C such that $D/2 \leq r^+(C) < \infty$, then G has no diameter-D outerplanar completion.*

Proof: Let us denote C_1, C_2, C_3, and C_4 the connected components such that $r^+(C_i) \geq D/2$, distinct from $\mathcal{C}_{max}$. Assume for contradiction that G admits a diameter-D outerplanar completion, denoted G'.

Claim 9 *For each C_i, C_j, either C_i and C_j are adjacent in G', or C_i and C_j have a common neighbor in G'.*

Proof: Assume for contradiction that C_i and C_j are not adjacent and do not have a common neighbor in G'. Let us now construct the graph G'' from G' as follows. For any component C of $G' \setminus (C_i \cup C_j)$ that is not adjacent to both C_i and C_j, contract C onto vertices of C_i or C_j (According to the one C is neighboring). As G'' is obtained from G' by contracting edges, G'' also is a diameter-D outerplanar completion (for some graph containing C_i and C_j). Let $N_i := N_{G''}(C_i)$, let $N_j := N_{G''}(C_j)$, and note that $C_i \cap N_j = \emptyset$, $N_i \cap C_j = \emptyset$, and $N_i \cap N_j = \emptyset$. Then, by Observation 2 (as $G'' \setminus C_i$ and $G'' \setminus C_j$ are connected), there are vertices $v_i \in C_i$ and $v_j \in C_j$ at distance at least $D/2$ to each vertex in N_i and N_j, respectively, in G''. Since N_i and N_j are at distance at least one, v_i and v_j are at distance at least $D+1$, contradicting G'' having diameter D. □

Claim 10 *There is a vertex $u \in \mathcal{C}_{max}$ that is adjacent in G' to 3 of the components C_1, C_2, C_3, and C_4.*

Proof: First, note that there is a vertex u and 3 components, say C_1, C_2, C_3, with $u \in N_{G'}[C_i]$ for all $1 \leq i \leq 3$, since otherwise, there would be internally vertex-disjoint paths between each two of the four components C_i, implying the existence of a K_4-minor in G'.

If u is neither in $\mathcal{C}_{max}$ nor in C_i, for $1 \leq i \leq 3$, then, since all the C_i are adjacent to $\mathcal{C}_{max}$ (by Corollary 3), G' would have a $K_{2,3}$-minor on the vertex sets $\{u, \mathcal{C}_{max}\}$ and $\{C_1, C_2, C_3\}$.

Hence, in the following, we assume that $u \in C_1$. Let z be a neighbor of C_1 in $\mathcal{C}_{max}$ and, for $i \in \{2, 3\}$ let w_i denote a neighbor of C_4 in $N[C_i]$ (here, $N[C_i]$ denotes the closed neighborhood of component C_i). We note that $w_2 \neq z$ and $w_3 \neq z$, since otherwise, the claim follows and we are done. Furthermore, $w_2 \neq u$ and $w_3 \neq u$, since otherwise there is a $K_{2,3}$-minor on the vertex sets $\{u, \mathcal{C}_{max}\}$ and $\{C_2, C_3, C_4\}$. Let $X := (C_4 \cup \{w_2, w_3\}) \setminus (C_2 \cup C_3)$ and note that X is adjacent to C_2 and C_3, respectively. Let Y be the connected component of $\mathcal{C}_{max} \setminus \{w_2, w_3\}$ containing z, and note that Y is adjacent to C_1 and X. Finally, since X, Y, C_1, C_2, and C_3 are pairwise disjoint, G' has a $K_{2,3}$-minor on the vertex sets $\{X, C_1\}$ and $\{C_2, C_3, Y\}$. □

Let v denote a vertex of $\mathcal{C}_{max}$ that is at distance at least $D/2 + 1$ to u in G' and consider the result $G' \setminus \{u\}$ of removing u from G'. Let C denote the connected component of $G' \setminus \{u\}$ that contains v. Towards a contradiction, assume there is a connected component C_i that is adjacent to u but not to C in G', then all paths between v and any vertex in C_i contain u. Since G' has diameter D, all vertices in C_i are at distance at most $D/2 - 1$ to u in G', contradicting $r^+(C_i) \geq D/2$. Thus there is a $K_{2,3}$-minor in G' on the vertex sets $\{C_1, C_2, C_3\}$ and $\{u, X\}$ where X is the connected component of $G' \setminus (C_1 \cup C_2 \cup C_3 \cup \{u\})$ containing v. This concludes the proof of the lemma. □

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Hence, assume G has $q = 0, 1, 2,$ or 3 connected components C such that $r^+(C) = D/2$. By Corollary 3 these q components are adjacent to each of the p components such that $r^+(C) > D/2$. Note that with $O(n^{2q})$ tries, we can guess q edges connecting each of the q components to one of the p component. Then we are left with a connected graph, and we can call `OPDI-Connected`.

The algorithm itself. We now describe a polynomial-time algorithm that solves the OUTERPLANAR DIAMETER IMPROVEMENT problem when the input contains a disconnected outerplanar graph. Algorithm 2 described below receives a (disconnected) outerplanar graph G, and a positive integer D.

Algorithm 2: `OPDI-Disconnected`

Input : A disconnected outerplanar graph $G = (V, E)$ and an integer D.
Output: 'TRUE' if and only if G has a diameter-D outerplanar completion.

1 **foreach** *connected component C of G* **do**
2     $r^+(C) \leftarrow \infty$
3     $r^*(C) \leftarrow \infty$
4   **foreach** $u \in V(C)$ **do**
5   Ecc$\leftarrow$ `OPDI-Connected`(C, u, D)
6        $r^*(C) \leftarrow \min\{r^*(C), \text{Ecc}\}$
7        $C' \leftarrow C$ with added vertex v and added edge uv
8   Ecc$\leftarrow$ `OPDI-Connected`(C', v, D)
9        $r^+(C) \leftarrow \min\{r^+(C), \text{Ecc}\}$
10   **if** $r^+(C) = \infty$ **then return** FALSE
11   **if** $r^+(C) < D/2$ **then** Remove C from G
12 **if** $r^+(C) \leq D/2$ *for every C* **then return** TRUE
13 **if** $r^+(C) \leq D/2$ *for every C except one, $\mathcal{C}_{max}$, and $r^*(\mathcal{C}_{max}) \leq D/2$* **then**
14   **return** TRUE
15 **if** $r^+(C) > D/2$ *for at least 4 conn. components C* **then return** FALSE
16 **foreach** *choice of edges interconnecting these $p = 1, 2,$ or 3 connected components* **do**
    // choose $p - 1$ edges
17   Let $\mathcal{C}_{max}$ be this new conn. component
18   **if** `OPDI-Connected`$(\mathcal{C}_{max}, v, D) < \infty$ **then**
19   **if** *D is odd* **then return** TRUE
20   **if** *G has more than $5 - p$ conn. comp.* **then**
          // $\mathcal{C}_{max}$ and q connected comp. such that $r^+(C) = D/2$
21   **return** FALSE
22   **else**
23   **foreach** *choice of q edges connecting G* **do**
24   **if** `OPDI-Connected`$(G, v, D) < \infty$ **then return** TRUE
25   **return** FALSE
26 **return** FALSE

At the beginning, the algorithm computes $r^+(C)$ and $r^*(C)$ for each connected component C of G. For computing $r^+(C)$ the algorithm adds a vertex v, guessing (with $O(n)$ tries) an edge connecting v to C, and then calls `OPDI-Connected` for this component and root v. For computing $r^*(C)$ the algorithm guesses a root u (with $O(n)$

tries), and then calls `OPDI-Connected` for C and root u.

If $r^*(C) = \infty$ for some component C then, as $r^*(G) \geq r^*(C)$, G has no diameter-D outerplanar completion.

Then, as they could be added in a diameter-D outerplanar completion (by Lemma 6), the algorithm removes the components C with small escalated eccentricity, that is those such that $r^+(C) < D/2$.

Then the algorithm tests if there is no component C such that $r^+(C) > D/2$, or if there is only one component C such that $r^+(C) > D/2$, and if $r^*(C) \leq D/2$. In both cases by Lemma 7 and Lemma 8, G is a positive instance.

Then the algorithm tests if there are more than 3 components C such that $r^+(C) > D/2$. In this case, by Lemma 7, G is a negative instance. Otherwise, G has $p = 1$, 2, or 3 such connected components, and the algorithm guesses $p - 1$ edges (in time $O(n^{2p-2})$) to connect them (as they should be by Corollary 3). For each such graph we call algorithm `OPDI-Connected` to check that this graph has a diameter-D outerplanar completion.

Then the algorithm proceeds differently according to D's parity. If D is odd, then G is a positive instance (By Lemma 9). If D is even, if G has (still) more than $5 - p$ connected components (by Lemma 10 and Lemma 11), then G is a negative instance. Then we are left with a graph G with $1 + q$ connected components, and again the algorithm guesses q edges (in time $O(n^{2q})$), connecting G. For each of these graphs the algorithm calls `OPDI-Connected`(G, v, D) (for any v) to check whether this graph admits a diameter-D outerplanar completion.

Finally if none of these "guessed" connected graphs has a diameter-D outerplanar completion, then the algorithm concludes that G is a negative instance.

Theorem 2 *Algorithm 2 solves* OUTERPLANAR DIAMETER IMPROVEMENT *for disconnected input graphs in polynomial time. For odd D the running time is $O(n^7)$, while it is $O(n^9)$ for even D.*

Proof: Indeed, the algorithm runs in time $O(n^7)$ for odd D (at most $O(n^4)$ at line 16, times $O(n^3)$ for the call to `OPDI-Connected` in line 18). The algorithm runs in $O(n^{2p+2q+1})$ time for even D ($O(n^{2p-2})$ in line 16, times $O(n^{2q})$ in line 23, times $O(n^3)$ for the call to `OPDI-Connected` in line 24), where p and q respectively denote the number of connected components C such that $r^+(C) > D/2$ and $r^+(C) = D/2$. As $p + q \leq 4$, we are done. $\square$

3 Conclusions and further research

Our algorithm for OPDI runs in time $O(n^3)$ for connected input graphs, and in time $O(n^7)$ or $O(n^9)$ for disconnected input graphs, depending on whether D is odd or even, respectively. The main contribution of our work is to establish the computational complexity of OPDI and there is room for improvement of the running time.

We believe that our approach might be interesting for generalizations or variations of the OPDI problem, such as the one where we demand that the completed graph has fixed outerplanarity or is series-parallel.

By the Graph Minors series of Robertson and Seymour [25, 26], we know that for each fixed integer D, the set of *minor obstructions*[2] of OPDI is *finite*. We have some preliminary results in this direction, but we managed to obtain a complete list only for

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[2]The *minor obstruction set* of OPDI for some D is the smallest family $\mathcal{F}$ of graphs such that a graph G has an outerplanar completion of diameter D if and only if no graph of $\mathcal{F}$ is a minor of G.

small values of D. Namely, we obtained a partial list of forbidden substructures (not necessarily minimal), by using the notion of *parallel matching*. These partial results can be found in the arXiv version of this article, see [6].

Settling the computational complexity of PDI remains the main open problem in this area. An explicit FPT-algorithm, or even an XP-algorithm, would also be significant. Again, we have some partial NP-completeness result in this direction, but for a problem which is slightly more complicated than PDI, in particular involving edge weights. This reduction can be also found in the arXiv version [6].

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