Problems and conjectures on parameterized $H$-coloring

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Recall that given two directed or undirected graphs $G = (V(G), E(G))$ and $G' = (V(G'), E(G'))$ an homomorphism of $G$ into $G'$ is a map $\theta : V(G) \rightarrow V(G')$ with the property that $\{v, w\} \in E(G) \Rightarrow \{\theta(v), \theta(w)\} \in E(G')$. The $H$-coloring problem is the problem of checking whether, for fixed $H$, there exists an homomorphism (H-coloring) from an input graph $G$ to $H$.

It is known that if $H$ is bipartite or it has a loop the $H$-coloring problem can be trivially solved in polynomial time, but in the case that $H$ is loop less and not-bipart ite the problem is known to be NP-complete [HN90]. An interesting generalization of the $H$-coloring problem is the list $H$-coloring problem where each vertex of $G$ carries a list of the vertices of $H$ where it is allowed to be mapped [FH98, FHH99]. In [DST01] we consider the following parameterized version of the $H$-coloring problem:

Set up a weighting $K = \{k_j | j \in C\}$ of $C \subseteq V(G)$ with non negative integers, we say that an input graph $G$ has a $(H, C, K)$-coloring if there exists an $H$-homomorphism $\chi : V(G) \rightarrow V(H)$ such that $\forall v \in C, |\chi^{-1}(v)| = k_v$. Denote $(H, C, K)$ a partial weighted assignment. If we additionally assign to each vertex of $G$ a list permissible images when we have a more general version of the $(H, C, K)$-coloring problem that we call list $(H, C, K)$-coloring. We can consider the integers in $K$ to be small fixed constants, which constitutes a parameterization of the above problems problems. A weighted assignment $(H, C, K)$ is a weighted extension of a graph $F$ if $H - C = F$.

In [DST01] we prove that if $F$ is a graph where the $F$-coloring is NP-complete then for any weighted extension $(H, C, K)$ of $F$ the $(H, C, K)$-coloring is also NP-complete. On the other hand, if $F$-coloring is in P then there exist a weighted extension $(H, C, K)$ of $F$ so that the $(H, C, K)$-coloring is in P.

In the same reference we also prove that if $F$ is a graph where the list $F$-coloring is in P then for any weighted extension $(H, C, K)$ of $F$, the

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\((H, C, K)\)-coloring is also in P and from the other hand, we provided some examples where the \(F\)-list coloring is NP-complete and \(F\) has a weighted extension \((H, C, K)\) such that \((H, C, K)\)-coloring is also NP-complete.

The above results seem to imply that there are graphs that for different weighted extensions produce parameterized coloring problems with different complexities. It seems to be a hard problem to achieve a dichotomy distinguishing those parameterized assignments \((H, C, K)\) of a given graph \(F\) for which the \((H, C, K)\)-coloring problem is NP-complete or in P. We conjecture the following.

**Conjecture 1.** For any graph \(F\) such that the list \(F\)-coloring is NP-complete, there is a weighted extension \((H, C, K)\) of \(F\) such that the \((H, C, K)\)-coloring is also NP-complete.

All of our results on NP-completeness indicate that this frontier depends not only on \(H - C\) but also on the structure imposed by the weighted vertices. However, we observe that in all our complexity results – positive or negative – this dichotomy does not depend on the choice of the numbers in \(K\) when \(K\) is positive.

Another conjecture, let \((H, C, K)\) be a weighted assignment. We call it *compact* when each connected component \(H_i\) of \(H\) satisfies one of the following:

1. \(E(H_i - C) = \emptyset\),
2. \(H_i[C]\) is a non-empty reflexive clique with all its vertices adjacent with one looped vertex of \(H_i - C\), or
3. \(V(H_i) \cap C = \emptyset\) and \(H_i\) contains at least one looped vertex.

**Conjecture 2.** For any partial weighted assignment \((H, C, K)\), if \((H, C, K)\) is compact then the \((H, C, K)\)-coloring problem, with the numbers in \(K\) as parameters, is in FPT, otherwise it is \(W[1]\)-hard.

In support of our dichotomy conjecture we recall that the parameterized independent set problem, that is known to be \(W[1]\)-complete [DF99] falls in this area.

We mention that our second conjecture is quite general to express several open problems in parameterized complexity such as

1. (Parameter: \(k\)) Does \(G\) contain an independent set \(S\), where \(|S| = k\), and such that \(G[V(G) - S]\) is bipartite? (This problem can be seen as a parameterization of 3-coloring where some color should be used exactly \(k\) times.)

2. (Parameter: \(k\)) Is there a set \(S \subseteq V(G)\), where \(|S| = k\) and \(G[V(G) - S]\) is bipartite?
Figura 1: Three weighted extensions conjectured to be W[1]-hard when $K$ is parameterized

3. (Parameters: $k, l$) Does $G$ contain $K_{k,l}$ as subgraph?

This problems correspond to the the parameterized colorings given in figure 1. The $(H_3, C_2, K_3)$-problem is equivalent with asking if the complement of $G$ contains $K_{k_1, k_2}$ as a subgraph.

Referencias


