

# Odd cyclic surface separators in planar graphs

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Given a plane graph  $G$  and two vertices  $s$  and  $t$  in  $G$ , an  $s$ - $t$  *surface separator* is a closed curve  $C$  such that  $s$  belongs to the other region of  $\mathbb{R}^2 \setminus C$  than  $t$ . A cycle  $C$  is called a *cyclic surface separator* if  $C$  is an  $s$ - $t$  surface separator, and an *odd cyclic surface separator* if  $C$  is an odd cycle. We consider the problem of finding odd cyclic surface separators in planar graphs and show that the problem can be solved in polynomial time.

## 1 Introduction

Given two vertices  $s, t \in V(G)$ , we are interested in finding a cycle in  $G$  such that one of the two regions of  $\mathbb{R}^2 \setminus C$  contains  $s$  and the other contains  $t$ . We show that the problem is solvable in polynomial time, even when we impose a parity condition on the length of the cycle.

Our proof is based on the *irrelevant vertex technique* introduced by Robertson and Seymour in Graph Minors: we prove that either the treewidth of the input graph  $G$  is small, and then the problem can be solved by dynamic programming, or when it is large, the  $G$  contains a vertex  $v$  such that  $G - v$  is a yes-instance if and only if  $G$  is a yes-instance. Moreover, such an irrelevant vertex can be found in polynomial time. We recursively remove irrelevant vertices, each time reducing the size of the graph. Finally, the treewidth is small and we apply the dynamic programming approach.

For a cycle  $C$  in a plane graph  $G$ , there are two open regions of  $\mathbb{R}^2 \setminus C$ . The one containing the outerface of  $G$  is the *exterior* of  $C$ . The other one is the *interior* of  $C$  and we denote it by  $\text{int}(C)$ . If a path contains a vertex from the interior and a vertex from the exterior of a cycle, we say that it *crosses* the cycle.

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## 2 Results

### 2.1 The bipartite case

**Lemma 1** *Let  $G$  be a plane graph,  $s, t \in V(G)$ ,  $C$  be a cycle in  $G$  and  $G'$  be the graph induced by the vertices in the interior of  $C$ . If  $G[V(G')]$  is bipartite,  $s, t \notin V(G' \cup C)$  and exists a vertex  $v \in \text{int}(C)$ , then if  $G$  has an odd cyclic surface separator, so does  $G - v$ .*

*Proof.* Let us consider an odd cyclic surface separator in  $G$ . If it does not cross  $C$ , it contains no vertex of  $V(G')$ . (Notice that  $s$  and  $t$  lie in the exterior of  $C$ .) Assume that every odd cyclic surface separator crosses  $C$  and let  $C^*$  be one which crosses  $C$  the minimum number of times. Exactly one of the terminals, without loss of generality we assume it is  $s$ , lies in the interior of  $C^*$ .

Let  $P$  be the set of edges of  $C^*$  that lie in the interior of  $C$ , and  $Q$  be the set of edges of  $C$  that lie in the interior of  $C^*$ . ( $P$  and  $Q$  are disjoint.) Then,  $C \cup P - Q$  is a collection of cycles. Terminal  $s$  lies in the interior of one of the cycles in the collection; let this cycle be called  $C_s$ .

Notice that  $C_s$  does not cross  $C$ . If  $C_s$  was an odd cycle, it would be an odd cyclic surface separator; hence,  $C_s$  is even. Hence, there must be another cycle in  $C \cup P - Q$  which is of odd parity, since  $C^*$  was odd. Let us call the odd cycle  $C_{odd}$ .

Now we will investigate how  $C_s$  and  $C_{odd}$  are related. Suppose there is a sub-path  $v_L P v_R$  of  $C_s$  such that  $v_L, v_R \in C$  but no vertex of  $P$  belongs to  $C$ . Now,  $C$  can be split into two disjoint paths between vertices  $v_L$  and  $v_R$ . Exactly one of those two paths has the property that the cycle formed by  $v_L P v_R$  and this path does not contain the other path inside. Let us call that path  $P'$ . The cycle formed by  $v_L P v_R$  and  $P'$  will be called a *pouch*.

Clearly,  $C_{odd}$  lies in one of the pouches of  $C_s$ ; we will look at this pouch now. Let  $v_L, v_R, P$  and  $P'$  be as in the definition of the pouch. Vertices  $v_L$  and  $v_R$  split  $C_s$  into two paths; one is  $P$  and let the other be called  $Q$ . We investigate two cases:

**CASE 1.** First let us assume that the other terminal  $t$  **does not lie in the interior of the pouch**. Let  $v'_L$  and  $v'_R$  be the first and last vertex along  $P'$  that belong to  $C_{odd}$ . Notice that since  $C_{odd}$  is odd, vertices  $v'_L$  and  $v'_R$  split it into two paths of different parity  $P_{even}$  and  $P_{odd}$ . Consider the two cycles formed by the path  $v'_L P' v'_R$  and either  $P_{even}$  or  $P_{odd}$ . They both are  $s, t$ -separators and they do not cross  $C$ . However, exactly one of them is odd; a contradiction.

**CASE 2.** Now let us assume that  $t$  **lies in the interior of the pouch** of  $C_s$  in which  $C_{odd}$  also lies. We define pouches for  $C_{odd}$  analogously. If  $t$  does not lie in the interior of the pouch of  $C_{odd}$  that contains  $C_s$ , then the same argument as in CASE 1 can be applied to show a contradiction.

Let us assume that  $t$  lies in the interior of the pouch of  $C_{odd}$  that contains  $C_s$ . Also let  $v'_L, v'_R, P_{odd}$ , and  $P_{even}$  be as in CASE 1. Consider the two cycles formed by the path  $v'_L P' v_L P v_R P' v'_R$  and either  $P_{even}$  or  $P_{odd}$ . They both are  $s, t$ -separators and they do not cross  $C$ . However, exactly one of them is odd; a contradiction.  $\square$

## 2.2 Odd cycle

**Lemma 2** *Let  $G$  be a 2-connected, plane graph,  $s, t \in V(G)$ ,  $C$  be a cycle in  $G$  and  $G'$  be the graph induced by the vertices in the interior of  $C$ . If  $G[V(G')]$  is non-bipartite and contains a odd cycle  $C'$  such that exists a vertex  $v \in \text{int}(C')$ ,  $s, t \notin V(G' \cup C)$ , then if  $G$  has an odd surface separator, so does  $G - v$ .*

*Proof.* As in the proof of Lemma 1, we can assume that there is a cycle  $C_s$  whose interior contains  $s$ . Also,  $|C \cap C_s| \geq 2$ . From 2-connectivity, there are two paths  $P_1$  and  $P_2$  between  $C \cap C_s$  and  $C'$ . The endpoints of  $P_1$  and  $P_2$  on  $C'$  split  $C'$  into two paths  $P'_1$  and  $P'_2$ . Similarly, the endpoints of  $P_1$  and  $P_2$  split  $C_s$  into two paths; let  $P_s$  be one of these paths, the one that contains vertices from the exterior of  $C$ . Then, both  $P_s P_1 P'_1 P_2$  and  $P_s P_1 P'_2 P_2$  are cyclic separators and they are of different parity.

Clearly, if  $G$  has an odd surface separator, so does  $G - v$ .  $\square$

## 2.3 Algorithm

Let  $G$  be a plane input graph. The problem easily reduces to 2-connected graphs so we can assume that  $G$  is 2-connected. There exists a constant  $c$  such that if the treewidth of  $G$  is at least  $c$ , then (1) it contains a cycle  $C$  with a vertex in its interior and (2) if the graph induced by the vertices in the interior of  $C$  is not bipartite, the interior of  $C$  contains an odd cycle whose interior contains a vertex. Therefore, if the treewidth of  $G$  is at least  $c$ , then we can find using Lemmas 1 and 2 an irrelevant vertex  $v$ . Removing  $v$  reduces the size of the graph and we recurse. If the treewidth of  $G$  is at most  $c$ , then the problem can be solved efficiently by dynamic programming.

**Theorem 3** *There exists a polynomial-time algorithm deciding given a plane graph  $G$  and two of its vertices  $s, t$ , whether  $G$  contains an odd cyclic surface separator.*