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Subexponential Parameterized Algorithms for Bounded-Degree Connected Subgraph Problems on Planar Graphs

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Abstract

We present subexponential parameterized algorithms on planar graphs for a family of problems that consist in, given a graph G, finding a connected (induced) subgraph H with bounded maximum degree, while maximising the number of edges (or vertices) of H. These problems are natural generalisations of LONGEST PATH. Our approach uses bidimensionality theory combined with novel dynamic programming techniques over branch decompositions of the input graph. These techniques can be applied to a more general family of problems that deal with finding connected subgraphs under certain degree constraints.

Keywords: Parameterized complexity, planar graphs, subexponential algorithm, branch decomposition, graph minors, bidimensionality, Catalan structures.

1 Introduction

During the last years a considerable amount of work has been devoted to design subexponential parameterized algorithms for NP-hard optimisation problems on planar graphs and, more generally, on sparse classes of graphs [2, 3, 4, 5].

In this paper, we apply the general approach of [2, 3, 4, 5] to a family of problems dealing with finding maximum connected subgraphs under degree

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constraints. Along the way, we introduce novel dynamic programming techniques over branch decompositions that can be applied to more general classes of problems.

All the graphs considered in this article are simple and undirected. Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and replace all multiple edges by simple edges. A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. H is a *minor* of G if H is a subgraph of a contraction of G. The *maximum degree* of a graph G is denoted by $\Delta(G)$. We define the following family of problems for $d \geq 2$.

MAXIMUM d-DEGREE-BOUNDED CONNECTED SUBGRAPH (MDBCS$_d$)
Input: A graph G and a non-negative integer k.
Question: Does G contain a connected subgraph H with $\Delta(H) \leq d$ and $\lvert E(H)\rvert \geq k$?

If $d = 2$ the problem is equivalent to the LONGEST PATH problem (or CYCLE, if G is Hamiltonian), hence MDBCS$_d$ is a generalisation of it. MDBCS$_d$ is one of the classical NP-hard problems listed in [9] and it has been recently proved that it is not in APX for any $d \geq 2$ [1]. Without the connectivity constraint, the problem is known to be in P using matching techniques [12]. When the problem is parameterized by k we denote it by k-MDBCS$_d$. (We refer to [7] for an introduction to parameterized complexity.) Our target is to find $2^{\mathcal{O}(\sqrt{k})} \cdot \mathcal{O}(n)$ step algorithms to solve this problem and its variants when the input is restricted to planar graphs.

The paper is organized as follows. Section 2 is devoted to obtain combinatorial bounds using bidimensionality theory. Section 3 presents new dynamic programming techniques that can be applied to general graphs. In Section 4 we see how to speed-up these algorithms when the input is restricted to planar graphs, using Catalan structures. This strategy can be extended to several related problems asking for a maximum connected subgraph satisfying certain degree constraints, as discussed in Section 5. An extended version of our results can be found in [14].

2 Bounds for Branchwidth

We say that a parameter $\mathbf{p}$ defined on simple undirected graphs is *closed under taking of minors* (or simply *minor closed*) if $G' \preceq G \Rightarrow \mathbf{p}(G') \leq \mathbf{p}(G)$ (here "$\preceq$" denotes the minor relation). We define the following parameter on simple undirected graphs.

$$\mathbf{medbcs}_d(G) = \max\{|E(H)| \mid H \subseteq G \wedge H \text{ is connected} \wedge \Delta(H) \leq d\}.$$

For the proof of the following lemma, see [14].

Lemma 2.1 *For any integer $d \geq 1$, the parameter $\mathbf{medbcs}_d$ is minor closed.*

Let G be a graph on n vertices. A *branch decomposition* (T, μ) of a graph G consists of an unrooted ternary tree T (i.e., all internal vertices are of degree three) and a bijection $\mu : L \rightarrow E(G)$ from the set L of leaves of T to the edge set of G. We define for every edge e of T the *middle set* $\mathbf{mid}(e) \subseteq V(G)$ as follows: Let T_1 and T_2 be the two connected components of $T \setminus \{e\}$. Then let G_i be the graph induced by the edge set $\{\mu(f) : f \in L \cap V(T_i)\}$ for $i \in \{1, 2\}$. The *middle set* is the intersection of the vertex sets of G_1 and G_2, i.e., $\mathbf{mid}(e) := V(G_1) \cap V(G_2)$. The *width* of (T, μ) is the maximum order of the middle sets over all edges of T, i.e., $\mathbf{w}(T, \mu) := \max\{|\mathbf{mid}(e)| \colon e \in T\}$. An optimal branch decomposition of G is defined by a tree T and a bijection μ which give the minimum width, the *branchwidth*, denoted by $\mathbf{bw}(G)$.

Theorem 2.2 (Robertson, Seymour, and Thomas [13]) *Let $\ell \geq 1$ be an integer. Every planar graph of branchwidth at least ℓ contains an $(\lfloor \ell/4 \rfloor \times \lfloor \ell/4 \rfloor)$-grid as a minor.*

A parameter P is *minor bidimensional* [2] with *density* δ if P is minor closed and for the $(r \times r)$-grid R, $P(R) = (\delta r)^2 + o((\delta r)^2)$. Theorem 2.2 implies the following useful property.

Lemma 2.3 (Demaine et al. [2]) *If P is a bidimensional parameter with density δ then for any planar graph G, $\mathbf{bw}(G) \leq \frac{4}{\delta} \cdot \sqrt{P(G)} + \mathcal{O}(1)$.*

Using Lemmas 2.1 and 2.3 we can obtain a combinatorial bound of the parameter $\mathbf{medbcs}_d$ in terms of the branchwidth of the planar graph G, as stated in Lemma 2.4. For a proof, see [14].

Lemma 2.4 *For any $d \geq 2$ and for any planar graph G it holds that $\mathbf{bw}(G) \leq \frac{4}{\delta} \cdot \sqrt{\mathbf{medbcs}_d(G)} + \mathcal{O}(1)$, with $\delta = 1$ if $d = 2$, $\delta = \sqrt{3/2}$ if $d = 3$, and $\delta = \sqrt{2}$ if $d \geq 4$.*

3 The Algorithms

Let G be in this section a (not necessarily planar) graph on n vertices. We denote the *empty set* by $\emptyset$ and the *empty function* by $\varnothing$. Let (T, μ) be a branch decomposition of width $\leq \ell$ of G. We pick an arbitrary edge $e^* \in E(T)$, we subdivide it by adding a new vertex v_{new} and then add a new vertex r and make it adjacent to v_{new}. We extend μ by setting $\mu(r) = \emptyset$ and we root T at vertex r. For each $e \in E(T)$ let T_e be the tree of the forest $T \backslash e$ that does not contain r as a leaf (i.e., the tree that is "below" e in the rooted tree T) and let E_e be the edges that are images, via μ, of the leaves of T that are also leaves of T_e. We denote $G_e = G[E_e]$. Observe that, if $e_r = \{v_{\text{new}}, r\}$, then $G_{e_r} = G$.

Given a set A, we define a *d-weighted packing* of A as any pair $(\mathcal{A}, \psi)$ where $\mathcal{A}$ is a (possible empty) collection of mutually disjoint nonempty subsets of A and $\psi : A \to \{0, \ldots, d\}$ is a mapping corresponding integers from 0 to d to the elements of A. It will be convenient to think of such a packing $\mathcal{A}$ of A as a hypergraph $\mathcal{G} = (A, \mathcal{A})$. Note that, by definition, $\mathcal{A}$ is a matching in $\mathcal{G}$.

Let $(\mathcal{A}, \psi)$ and $(\mathcal{A}', \psi')$ be two d-weighted packings of two sets A and A'. We define $(\mathcal{A}, \psi) \oplus (\mathcal{A}', \psi')$ as the $2d$-weighted packing $(\mathcal{A}'', \psi'')$ of $A'' = A \cup A'$ where $\mathcal{A}''$ is the packing of A'' defined by the connected components of the hypergraph $(A \cup A', \mathcal{A} \cup \mathcal{A}')$ (i.e., the nonempty subsets of the packing $\mathcal{A}''$ are the vertex sets corresponding to the connected components of the hypergraph $(A \cup A', \mathcal{A} \cup \mathcal{A}')$) and where for any $x \in A \cup A'$, $\psi''(x) = \psi(x)$ (resp. $\psi'(x)$) if $x \in A - A'$ (resp. $x \in A' - A$) and $\psi''(x) = \psi(x) + \psi'(x)$ if $x \in A \cap A'$. If $(\mathcal{A}, \psi)$ is a d-weighted packing of a set A and $A' \subseteq A$, we define $(\mathcal{A}, \psi)|_{A'}$ as the d-weighted packing $(\mathcal{A}', \psi')$ of the set A' where $\mathcal{A}' = \{X \cap A' \mid X \in \mathcal{A}\}$ and $\psi' = \{(x, \psi(x)) \mid x \in A'\}$.

Let $\mathscr{P}_e$ be the collection of all d-weighted packings $(\mathcal{A}, \psi)$ of $\mathbf{mid}(e)$, and let $\ell = |\mathbf{mid}(e)|$. Observe that if $e_r = \{v_{\text{new}}, r\}$, then $\mathscr{P}_{e_r} = \{(\emptyset, \varnothing)\}$. We use the notation $\mathcal{C}(H)$ for the set of connected components of a graph (or hypergraph) H. Given $(\mathcal{A}, \psi) \in \mathscr{P}_e$ we define

$$
\begin{aligned}
\mathbf{opt}_e(\mathcal{A}, \psi) = \max\{\{0\} \cup \{|E(H)| \ :\ \exists\, H \subseteq G_e : \Delta(H) \leq d \wedge \\
&\text{if } (\mathcal{A} \neq \emptyset) \text{ then} \\
&\quad \{V(H') \cap \mathbf{mid}(e) \mid H' \in \mathcal{C}(H)\} = \mathcal{A} \ \wedge \\
&\quad \{(v, \mathbf{deg}_H(v)) \mid v \in \cup_{A \in \mathcal{A}} A\} = \psi \\
&\text{else if } (\mathcal{A} = \emptyset) \text{ then} \\
&\quad |\mathcal{C}(H)| \leq 1 \ \wedge\ V(H) \cap \mathbf{mid}(e) = \emptyset \ \}\}
\end{aligned}
$$

Clearly, $\mathbf{opt}_{e_r}(\emptyset, \varnothing) = \mathbf{medbcs}_d(G)$. Let us now see how these values of $\mathbf{opt}_e(\mathcal{A}, \psi)$ can be explicitly computed using dynamic programming over a branch decomposition of G. Let e, e_1, e_2 be three edges of T that are incident to the same vertex and such that e is closer to the root of T than the other two. To perform the *join/forget* operations in the middle set $\mathbf{mid}(e)$, we distinguish two cases: (1) $\mathcal{A} \neq \emptyset$; and (2) $\mathcal{A} = \emptyset$.

$$
\begin{aligned}
(1)\ \mathbf{opt}_e(\mathcal{A},\psi) = \max\{\{0\} \cup \{l : \exists(\mathcal{A}_i,\psi_i) \in \mathscr{P}_{e_i}, i=1,2, \text{ such that}\\
&(\mathcal{A}_1,\psi_1) \oplus (\mathcal{A}_2,\psi_2) \text{ is a } d\text{-weighted packing of}\\
&\qquad \mathbf{mid}(e_1) \cup \mathbf{mid}(e_2) \ \wedge\\
&(\mathcal{A},\psi) = ((\mathcal{A}_1,\psi_1) \oplus (\mathcal{A}_2,\psi_2))|_{\mathbf{mid}(e)} \ \wedge\\
&\text{if } (\mathcal{A}_1 = \emptyset) \text{ then } l = \mathbf{opt}_{e_2}(\mathcal{A}_2,\psi_2)\\
&\text{if } (\mathcal{A}_2 = \emptyset) \text{ then } l = \mathbf{opt}_{e_1}(\mathcal{A}_1,\psi_1)\\
&\text{else } l = \mathbf{opt}_{e_1}(\mathcal{A}_1,\psi_1) + \mathbf{opt}_{e_2}(\mathcal{A}_2,\psi_2)\ \}\}
\end{aligned}
$$

$$
\begin{aligned}
(2)\ \mathbf{opt}_e(\emptyset,\psi) = \max\{\{0\} \cup \{l : \exists(\mathcal{A}_i,\psi_i) \in \mathscr{P}_{e_i}, i=1,2, \text{ such that}\\
&(\mathcal{A}_1,\psi_1) \oplus (\mathcal{A}_2,\psi_2) \text{ is a } d\text{-weighted packing of}\\
&\qquad \mathbf{mid}(e_1) \cup \mathbf{mid}(e_2) \ \wedge\\
&(\emptyset,\psi) = ((\mathcal{A}_1,\psi_1) \oplus (\mathcal{A}_2,\psi_2))|_{\mathbf{mid}(e)} \ \wedge\\
&\text{if } (\mathcal{A}_1 = \emptyset \ \wedge \ \mathcal{A}_2 = \emptyset) \text{ then } l = \max\{\mathbf{opt}_{e_1}(\mathcal{A}_1,\psi_1), \mathbf{opt}_{e_2}(\mathcal{A}_2,\psi_2)\}\\
&\text{if } (\mathcal{A}_1 \neq \emptyset \ \wedge \ \mathcal{A}_2 = \emptyset) \text{ then}\\
&\qquad l = \max\{\mathbf{opt}_{e_2}(\mathcal{A}_2,\psi_2), \{\mathbf{opt}_{e_1}(\mathcal{A}_1,\psi_1)|_X \colon X \in \mathcal{A}_1\}\}\\
&\text{if } (\mathcal{A}_1 = \emptyset \ \wedge \ \mathcal{A}_2 \neq \emptyset) \text{ then}\\
&\qquad l = \max\{\mathbf{opt}_{e_1}(\mathcal{A}_1,\psi_1), \{\mathbf{opt}_{e_2}(\mathcal{A}_2,\psi_2)|_X \colon X \in \mathcal{A}_2\}\}\\
&\text{if } (\mathcal{A}_1 \neq \emptyset \ \wedge \ \mathcal{A}_2 \neq \emptyset) \text{ then}\\
&\qquad l = \max\{\mathbf{opt}_{e_1}(X,\psi_1)|_{\mathbf{mid}(e_1)} + \mathbf{opt}_{e_2}(X,\psi_2)|_{\mathbf{mid}(e_2)} :\\
&\qquad\qquad X \in \mathcal{C}(\mathbf{mid}(e_1) \cup \mathbf{mid}(e_2), \mathcal{A}_1 \cup \mathcal{A}_2)\}\ \}\}
\end{aligned}
$$

Finally, suppose that $e_{\text{leaf}} = \{x,y\} \in E(T)$ is an edge such that x is a leaf of T. Let $\{v_1,v_2\} \in E(G)$ be the image of x under μ. Then $\mathbf{opt}_{e_{\text{leaf}}}(\mathcal{A},\psi) = 1$ if $\mathcal{A} = \{\{v_1,v_2\}\}$ and $\psi = \{(v_1,1),(v_2,1)\}$, otherwise $\mathbf{opt}_{e_{\text{leaf}}}(\mathcal{A},\psi) = 0$.

Running time. The size of the tables of the dynamic programming over the branch decomposition of the input graph, namely $|\mathscr{P}_e|$, determines the running time of our algorithms. The number of ways a set of ℓ elements can be partitioned into nonempty subsets is well-known as the ℓ-th *Bell number* [6] and is denoted by B_ℓ. We can express $|\mathscr{P}_e|$ in terms of the Bell numbers:

$$|\mathscr{P}_e| \ = \ (d+1)^\ell \cdot \sum_{i=0}^{\ell} \binom{\ell}{i} B_{\ell-i} \ \leq \ (d+1)^\ell \cdot 2^{2\ell \cdot \log \ell}, \qquad (1)$$

where the last inequality is an easy exercise using that $B_\ell \leq \frac{e^\ell - 1}{(\log \ell)^\ell}\ell!$ [6]. At each edge e of the branch decomposition, to compute all the values $\mathbf{opt}_e(\mathcal{A},\psi)$ we test all the possibilities of combining d-weighted packings of the two middle sets $\mathbf{mid}(e_1)$ and $\mathbf{mid}(e_2)$. The operations $(\mathcal{A}_1,\psi_1) \oplus (\mathcal{A}_2,\psi_2)$ and $(\mathcal{A},\psi)|_{A'}$ take $\mathcal{O}(|\mathbf{mid}(e)|)$ time. Let $m = |E(G)|$. Hence, by Eq. (1), given a branch decomposition of a general graph G of width at most ℓ, the value of $\mathbf{medbcs}_d(G)$

can be computed in $(d+1)^{2\ell} \cdot 2^{4\ell \cdot \log \ell} \cdot \ell \cdot m$ steps.

4 Speed-up for Planar Graphs using Catalan Structures

In this section we will see that when the input is restricted to planar graphs the term $2^{\mathcal{O}(\ell \cdot \log \ell)}$ in Eq. (1) can be reduced to $2^{\mathcal{O}(\ell)}$.

Let G be a planar graph embedded on a sphere $\mathbb{S}$. A *noose* is a Jordan curve in $\mathbb{S}$ not intersecting the edge set of G. A *sphere cut decomposition* or *sc-decomposition* (T, μ, π) of G is a branch decomposition of G with the following property: for every edge e of T, there exists a noose O_e meeting every face at most once and bounding the two open discs Δ_1 and Δ_2 such that $G_i \subseteq \Delta_i \cup O_e$, $1 \leq i \leq 2$. Thus O_e meets G only in $\mathbf{mid}(e)$ and its length is $|\mathbf{mid}(e)|$. A *clockwise traversal* of O_e in the embedding of G defines the cyclic ordering π of $\mathbf{mid}(e)$. We always assume that the vertices of every middle set $\mathbf{mid}(e) = V(G_1) \cap V(G_2)$ are enumerated according to π.

Theorem 4.1 (Seymour and Thomas [15]) *Let G be a planar graph of branchwidth at most ℓ without vertices of degree one embedded on a sphere. Then there exists an sc-decomposition of G of width at most ℓ.*

In addition, such an sc-decomposition can be constructed in time $\mathcal{O}(n^3)$ [10]. The size of the tables of the dynamic programming algorithm is given by in how many ways a solution of k-MDBCS$_d$ in G_e can intersect $\mathbf{mid}(e)$. Let (T, μ, π) be a sphere cut decomposition of width $\leq \ell$, and we can assume $\ell \leq \mathbf{bw}(G)$ by Theorem 4.1. Then the vertices of $\mathbf{mid}(e)$ are situated around a noose. A *non-crossing partition (ncp)* is a partition $P(n) = \{P_1, \ldots, P_m\}$ of the set $S = \{1, \ldots, n\}$ such that there are no numbers $a < b < c < d$ where $a, c \in P_i$, and $b, d \in P_j$ with $i \neq j$.

When we restrict the input graph G to be planar, then the subgraph given by the intersection of a partial solution of k-MDBCS$_d$ in G_e with $\mathbf{mid}(e)$ is also planar. The reduction from $2^{\mathcal{O}(\ell \cdot \log \ell)}$ to $2^{\mathcal{O}(\ell)}$ is based on calculating in how many ways we can draw hyperedges inside a cycle such that they touch the cycle on its vertices and they do not share common internal points in the plain (they do not intersect).

The number of such configurations is closely related to the number of *non-crossing partitions* over ℓ vertices, which is equal to the ℓ-th *Catalan number* $\mathrm{CN}(\ell) = \frac{1}{\ell+1}\binom{2\ell}{\ell} \sim \frac{4^\ell}{\sqrt{\pi}\ell^{3/2}} \leq 4^\ell$ [11]. Indeed, in the same spirit of Eq. (1), $|\mathscr{P}_e| = (d+1)^\ell \cdot \sum_{i=0}^{\ell} \binom{\ell}{i} \mathrm{CN}(\ell - i) \leq (d+1)^\ell \cdot \sum_{i=0}^{\ell} \binom{\ell}{i} 4^{\ell-i} = (d+1)^\ell 4^\ell \cdot \sum_{i=0}^{\ell} \binom{\ell}{i} \left(\frac{1}{4}\right)^i = (d+1)^\ell 4^\ell \cdot \left(1 + \frac{1}{4}\right)^\ell = (d+1)^\ell \cdot 5^\ell$.

Since G is planar, $|E(G)| = \mathcal{O}(|V(G)|)$, hence so is the number of middle sets in any branch decomposition of G. Therefore,

Proposition 4.2 *For every planar graph G and given a sphere cut decomposition (T, μ, π) of G of width $\leq \ell$, the value of $\mathbf{medbcs}_d(G)$ can be computed in $\mathcal{O}\left((d+1)^{2\ell} \cdot 5^{2\ell} \cdot \ell \cdot n\right)$ steps.*

Let δ be the constant defined in Lemma 2.4. Summarizing,

Theorem 4.3 *k-Planar Maximum d-Degree-Bounded Connected Subgraph is solvable in time $\mathcal{O}\left(2^{\log(5(d+1))8\sqrt{k}/\delta}\sqrt{k} \cdot n + n^3\right)$.*

Proof. First, using Theorem 4.1, we construct in time $\mathcal{O}(n^3)$ an optimal sphere cut decomposition of G of width $\mathbf{bw}(G)$. We distinguish two cases according to $\mathbf{bw}(G)$. If $\mathbf{bw}(G) > 4/\delta \cdot \sqrt{k}$, then by Lemma 2.4 the answer to the parameterized problem is automatically YES. Otherwise, if $\mathbf{bw}(G) \leq 4/\delta \cdot \sqrt{k}$, the value of the parameter $\mathbf{medbcs}_d(G)$ can be computed by Proposition 4.2 in time $\mathcal{O}\left((d+1)^{8\sqrt{k}/\delta} \cdot 5^{8\sqrt{k}/\delta} \cdot 4/\delta\sqrt{k} \cdot n\right) = \mathcal{O}\left(2^{\log(5(d+1))8\sqrt{k}/\delta}\sqrt{k} \cdot n\right)$. □

5 Extensions and Conclusions

In this article we obtained a $2^{\mathcal{O}(\sqrt{k})}n^{O(1)}$ algorithm for k-MDBCS$_d$ on planar graphs. Appropriate modifications of the dynamic programming algorithm of Section 3 allow us to obtain subexponential parameterized algorithms for the variant of the problem in which the aim is to maximise the number of vertices of the subgraph H, as well as for the variant in which the output subgraph is required to be induced (for both the edge and vertex maximisation versions). Another variant for which subexponential parameterized algorithms exists is when the list of prescribed degrees of the vertices belongs to a subset of $\mathbb{Z}_q$ for a fixed integer q. The details can be found in [14]. The subexponential parameterized algorithms we have presented on planar graphs can be naturally transformed to *exact* subexponential algorithms by using that for any planar graph G, $\mathbf{bw}(G) \leq \sqrt{4.5 \cdot |V(G)|}$ [8].

Several interesting problems remain open. First, it seems natural to try to improve the worst-case running time of our algorithms. Much more challenging is to find subexponential parameterized algorithms for the edge- or node-weighted versions of the problem. Actually, the weighted versions of our parameters remain minor closed (by an easy modification of Lemma 2.1), however the fundamental difference is that the combinatorial bound of Lemma 2.4 does not hold anymore. Finally, the natural extension of this article would be to conceive subexponential parameterized algorithms for k-MDBCS$_d$ on other sparse graph classes, like graphs of bounded genus and, more generally, minor-free families of graphs.

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