Low Polynomial Exclusion of Planar Graph Patterns

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Abstract

The celebrated grid exclusion theorem states that for every h-vertex planar graph H, there is a constant c_h such that if a graph G does not contain H as a minor then G has treewidth at most c_h. We are looking for patterns of H where this bound can become a low degree polynomial. We provide such bounds for the following parameterized graphs: the wheel ($c_h = O(h)$), the double wheel ($c_h = O(h^2 \cdot \log^2 h)$), any graph of pathwidth at most 2 ($c_h = O(h^2)$), and the yurt graph ($c_h = O(h^4)$).

Keywords: Treewidth, Graph Minors

1 Introduction

Treewidth is one of the most important graph invariants in modern graph theory. It has been introduced in [16] by Robertson and Seymour as one of the cornerstones of their Graph Minors series. Apart from its huge combinatorial value, it has been extensively used in graph algorithm design (see [3] for an extensive survey on treewidth). In an intuitive level, treewidth expresses how close is the topology of the graph to the one of a tree and, in a sense, can be seen as a measure of the "global connectivity" of a graph.

Formally, a *tree decomposition* of a graph G is a pair $(T, \mathcal{X})$ where T is a tree and $\mathcal{X}$ a family $(X_t)_{t\in V(T)}$ of subsets of $V(G)$ (called *bags*) indexed by elements of $V(T)$ and such that

(i) $\bigcup_{t\in V(T)} X_t = V(G)$;

(ii) for every edge e of G there is an element of $\mathcal{X}$ containing both ends of e;

(iii) for every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in X_t\}$ is connected.

The *width* of a tree decomposition is equal to $\max_{t\in V(T)} \ |X_t| - 1$, while the *treewidth* of G, written $\mathbf{tw}(G)$, is the minimum width of any of its tree decompositions. Similarly one may define the notions of path decomposition and pathwidth by additionally asking that T is a path (see Section 2).

We say that a graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained by a subgraph of G after applying a series of edge contractions and we denote this fact by $H \leqslant_{\mathrm{m}} G$.

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The grid exclusion theorem. One of the most celebrated results from the Graph Minors series of Robertson and Seymour is the following result, also known as the *grid exclusion theorem*.

Proposition 1 ([17])**.** *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every for every planar graph H on h vertices, every graph G that does not contain a minor isomorphic to H has treewidth at most $f(h)$.*

The original proof the the above result in [17] did not provided any explicit estimation for the function f. Later, in [18], Robertson, Seymour, and Thomas proved the same result for $f(h) = 2^{O(h^5)}$, while a less complicated proof appeared in [9]. For a long time, the insisting open problem was whether this result can be proved for a polynomial f. In [18], an $\Omega(h^2 \cdot \log h)$ lower bound was provided for the best possible estimation of f and was also conjectured that the optimal estimation should not be far away from this lower bound. In fact, a more precise variant of the same conjecture was given by Demaine, Hajiaghayi, and Kawarabayashi in [8] where they conjectured that Proposition 1 holds for $f(h) = O(h^3)$. The estimation of [18] was recently improved by Kawarabayashi and Kobayashi [13], where they show that Proposition 1 holds for $f(h) = 2^{O(h^2 \cdot \log h)}$. Until recently, the best known estimation of f followed by be the result of Leaf an Seymour [14] who proved Proposition 1 for $f(h) = 2^{O(h \cdot \log h)}$.

Very recently, in a breakthrough result [6], Chekuri and Chuzhoy proved that Propostion 1 holds for $f(h) = O(h^{228})$. The remaining open question is whether the degree of this polynomial bound can be substantially reduced in general. In this direction, one may still consider restrictions either on the graph G or on the graph H that yield a low polynomial dependence between the treewidth and the size of the excluded minor. In the first direction, Demaine and Hajiaghayi proved in [7] that, when G is restricted to belong in some graph class excluding some fixed graph R as a minor, then Proposition 1 (optimally) holds for $f(h) = O(h)$. Similar results have been proved by Fomin, Saurabh, and Lokshtanov, in [12], for the case where G is either a unit disk graph or a map graph that does not contain a clique as a subgraph.

In the second direction, one may consider H to be some specific planar graph and find a good upper bound for the treewidth of the graphs that exclude it as a minor. More generally, we can consider a parametrized class of planar graphs $\mathcal{H}_k$ where each graph in $\mathcal{H}_k$ has size bounded by a polynomial on k, and prove that the following fragment of Proposition 1 holds for some low degree polynomial function $f : \mathbb{N} \to \mathbb{N}$:

$$\forall k \geqslant 0\ \forall H \in \mathcal{H}_k,\ if H \not\leqslant_{\mathrm{m}} G \text{ then } \mathbf{tw}(G) \leqslant f(k). \tag{1}$$

The question can be stated as follows: find pairs $(\mathcal{H}_k, f)$ for which (1) holds where H_k is as general as possible and f is as small as possible (and certainly polynomial). It is known, for example, that (1) holds for the pair $(\{C_k\}, k)$, where C_k is the cycle or a path of k vertices (see e.g. [2, 11]), and for the pair $(\{K_{2,k}\}, 2k-2)$, (see [5]). Two more results in the same direction that appeared recently are the following: according to the result of Birmele, Bondy, and Reed in [1], (1) holds for the pair $(\mathcal{P}_k, O(k^2))$ where $\mathcal{P}_k$ contains all minors of $K_2 \times C_k$ (we denote by $K_2 \times C_k$ the Cartesian product of K_2 and the cycle of k vertices, also known as the *k-prism*). Finally, one of the consequences of the recent results of Leaf and Seymour in [14], implies that (1) holds for the pair $(\mathcal{F}_r, O(k))$, where $\mathcal{F}_r$ contains every graph on r vertices that contains a vertex that meets all its cycles.

Our results. In this paper we provide polynomially bounded minor exclusion theorems for the following parameterized graph classes:

$\mathcal{H}_k^0$: containing all graphs on k vertices that have pathwidth at most 2.

$\mathcal{H}_k^1$: containing all minors of a wheel on $k+1$ vertices – see Figure 1.

$\mathcal{H}_k^2$: containing all minors of a double wheel on $k+2$ vertices – see Figure 1.

$\mathcal{H}_k^3$: containing all minors of the yurt graph on $2k+1$ vertices (i.e. the graph obtained it we take a $(2 \times k)$-grid and add a new vertex adjacent with all the vertices of its "upper layer" – see Figure 4).

Notice that none of the above classes is minor comparable with the classes $\mathcal{P}_k$ and $\mathcal{F}_k$ treated in [1] and [14]. Moreover, $\mathcal{H}_k^1 \subset \mathcal{H}_k^2 \subset \mathcal{H}_k^3$, while $\mathcal{H}_k^0$ is not minor comparable with the other three. In this paper we prove that (1) holds for the pairs:

- $(\mathcal{H}_k^0, O(k^2))$,
- $(\mathcal{H}_k^1, O(k))$,
- $(\mathcal{H}_k^2, O(k^2 \log^2 k))$, and
- $(\mathcal{H}_k^3, O(k^4))$.

The above results are presented in detail, without the O-notation, in Section 3. All of our proofs use as a departure point the results of Leaf and Seymour in [14].

2 Definitions

All graphs in this paper are finite and simple, *i.e.*, do not have loops nor multiple edges. We use the notation $V(G)$ (resp. $E(G)$) for the sets of vertices (resp. edges) of G. Logarithms are binary.

Definition 1 (path decomposition, pathwidth)**.** A *path decomposition* of a graph G is a tree decomposition T of G such that T is a path. Its width is the width of the tree decomposition T and the *pathwidth* of G, written $\mathbf{pw}(G)$, is the minimum width of any of its path decompositions.

Definition 2 (minor model)**.** A *minor model* (sometimes abbreviated *model*) of a graph H in a graph G is a pair $(\mathcal{M}, \varphi)$ where $\mathcal{M}$ is a collection of disjoint subsets of $V(G)$ such that $\forall X \in \mathcal{M}$, $G[X]$ is connected and $\varphi\colon V(H) \to \mathcal{M}$ is a bijection that satisfies $\forall \{u,v\} \in E(H), \exists u' \in \varphi(u), \exists v' \in \varphi(v),\ \{u',v'\} \in E(G)$. We say that a graph H is a *minor* of a graph G ($H \leqslant_{\mathrm{m}} G$) if there is a minor model of H in G. Notice that H is a minor of G if H can be obtained by a subgraph of G after contracting edges.

Definition 3 (linked sets)**.** Let G be a graph and $S \subseteq V(G)$. The set S is said to be *linked* in G if for every two subsets X_1, X_2 of S (not necessarily disjoint) such that $|X_1| = |X_2|$, there is a set Q of $|X_1|$ (vertex-)disjoint paths between X_1 and X_2 in G whose length is not one (but can be null) and whose endpoints only are in S.

Definition 4 (separation)**.** A pair (A, B) of subsets of $V(G)$ is a called a *separation of order k in G* if $k = |A \cap B|$ and there is no edge of G between $A \setminus B$ and $B \setminus A$.

Definition 5 (left-contains, [14])**.** Let H be a graph on r vertices, G a graph and (A, B) a separation of order r in G. We say that (A, B) *left-contains* H if $G[A]$ contains a minor model $\mathcal{M}$ of H such that $\forall M \in \mathcal{M},\ |M \cap (A \cap B)| = 1$

Definition 6 (Trees and cycles)**.** Given a tree T we denote by $L(T)$ the set of its leaves, *i.e.* vertices of degree 1 and by $\mathbf{diam}(T)$ its diameter, that is the maximum length (in number of edges) of a path in T.

For every two vertices $u, v \in V(T)$, there is exactly one path in T between u and v, that we denote by uTv. Also, given that uTv has at least 2 vertices, we denote by $\mathring{u}Tv$ (resp. $uT\mathring{v}$) the path uTv without vertex u (resp. v).

Let C be a cycle on which we fixed some orientation. Then, there is exactly one path following this orientation between any two vertices $u, v \in V(C)$. Similarly, we denote this path by uCv and we define $\mathring{u}Cv$ and $uC\mathring{v}$ as we did for the tree.

In a rooted tree T of root r, the *least common ancestor* of two vertices u and v, written $\mathbf{lca}_T(u, v)$ is the first common vertex of the paths uTr and vTr. We refer to the root of T by the notation $\mathbf{root}(T)$.

For every integer $h > 0$, we denote by B_h the complete binary tree of height h.

3 Results

We present in this paper bounds on the treewidth of graphs not containing the following parametrized graphs: the wheel of order k (section 5), the double wheel of order k (section 6), any graph on k vertices and pathwidth at most 2 (section 7) and the yurt graph of order k (section 8). The definitions of these graphs can be found in the corresponding sections. In section 4, we recall some propositions that we will use and we prove two lemmata which will be useful later. The theorems we then prove are the following.

Theorem 1. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 36k - \frac{5}{2}$, then G contains a wheel of order k as minor.*

Theorem 2. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 12(8k \log(8k) + 2)^2 - 4$, then G contains a double wheel of order at least k as minor.*

Theorem 3. *Let $k > 0$ be an integer, G be a graph and H be a graph on k vertices and of pathwidth at most 2. If $\mathbf{tw}(G) \geqslant 3k(k - 4) + 8$ then G contains H as minor.*

Theorem 4. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 6k^4 - 24k^3 + 48k^2 - 48k + 23$, then G contains the yurt graph of order k as minor.*

4 Preliminaries

Proposition 2 ([14, (4.3)]). *Let $k > 0$ be an integer, let T be a tree on k vertices and let G be a graph. If $\mathbf{tw}(G) \geqslant \frac{3}{2}k - 1$, then G has a separation (A, B) of order k such that*

- *$G[B \setminus A]$ is connected;*
- *$A \cap B$ is linked in $G[B]$;*
- *(A, B) left-contains T.*

Proposition 3 ([15]). *If G is an n-vertex graph of pathwidth at most 2 then G is a minor of Ξ_n.*

Proposition 4 (Erdős–Szekeres Theorem, [10]). *Let k and ℓ be two positive integers. Then any sequence of $(\ell - 1)(k - 1) + 1$ distinct integers contains either an increasing subsequence of length k or a decreasing subsequence of length ℓ.*

Lemma 1. *For every tree T, $|L(T)| \cdot \mathbf{diam}(T) + 1 \geqslant |V(G)|$.*

Proof. Root T to an arbitrarily chosen vertex $r \in V(T)$. For each leaf $x \in L(T)$, we know that $|V(xT\mathring{r})| \leqslant \mathbf{diam}(T)$. Observe that $V(T) = \{r\} \cup \bigcup_{x \in L(T)} V(xT\mathring{r})$. Therefore,

$$|V(G)| \leqslant \sum_{x \in L(T)} |V(xT\mathring{r})| + 1$$
$$|V(G)| \leqslant |L(T)| \cdot \mathbf{diam}(T) + 1$$

as required. □

Definition 7 (The set $\Lambda(T)$)**.** Let T be a tree. We denote by $\Lambda(T)$ the set containing every graph obtained as follows: take the disjoint union of T, a path P where $|V(P)| \geqslant \sqrt{|L(T)|}$, and an extra vertex v_{new}, and add edges such that

(i) there is an edge between v_{new} and every vertex of P;

(ii) there are $|V(P)|$ disjoint edges between P and $L(T)$;

(iii) there are no more edges than the edges of A and P and the edges mentioned in (i) and (ii).

Lemma 2. *Let $n \geqslant 1$ be an integer, T be a tree on n vertices an let G be a graph. If $\mathbf{tw}(G) \geqslant 3n - 1$, then $H \leqslant_{\text{m}} G$ for some $H \in \Lambda(T)$.*

Proof. Let n, T, and G be as in the statement of the lemma. Let P be a path on n vertices. We denote by H the disjoint union of P and T and let H^* be the tree obtained from H after adding an edge with endpoints in P and T.

The graph G has treewidth at least $3n-1$, then by Proposition 2, G has a separation (A, B) of order $2n$ such that

(i) $G[B \setminus A]$ is connected;

(ii) $A \cap B$ is linked in $G[B]$;

(iii) (A, B) left-contains the graph H^*.

Let $(\mathcal{M}, \varphi)$ be the a model of H^* in $G[A]$ that witnesses (iii). According to (ii), there is a family $\mathcal{P}$ of n disjoint paths in G such that for every path $Q \in \mathcal{P}$ of this family:

- Q is of length at least 2 and its internal vertices are in $B \setminus A$;
- for the one endpoint, say x, of Q there exists some vertex x' of T such that $x \in \varphi(x')$ and for the other, say y, there exists some vertex y' of P such that $y \in \varphi(y')$.

We call $\mathcal{P}'$ the subset of $\mathcal{P}$ containing all paths whose T-endpoint is mapped to a leaf of T *via* φ.

We set $U = G[B \setminus A]$ and let Y be the graph obtained from $G[B]$ after removing all edges in $G[A \cap B]$.

We define T_U be the tree obtained from U as follows:

1. contract all edges that belong to some path in $\mathcal{P}'$. From Condition (i), this transforms U to a connected graph U^*. That way each path of U that is a subpath of some path in $\mathcal{P}'$ is shrinked to a vertex of U^*. We denote the set of these vertices by I.
2. Let T^* be a minimum size tree of U^* that spans all vertices in I. Remove from U^* all edges that are not edges of T^*.
3. Create T_U by dissolving in T^* all vertices of degree 2 that do not belong to I.

We also define the graph T_Y by first removing from Y all edges incident to vertices in $A \cap B$ that are not edges of some path in $\mathcal{P}'$ and then applying the same steps as above to the remaining edges and vertices that belong to U. Notice that T_Y is a subtree of T_U andthat $|V(T_U)| \geqslant |\mathcal{P}'| = |L(T)|$.

Also observe that $G^- = T \cup P \cup T_Y$ is a minor of G and that G^- contains a collection $\mathcal{Q}$ of $|L(T)|$ disjoint paths, each between some vertex of P and some vertex of $L(T)$ and with only one internal vertex that is a vertex of the tree T_U.

Let $s = \sqrt{|V(T_U)|}$. We consider two cases:

Case 1. There is a path in T_U of length at least s in T_U.

As $s \geqslant \sqrt{|L(T)|}$ there exists a path R in T_U of length at least $\sqrt{|L(T)|}$. Every vertex v of R is of one of the following types.

1. A leaf of T_U. Then $v \in I$, by the minimality of T^* in the second step of the construction of T_U. We then mark v as one of the *privileged* vertices of I.

2. A vertex of degree 2 in T_U. This means that $v \in I$, because of the third step of the construction of T_U. Again we mark v as one of the privileged vertices of I.

3. A vertex of degree $\geqslant 3$ in T_U. In this case, such a vertex v is either a vertex in I or it is connected to a leaf $u \in I$ of T_U with a path of T_U in a way such that $\mathring{v}T_U u$ and R are disjoint. If $v \in I$, then we mark v as one of the privileged vertices of I, otherwise we yield this status to vertex u.

We call R' the path obtained from T_U by contracting in it every edge not belonging to R. We insist that, while applying a contraction of an edge with one privileged vertex v, the resulting vertex has the same name as v and heritages its privileged status. Notice that R and R' have the same length and that all vertices of R' are privileged and therefore are members of I. Let $\mathcal{Q}'$ be the subset of $\mathcal{Q}$ containing paths with privileged vertices.

We call G'' the minor of G' obtained as follows:

(1) We remove from G' all edges of the paths in $\mathcal{Q} \setminus \mathcal{Q}'$.

(2) Apply to the edges of T_U the same contractions that we applied before in order to create R' from T_U.

(3) In the resulting graph, we contract all edges of P to a single vertex v_{new}.

We also define T_Y^* as the graph obtained if we first apply on T_Y all operations one edges that we applied in steps (1) and (2), remove all edges of R', and then identify all remaining vertices that are vertices of P to a single vertex v_{new}.

Notice that $G'' = T \cup T_Y^* \cup R'$ where $|V(R')| \geqslant \sqrt{|L(T)|}$. This implies that $G'' \in \Lambda(T)$, therefore, by the transitivity of the minor relation, G contains a graph in $\Lambda(T)$ as a minor.

Case 2. All paths in T_U have length strictly less than s.

From Lemma 1, $|L(()T_U)| \geqslant \sqrt{|V(T_U)|}$. Observe that $L(()T_U) \subseteq I$ (this follows by the minimality of T^* in the second step of the construction of T_U). Let $\mathcal{Q}'$ be the subset of $\mathcal{Q}$ of the paths that contain an element of $L(()T_U)$. Clearly, $|\mathcal{Q}'| = |L(()T_U)|$. We create the graph G'' as follows:

1. Remove all internal vertices of the paths in $\mathcal{Q} \setminus \mathcal{Q}'$.

2. Remove every edges in T_Y that is incident to $L(T) \cup V(P)$ and does not belong in a path in $\mathcal{Q}'$.

3. For every path $Q \in \mathcal{Q}'$ contract the unique edge with one endpoint in P and the other in $L(()T_U)$.

4. Contract all edges of T_U that are not incident to one of its leaves to a single vertex v_{new}.

5. Dissolve each vertex in P that is the endpoint of a path in $\mathcal{Q} \setminus \mathcal{Q}'$. We denote by P' the path obtained by applying the same operations on P.

Notice that $G'' \leqslant_{\mathrm{m}} G'$. Moreover $|V(P')| = |\mathcal{Q}'| = |L(()T_U)| \geqslant \sqrt{|V(T_U)|}$ which implies that that $G'' \in \Lambda(T)$. By the transitivity of the minor relation, G contains a graph in $\Lambda(T)$ as a minor. □

5 Excluding a wheel with a linear bound on treewidth

Definition 8 (wheel)**.** Let $r > 2$ be an integer. The *wheel* of order r (denoted W_r) is a cycle of length r whose each vertex is adjacent to an extra vertex, in other words it is a the graph of the form

$$\begin{aligned}
V(G) &= \{o, w_1, \dots, w_r\} \\
E(G) &= \{\{w_1, w_2\}, \{w_2, w_3\}, \dots, \{w_{r-1}, w_r\}, \{w_r, w_1\}\} \cup \{\{o, w_1\}, \dots, \{o, w_r\}\}
\end{aligned}$$

(see Figure 1 for an example)

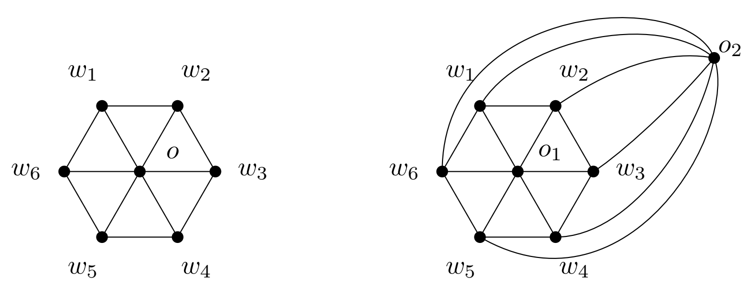

Figure 1: A wheel of order six (left) and a double wheel of order 6 (right)

Lemma 3. *Let $h > 2$ be an integer. Let G be a graph of the following form: the union of the tree $T = B_h$ and a path P such that for every $l \in L(T)$, $\{l, \psi(l)\} \in E(G)$, where $\psi\colon L(T) \to V(P)$ is a bijection. Then G contains a wheel of order $2^{h-2} + 1$.*

Proof. Let h, ψ, T, $P = p_1 \dots p_{2^h}$ and G be as above. Let r be the root of T.

In the sequel, if $t \in V(T)$, we denote by T_t the subtree of T rooted at t (*i.e.* the subtree of T whose vertices are all the vertices $t' \in V(T)$ such that the path $t'Tr$ contains t).

We consider the vertices $u = \psi^{-1}(p_1) \in L(T)$ and $v = \psi^{-1}(p_{2^h}) \in L(T)$ and $w = \mathbf{lca}_T(u, v) \in V(T) \setminus L(T)$.

Let τ be the biggest complete subtree of T which is disjoint from uTv. Let L_τ be the set of leaves of the subtree τ and let $Q = \psi(L_\tau) \subseteq P$. We first show that G contains a $W_{|Q|+1}$-model. We denote by $q_1, \dots, q_{|Q|}$ the elements of Q and we assume that these vertices appears in this order in P. We now present a $W_{|Q|+1}$-model $(\mathcal{M}, \varphi)$ in G by setting:

$$\begin{aligned}\forall i \in [\![1, |Q|-1]\!]\,,\ M_i &= V(q_i P \mathring{q}_{i+1})\\ M_{|Q|} &= \{q_{|Q|}\}\\ M_{|Q|+1} &= V(\mathring{q}_{|Q|} P p_{2^h-1}) \cup V(vTu) \cup V(p_0 P \mathring{q}_1)\\ M_{|Q|+2} &= V(\tau) \cup V(\mathbf{root}(\tau) T \mathring{w})\end{aligned}$$

and

$$\varphi\colon \left\{ \begin{array}{rcl} V(W_{|Q|+1}) & \to & \mathcal{M} \\ \forall i \in [\![1, |Q|+1]\!]\,,\ w_i & \mapsto & M_i \\ o & \mapsto & M_{|Q|+2} \end{array} \right.$$

Let us make some remarks on $(\mathcal{M}, \varphi)$.

Remark 1. Every element of $\mathcal{M}$ induces a subgraph of G which is connected:

- for every $i \in [\![1, |Q|]\!]$, M_i is defined as the set of vertices of a path or of meeting paths, thus it induces a connected graph in G;
- the set $M_{|Q|+2}$ contains the subtree τ (connected) and either the path from $\mathbf{root}(\tau)$ to w or the first vertices of this path.

Remark 2. Every two different elements of $\mathcal{M}$ are disjoint.

Remark 3. For all $x, y \in V(W_{|Q|+2})$ if $\{x, y\}$ is an edge in $W_{|Q|+2}$ then there is an edge in G between a vertex of $\varphi(x)$ and a vertex of $\varphi(y)$.

In fact, $W_{|Q|+2}$ has edges $\{\{w_i, w_{i+1}\}\}_{i \in [\![1, |Q|]\!]}$, $\{w_{|Q|+1}, w_1\}$ and $\{\{w_i, o\}\}_{i \in [\![1, |Q|+1]\!]}$.

1. For every $i \in [\![1, |Q|-1]\!]$, M_i contains the vertices of the path $q_i Q \mathring{q}_{i+1}$ and $q_{i+1} \in M_{i+1}$, so $\varphi(w_i) = M_i$ and $\varphi(w_{i+1}) = M_{i+1}$ are linked by an edge;
2. $M_{|Q|}$ contains the vertices of $vT\mathring{w}$ and $w \in M_{|Q|+1}$, so $\varphi(w_{|Q|}) = M_{|Q|}$ and $\varphi(w_{|Q|+1}) = M_{|Q|+1}$ are linked by an edge;
3. $M_{|Q|+1}$ contains the vertices of $wT\mathring{u}$ and $u \in M_1$, so $\varphi(w_{|Q|+1}) = M_{|Q|+1}$ and $\varphi(w_1) = M_1$ are linked by an edge;
4. for every $i \in [\![1, |Q|]\!]$, M_i contains a element of Q which is by definition of G and Q connected by an edge to a leaf of τ and $V(\tau) \supseteq M_{|Q|+2}$, so $\varphi(w_i) = M_i$ and $\varphi(o) = M_{|Q|+2}$ are linked by an edge;
5. $M_{|Q|+2}$ contains the vertices of the path to $rT\mathring{w}$ and $w \in M_{|Q|+1}$, so $\varphi(w_{|Q|+1}) = M_{|Q|+1}$ and $\varphi(o) = M_{|Q|+2}$ are linked by an edge.

According to the previous remarks, $(\mathcal{M}, \phi)$ is a model of $W_{|Q|+1}$.

Depending on G, $|Q|$ may take different values. However, we show that it is never less than 2^{h-2}. Remember, $|Q|$ is the number of leaves of the biggest complete subtree of T that is disjoint from uTv. The root r of T has two children r_1 and r_2, inducing two subtrees T_{r_1} and T_{r_2} of T.

Case 1. $w \neq r$. As $w \neq r$, w is a vertex of one of $\{T_{r_1}, T_{r_2}\}$, say T_{r_1}, which contains also u and v, and thus the path uTv. The other subtree T_{r_2} is then disjoint from uTv, it have height $h-1$ and is complete so it have 2^{h-1} leaves. Consequently, in this case $|Q| \geqslant 2^{h-1}$.

Case 2. $w = r$. In this case, the path uTv contains r (and $r \neq u$, $r \neq v$ as u and v are leaves) so u and v are not in the same subtree of $\{T_{r_1}, T_{r_2}\}$ and uTv contains the two edges $\{r, r_1\}$ and $\{r, r_2\}$. For every $i \in \{1, 2\}$, we denote by $r_{i,1}$ and $r_{i,2}$ the two children of r_i in T. We assume without loss of generality that $u \in V(T_{r_{1,1}})$ and $v \in V(T_{r_{2,1}})$ (if not, we just rename the r_i's ans $r_{i,j}$'s). Notice that the path uTv is the concatenation of the paths $uT_{r_1}r_1$, r_1Tr_2, $r_2T_{r_2}v$. Since the tree $T_{r_{1,2}}$ is disjoint from uTv, is complete and is of height $h-2$, it have 2^{h-2} leaves. Therefore we have $|Q| \geqslant 2^{h-2}$.

In both cases, $|Q| \geqslant 2^{h-2}$ and according to what we proved before, G contains a model of $\mathrm{W}_{|Q|+2}$. As every wheel contains a model of every smaller wheel, we have proved that G contains a wheel of order at least 2^{h-2}. □

Theorem 5. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 36k - \frac{5}{2}$, then G contains a W_k-model.*

Proof. Let $k > 0$ be an integer, G be a graph such that $\mathbf{tw}(G) \geqslant 36k - \frac{5}{2}$, and let $h = \lceil \log 4k \rceil$. Since every wheel contains a model of every smaller wheel, we have

$$\begin{aligned} \mathrm{W}_k &\leqslant_{\mathrm{m}} \mathrm{W}_{2^{\lceil \log k \rceil}+1} \\ &\leqslant_{\mathrm{m}} \mathrm{W}_{2^{\lceil (\log 4k)-2 \rceil}+1} \\ &\leqslant_{\mathrm{m}} \mathrm{W}_{2^{h-2}+1} \end{aligned}$$

Therefore, if we prove that G contains a $\mathrm{W}_{2^{h-2}+1}$-model, then we are done because the minor relation is transitive. Let Y_h^- be the graph of the following form: the disjoint union of the complete binary tree B_h of height h with leaves set Y_L and of the path Y_P on 2^h vertices, and let $\mathcal{Y}_h$ be the set of graphs of the same form, but with the extra edges $\{\{l, \phi(l)\}\}_{l \in Y_L}$, where $\phi : Y_L \to V(Y_P)$ is a bijection. As we proved in Lemma 3 that every graph of $\mathcal{Y}_h$ contains the wheel of order $2^{h-2}+1$ as minor, showing that G contains a graph of $\mathcal{Y}_h$ as minor suffices to prove this lemma. That is what we will do. Let H be a graph of $\mathcal{Y}_h$.

From our initial assumption, we deduce the following.

$$\begin{aligned} \mathbf{tw}(G) &\geqslant 36k - \frac{5}{2} \\ &\geqslant \frac{3}{2}(3 \cdot 2^{\log 8k} - 1) - 1 \\ &\geqslant \frac{3}{2}(3 \cdot 2^{\lfloor \log 4k \rfloor + 1} - 1) - 1 \\ \mathbf{tw}(G) &\geqslant \frac{3}{2}(3 \cdot 2^h - 1) - 1 \end{aligned}$$

According to Proposition 2, G has a separation (A, B) of order $3 \cdot 2^h - 1$ such that

(i) $G[B \setminus A]$ is connected;

(ii) $A \cap B$ is linked in $G[B]$;

(iii) (A, B) left-contains the graph Y_h^-.

By definition of *left-contains*, $G[A]$ contains a model $(\mathcal{M}^-, \varphi^-)$ of Y_h^- and every element of $\mathcal{M}^-$ contains exactly one element of $A \cap B$. For every $x \in A \cap B$, we denote by M_x^- the element of $\mathcal{M}^-$ that contains x. Let L (resp. R) be the subset of $A \cap B$ of vertices that belong to an element of M related to the leaves of B_h in Y_h^- (resp. to the path P). We remark that these sets are both of cardinality 2^h.

Since $A \cap B$ is linked in $G[B]$ (see (ii)), there is a set $\mathcal{P}$ of 2^h disjoint paths between the vertices of L and the elements of R. Let $\psi \colon L \to V(P)$ be the function that match each element

l of L with the (unique) element of R it is linked to by a path (that we call $\mathcal{P}_l$) of $\mathcal{P}$. Observe that ψ is a bijection. We set

$$\forall l \in L,\ M_l = M_l^- \cup V(l\mathcal{P}_l\mathring{\psi}(l))$$
$$\forall r \in (A \cap B) \setminus L,\ M_r = M_r^-$$

$$\mathcal{M} = \bigcup_{x \in A \cup B} M_x$$

and

$$\varphi\colon \begin{cases} V(H) & \to & \mathcal{M} \\ x & \mapsto & M_x \end{cases}$$

We claim that $(\mathcal{M}, \varphi)$ is a model of H. This is a consequence of the following remarks.

Remark 4. Every element of $\mathcal{M}$ is either an element of $\mathcal{M}^-$, or the union of a element M of $\mathcal{M}^-$ and of the vertices of a path that start in M, thus every element of $\mathcal{M}$ induces a connected subgraph of G.

Remark 5. The paths of $\mathcal{P}$ are all disjoint and are disjoint from the elements of $\mathcal{M}^-$. Every interior of path of $\mathcal{P}$ is in but one element of $\mathcal{M}$, therefore the elements of $\mathcal{M}$ are disjoint.

Remark 6. The elements $\{m_l\}_{l \in L}$ are in bijection with the elements of $\{m_r\}_{r \in R}$ (thanks to the function ψ) and every two vertices $l \in L$ and $\psi(l) \in R$ are such that there is an edge between m_l and $m_{\psi(l)}$ (by definition of $\mathcal{M}^+$).

We just proved that $(\mathcal{M}, \varphi)$ is a model of a graph of $\mathcal{Y}_h$ in G. Finally, we apply Lemma 3 to find a model of the wheel of order $2^{h-2} + 1 = 2^{\lceil \log k \rceil} + 1 \geqslant k$ in G and this concludes the proof. □

6 Excluding a double wheel with a $(l \log l)^2$ bound on treewidth

Definition 9 (double wheel)**.** Let $r > 2$ be an integer. The *double wheel* of order r (denoted W_r^2) is a cycle of length r whose each vertex is adjacent to two different extra vertices, in other words it is the graph of the form

$$\begin{aligned} V(G) =& \{o_1, o_2, w_1, \ldots, w_r\} \\ E(G) =& \{\{w_1, w_2\}, \{w_2, w_3\}, \ldots, \{w_{r-1}, w_r\}, \{w_r, w_1\}\} \\ & \cup \{\{o_1, w_1\}, \ldots, \{o_1, w_r\}\} \\ & \cup \{\{o_2, w_1\}, \ldots, \{o_2, w_r\}\} \end{aligned}$$

(see Figure 1 for an example)

Lemma 4. *Let G be a graph and $h > 0$ be an integer. If $tw(G) \geqslant 6 \cdot 2^h - 4$, then G contains as minor a double wheel of order at least $\frac{2^{\frac{h}{2}} - 2}{2h-3}$.*

Proof. Let h and G be as above. Observe that $\mathbf{tw}(G) \geqslant 3(2^{h+1} - 1) - 1$. As the binary tree $T = B_h$ has $2^{h+1} - 1$ vertices, G contains a graph $H \in \Lambda(B_h)$ as minor (by Lemma 2). Let us show that any graph $H \in \Lambda(B_h)$ contains a double wheel of order at least $\frac{2^{\frac{h}{2}} - 2}{2h-3}$ as minor.

Let P be the path of length at least $2^{\frac{h}{2}}$ in the definition of H. Let L be the set, of size at least $2^{\frac{h}{2}}$, of the leaves of T that are adjacent to P in H. Such a set exists by definition of $\Lambda(B_h)$. We also define u (resp. u') as the vertex of $L(T)$ that is adjacent to one end of P (resp. to the other end of P) and $Q = uTu'$.

As T is a binary tree of height h, Q has at most $2h-1$ vertices. Each vertex of Q is of degree at most 3 in T except the two ends which are of degree 1. Consequently, $T \setminus Q$ has at most $2h-3$ connected components that are subtrees of T. Notice that every vertex of the $2^{\frac{h}{2}}$ elements of L is either a leaf of one of these $2h-3$ subtrees, or one of the two ends of Q. By the pigeonhole principle, one of these subtrees, say T_1, has at least $\frac{2^{\frac{h}{2}}-2}{2h-3}$ leaves that are elements of L.

Let M_{o_1} be the set of vertices of this subtree T_1. We also set $M_{o_2} = \{o_H\}$. Let us consider the cycle C made by the concatenation of the paths of H uPu' and $u'Tu$.

By definition of M_{o_1}, there are at least $\frac{2^{\frac{h}{2}}-2}{2h-3}$ vertices of C adjacent to vertices of M_{o_1}. Let $J = \left\{j_1, \dots, j_{|J|}\right\}$ be the set of such vertices of C, in the same order as they appear in C (we then have $|J| \geqslant \frac{2^{\frac{h}{2}}-2}{2h-3}$).

We arbitrarily choose an orientation of C and define the sets of vertices $M_1, M_2, \dots, M_{|J|}$ as follows.

$$\begin{aligned} \forall i \in [\![1, |J|-1]\!]\,,\ M_i &= V(j_i C \mathring{j}_{i+1}) \\ M_{|J|} &= V(j_{|J|} C \mathring{j}_1) \end{aligned}$$

Let $\mathcal{M} = \left\{M_1, \dots, M_{|J|}, M_{o_1}, M_{o_2}\right\}$ and $\psi\colon V(\mathrm{W}^2_{|J|}) \to \mathcal{M}$ be the function defined by

$$\begin{aligned} \forall i \in [\![1, |J|]\!]\,,\ \psi(w_i) &= M_i \\ \psi(o_1) &= M_{o_1} \\ \psi(o_2) &= M_{o_2} \end{aligned}$$

Notice that ψ maps the vertices of $\mathrm{W}^2_{|J|}$ to connected subgraphs of H such that $\forall (v,w) \in E(\mathrm{W}^2_{|J|})$, there is a vertex of $\psi(v)$ adjacent in H to a vertex of $\psi(w)$. Therefore, $(\mathcal{M}, \psi)$ is a $\mathrm{W}^2_{|J|}$-model in H.

Since $|J| \geqslant \frac{2^{\frac{h}{2}}-2}{2h-3}$, H contains a double wheel of order at least $\frac{2^{\frac{h}{2}}-2}{2h-3}$, what we wanted to show. $\square$

Corollary 1. *Let $l > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 12l - 4$ then G contains a double wheel of order at least $\frac{\sqrt{l}-2}{2\log l - 5}$ as minor.*

Proof. Let l and G be as above. First remark that

$$\lceil \log l \rceil - 1 \leqslant \log l \leqslant \lceil \log l \rceil \tag{2}$$

Our initial assumption on $\mathbf{tw}(G)$ gives the following.

$$\begin{aligned} \mathbf{tw}(G) &\geqslant 12l - 4 \\ &\geqslant 6 \cdot 2^{\log(2l)} - 4 \\ &\geqslant 6 \cdot 2^{\log l + 1} - 4 \\ &\geqslant 6 \cdot 2^{\lceil \log l \rceil} - 4 && \text{by (2)} \end{aligned}$$

By Lemma 4, G contains a double wheel of order at least

$$\begin{aligned} q &= \frac{2^{\frac{\lceil \log l \rceil}{2}} - 2}{2\lceil \log l \rceil - 3} \\ &\geqslant \frac{2^{\frac{1}{2}\log l} - 2}{2(\log l - 1) - 3} && \text{by (2)} \\ &\geqslant \frac{\sqrt{l} - 2}{2\log l - 5} \end{aligned}$$

Therefore, G contains a double wheel of order at least $q \geqslant \frac{\sqrt{l}-2}{2\log l - 5}$, as required. $\square$

Theorem 6 (follows from Corollary 1)**.** *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 12(8k\log(8k)+2)^2 - 4$, then G contains a double wheel of order at least k as minor.*

Proof. Applying Corollary 1 for $l = (8k\log(8k)+2)^2$ yields that G contains a double wheel of order at least

$$\begin{aligned} q &\geqslant \frac{\sqrt{l} - 2}{2\log l - 5} \\ &\geqslant \frac{8k\log(8k)}{4\log(8k\log(8k)+2) - 5} \\ &\geqslant \frac{8k\log(8k)}{4\log(8k\log(8k)) - 1} \\ &\geqslant \frac{8k\log(8k)}{4(\log(8k) + \log\log(8k)) - 1} \\ &\geqslant \frac{8k\log(8k)}{8\log(8k) - 1} \\ &\geqslant k \end{aligned}$$

Consequeltly G contains a double wheel of order at least $q \geqslant k$ and we are done. $\square$

7 Excluding a graph of pathwidth at most 2 with a quadratic bound on treewidth

Definition 10 (graph Ξ_r)**.** We define the graph Ξ_r as the graph of the following form (see figure 2).

$$\begin{cases} V(G) = \{x_0, \ldots, x_{r-1}, y_0, \ldots, y_{r-1}, z_0, \ldots, z_{r-1}\} \\ E(G) = \{\{x_i, x_{i+1}\}, \{z_i, z_{i+1}\}\}_{i \in [\![1, r-1]\!]} \cup \{\{x_i, y_i\}, \{y_i, z_i\}\}_{i \in [\![0, r-1]\!]} \end{cases}$$

7.1 Graphs of pathwidth 2 in Ξ_r

Instead of proving that having a graph H of pathwidth 2 as minor forces a treewidth quadratic in $|V(H)|$, we prove that a Ξ_r-minor forces a treewidth quadratic in r and that every graph of pathwidth at most 2 on r vertices is minor of Ξ_{r-1}. For this, we first need somme lemmata and remarks about path decompositions.

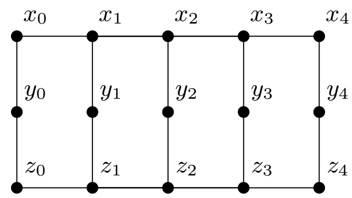

Figure 2: The graph Ξ_5

Definition 11 (nice path decompostion, [4])**.** A path decomposition $\left(p_1p_2\dots p_k, \{X_{p_i}\}_{i\in[\![1,k]\!]}\right)$ of a graph G is said to be *nice* if $|X_{p_1}| = 1$ and

$$\forall i \in [\![2,k]\!]\,,\ \left|(X_{p_i} \setminus X_{p_{i-1}}) \cup (X_{p_{i-1}} \setminus X_{p_i})\right| = 1$$

It is known [4] that every graph have an optimal path decomposition which is nice and that in such decomposition, every node X_i is either an *introduce node* (*i.e.* either $i = 1$ or $|X_i \setminus X_{i-1}| = 1$) or a *forget node* (*i.e.* $|X_{i-1} \setminus X_i| = 1$).

Remark 7. It is easy to observe that for every graph G on n vertices, there is an optimal path decomposition with n introduce nodes and n forget nodes (one of each for each vertex of G), thus of length $2n$.

Remark 8. Let G be a graph and $(p_1p_2\dots p_k, \mathcal{X})$ with $\mathcal{X} = \{X_{p_i}\}_{i\in[\![1,k]\!]}$ be a nice path decomposition of G.

For every $i \in [\![2, k-1]\!]$, if p_i is a forget node and p_{i+1} an introduce node, then by setting

$$\begin{aligned} X'_i &= X_{i-1} \cup X_{i+1} \\ \forall j \in [\![1,k]\!]\,,\ j \neq i,\ X'_j &= X_j \\ \mathcal{X}' &= \left\{X'_j\right\}_{j\in[\![1,k]\!]} \end{aligned}$$

we create from $(p_1p_2\dots p_k, \mathcal{X}')$ a valid path decomposition of G, where p_i is now an introduce node and p_{i+1} a forget node.

Remark 9. Let G be a graph and $P = (p_1p_2\dots p_k, \mathcal{X})$ be a nice path decomposition of G. For every $i \in [\![1, k]\!]$, the path $p_1\dots p_i$ contains at most as much forget nodes as introduce nodes and the difference between these two numbers is at most $w + 1$ where w is the width of P.

Lemma 5. *Let G be a graph on n vertices . Then G has an optimal path decomposition P such that*

(i) every bag of P has size $\mathbf{pw}(G) + 1$*;*

(ii) every two ajacent bags differs by exactly one element, i.e. *for every two adjacent vertices u and v of P, $|X_u \setminus X_v| = |X_v \setminus X_u| = 1$.*

Proof. Let $P = (p_1p_2\dots p_k, \mathcal{X})$ with $\mathcal{X} = \{X_{p_i}\}_{i\in[\![1,2k]\!]}$ be a nice optimal path decomposition of G with as much introduce nodes (resp. forget nodes) as there are vertices in G.

Let $s = \mathbf{pw}(G) + 1$. According to Remarks 8 and 9, P can be modified into a path decomposition of G of the same width and such that

(a) the s first vertices of P are introduce nodes and p_{s+1} is a forget node;

(b) the s last vertices of P are forget nodes and p_{2k-s} is an introduce node;

(c) for every $i \in [\![s, 2k-s]\!]$, p_i and p_{i+1} are nodes of different type.

In the sequel, we assume that P satisfies this property.

Remark 10. Introduce nodes all have bags of cardinal s.

Remark 11. For every $i \in [\![0, k-s]\!]$, the node p_{s+2i} is an introduce node and the node p_{s+2i+1} is a forget node, what implies $X_{p_{s+2i}} \subsetneq X_{p_{s+2i+1}}$. Also note that for every $i \in [\![1, s-1]\!]$, $X_i \subsetneq X_s$ and for every $i \in [\![2k-s+1, 2k]\!]$, $X_i \subsetneq X_{2k-s}$.

Intuitively, every bag X that is included in one of its adjacent bags X' contains no more information than what X' already contains, so we will just remove it.

We thus define $P' = p_s p_{s+2} \dots p_{s+2i} \dots p_{2k-s}$ (a path made of all introduce nodes of P). Clearly, P and P' have the same width and as we deleted only redundant nodes, P' is still a valid path decomposition of G.

Since every two ajacent nodes of P' were introduce nodes separated by a forget node in P, they only differ by one element. According to Remark 10 and since every node of P' was an introduce node in P, every bag of P' have size $\mathbf{pw}(G)+1$. Consequently, P' is an optimal path decomposition that satisfies the conditions of the lemma statement. □

Remark 12. The path decomposition of Lemma 5 has length $V(G) - \mathbf{pw}(G)$.

Proof. Let $(P, \mathcal{X})$ be such a path decomposition. Remember that the first node of P has a bag of size $\mathbf{pw}(G)+1$ and that every two adjacent nodes of P have bags which differs by exactly one element. Since every vertex of G is in a bag of P, in addition to the first bag containing $\mathbf{pw}(G)+1$ vertices of G, P must have $V(G) - \mathbf{pw}(G) - 1$ other bags in order to contain all vertices of G. Therefore P has length $V(G) - \mathbf{pw}(G)$. □

Lemma 6. *For every graph G on n vertices and of pathwidth at most 2, there is a minor model of G in Ξ_{n-1}.*

Proof. Let G be as in the statement of the lemma. We assume that $\mathbf{pw}(G) = 2$ (if this is not the case we add edges to G in order to obtain a graph of pathwidth 2 whose G is minor). Let $r = V(G) - \mathbf{pw}(G) = n-2$.

Let $P = (p_1 \dots p_r, \{X_{p_1}, \dots, X_{p_r}\})$ be an optimal path decomposition of G satisfying the properties of Lemma 5, of length r. Such decomposition exists according to Lemma 5 and Remark 12).

Using this decomposition, we will now define a labeling λ of the vertices of Ξ_{r+1}. When dealing with the vertices of Ξ_{r+1} we will use the notations defined in Definition 10. Let $\lambda\colon V(\Xi_{r+1}) \to V(G)$ be the function defined as follows:

(a) $\lambda(x_0)$ and $\lambda(y_0)$ are both equal to one (arbitrarly choosen) element of the $X_{p_1} \cap X_{p_2}$;

(b) $\lambda(z_0)$ is equal to the only element of $X_{p_1} \cap X_{p_2} \setminus \{\lambda(x_1)\}$;

(c) $\forall i \in [\![2, r]\!]$, $\lambda(y_i) = X_{p_i} \setminus X_{p_{i-1}}$ and we consider two cases:

Case 1: $X_{p_{i-1}} \cap X_{p_i} = X_{p_i} \cap X_{p_{i+1}}$
$\lambda(x_i) = \lambda(x_{i-1})$ and $\lambda(z_i) = \lambda(z_{i-1})$;

Case 2: $X_{p_{i-1}} \cap X_{p_i} \neq X_{p_i} \cap X_{p_{i+1}}$
if $X_{p_{i-1}} \cap X_{p_i} \cap X_{p_{i+1}} = \lambda(x_{i-1})$,
then $\lambda(x_i) = \lambda(x_{i-1})$ and $\lambda(z_i) = X_{p_i} \setminus X_{p_{i-1}}$;
else $\lambda(x_i) = X_{p_i} \setminus X_{p_{i-1}}$ and $\lambda(z_i) = \lambda(z_{i-1})$.

Thanks to this labeling, we are now able to present a minor model of G in Ξ_{r+1} :

$$\forall v \in V(G),\ M_v = \{u \in V(\Xi_{r+1}),\ \lambda(u) = v\}$$
$$\mathcal{M} = \{M_v\}_{v \in V(G)}$$

$$\varphi\colon \begin{cases} V(G) & \to & \mathcal{M} \\ u & \mapsto & M_u \end{cases}$$

To show that $(\mathcal{M}, \varphi)$ is a G-model in Ξ_{r+1}, we now check if it matches the definition of a minor model.

By definition, every element of $\mathcal{M}$ is a subset of $V(\Xi_{r+1})$. To show that every element of $\mathcal{M}$ induces a connected subgraph in G, it suffices to show that nodes of Ξ_{r+1} which have the same label induces a connected subgraph in G (by construction of the elements of $\mathcal{M}$). This can easily be seen by remarking that for every $i \in [\![2, r]\!]$, every vertex y_i of Ξ_{r+1} gets a new label and that every vertex x_i (resp. z_i) of Ξ_{r+1} receive either the same label as y_i, or the same label as x_{i-1} (resp. z_{i-1}).

Let us show that this labeling ensure that if two vertices u and v of G are in the same bag of P, there are two adjacent vertices of Ξ_{r+1} that respectively gets labels u and v. Let u, v be two vertices of G which are in the same bag of P. Let i be such that X_i is the first bag of P (with respect to the subsripts of the bags of P) which contains both u and v. The case $i = 1$ is trivial so we assume that $i > 1$. We also assume without loss of generality that $X_i \setminus X_{i-1} = \{v\}$, what gives $\lambda(y_i) = v$. Depending on in what case we are, either either $\lambda(x_i) = u$ (c1) or $\lambda(z_i) = u$ (c1 and c2). In both cases, u and v are the labels of two adjacent nodes of Ξ_{r+1}. By construction of the elements of $\mathcal{M}$, this implies that if $\{u, v\} \in E(G)$, then there are vertices $u' \in \varphi(u)$ and $v' \in \varphi(v)$ such that $\{u', v'\} \in E(\Xi_{r+1})$.

Therefore, $(\mathcal{M}, \varphi)$ is a G-model in Ξ_{n-1}, what we wanted to find.

□

7.2 Exclusion of Ξ_r

Lemma 7. *For any graph, if* $\mathbf{tw}(G) \geqslant 3\ell - 1$ *then* G *contains as minor the following graph: a path* $P = p_1 \dots p_{2\ell}$ *of length* 2ℓ *and a family* Q *of* ℓ *paths of length 2 such that every vertex of* P *is the end of exactly one path of* Q *and every path of* Q *has one end in* $p_1 \dots p_l$ *(the first half of* P*) and the other end in* $p_{l+1} \dots p_{2l}$ *(the second half of* P*) (see figure 3).*

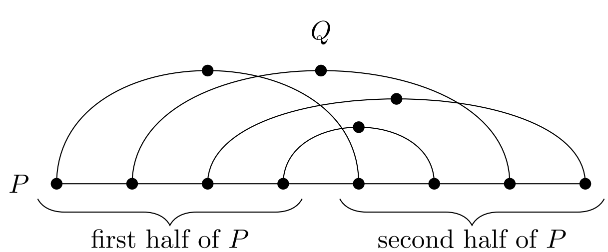

Figure 3: Example for Lemma 7

Proof. Let $\ell > 0$ be an integer and G be a graph of treewidth at least $3\ell - 1$. According to Proposition 2, G has a separation (A, B) of order 2ℓ such that

(i) $G[B \setminus A]$ is connected;

(ii) $A \cap B$ is linked in $G[B]$;

(iii) (A, B) left-contains a path $P = p_1 \dots p_{2\ell}$ of length 2ℓ.

Let $(\mathcal{M}, \varphi)$ be a model of P in $G[A]$, with $\mathcal{M} = \{M_1, \dots, M_{2\ell}\}$. We assume without loss of generality that φ maps p_i on M_i for every $i \in [\![1, 2\ell]\!]$.

As $A \cap B$ is linked in $G[B]$, there is a set Q of ℓ disjoint paths in $G[B]$ of length at least 2 and such that every path $q \in Q$ has one end in $(A \cap B) \cap \bigcup_{i \in [\![1,\ell]\!]} M_i$, the other end in $(A \cap B) \cap \bigcup_{i \in [\![\ell+1,2\ell]\!]} M_i$ and its internal vertices are not in $A \cap B$.

Let G' be the graph obtained from $G\left[(\cup_{q \in Q} V(q)) \cup \left(\bigcup_{M \in \mathcal{M}} M\right)\right]$ after the following operations.

1. iteratively contract the edges of every path of Q until it reaches a length of 2. The paths of Q have length at least 2, so this is always possible.
2. for every $i \in [\![1, 2\ell]\!]$, contract M_i to a single vertex. The elements of a model are connected (by definition) thus this operation can always be performed.

As one can easily check, the graph G' is the graph we were looking for and it has been obtained by contracting some edges of a subgraph of G, therefore $G' \leqslant_{\mathrm{m}} G$. ☐

Theorem 7. *Let $k > 0$ be an integer, G be a graph and H be a graph on h vertices satisfying $\mathbf{pw}(H) \leqslant 2$. If $\mathbf{tw}(G) \geqslant 3h(k-4) + 8$ then G contains H as minor.*

Proof. From Proposition 3, every graph of pathwidth at most two on r vertices is minor of Ξ_r, so if we show that G contains Ξ_{h-1} as minor then it contains H and we are done.

Let $k > 0$ be an integer. We prove the following statement: for every graph G, $\mathbf{tw}(G) \geqslant 3k(k-2) - 1$ implies that $G \geqslant_{\mathrm{m}} \Xi_k$. Let G be a graph of treewidth at least $3k(k-2) - 1$. According to Lemma 7, G contains as minor two paths $P = p_1 \dots p_{k(k-2)}$ and $R = r_1 \dots r_{k(k-2)}$ and a family Q of $k(k-2)$ paths of length 2 such that every vertex of P or R is the end of exactly one path of Q and every path of Q has one end in P and the other end in R. For every $p \in P$, we denote by $\varphi(p)$ the (unique) vertex of R to which p is linked to by a path of Q. Remark that φ is a bijection. By Proposition 4, there is a subsequence $P' = (p'_1, p'_1, \dots, p'_k)$ of P such that the vertices $\varphi(p'_1), \varphi(p'_1), \dots, \varphi(p'_k)$ appears in this order in R. Let $R' = (\varphi(p'_1), \varphi(p'_1), \dots, \varphi(p'_k))$ and Q' be the set of inner vertices of the paths from p'_i to $\varphi(p'_i)$ for all $i \in [\![1, k]\!]$.

Iteratively contracting in G the edges of P (resp. R) whose one of the ends is not in P' (resp. not in R') and removing the vertices that are not in P, R or Q gives the graph Ξ_k. The operations used to obtain it are vertices deletions and edge contractions, thus Ξ_k is a minor of G.

What we just proved is that for every graph G, if $\mathbf{tw}(G) \geqslant 3h(h-4) + 8$ then $\Xi_{h-1} \leqslant_{\mathrm{m}} G$. Since $H \leqslant_{\mathrm{m}} \Xi_{h-1}$ and by transitivity of the minor relation, we also have $\mathbf{tw}(G) \geqslant 3h(h-4)+8 \Rightarrow H \leqslant_{\mathrm{m}} G$, what we wanted to prove. ☐

8 Excluding a yurt graph

Definition 12 (yurt graph of order r)**.** Let $r > 0$ be an integer. In this paper, we call *yurt graph* of order r the graph Y_r of the form

$$V(Y_r) = \{x_1, \dots, x_r, y_1, \dots, y_r, o\}$$
$$E(Y_r) = \{(x_i, y_i)\}_{i \in [\![1,r]\!]} \cup \{(y_i, o)\}_{i \in [\![1,r]\!]}$$

(see Figure 4 for an example.)

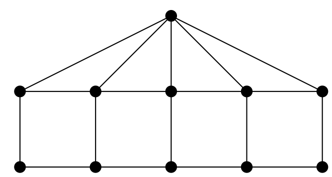

Figure 4: The yurt graph of order 5, Y_5

For every $r > 0$, we define the *comb of order* r as the tree made from the path $p_1 p_2 \dots p_r$ and the extra vertices $v_1, v_2, \dots, v_r$ by adding an edge between p_i and v_i for every $i \in [\![1, r]\!]$.

Theorem 8. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 6k^4 - 24k^3 + 48k^2 - 48k + 23$, then G contains Y_k as minor.*

Proof. Let $k > 0$ be an integer and G be a graph such that $\mathbf{tw}(G) \geqslant 6k^4 - 24k^3 + 48k^2 - 48k + 23$. Let C be the comb with $l = k^4 - 4k^3 + 8k^2 - 8k + 4$ teeth. As $\mathbf{tw}(G) \geqslant 3|V(C)| - 1$, G contains some graph of $\Lambda(C)$ by Lemma 2.

Let us prove that every graph of $\Lambda(C)$ contains the yurt graph of order k. Let H be a graph of $\Lambda(C)$. We respectively call T, P and o the tree, path and extra vertex of $\Lambda(C)$. Let F be the subset of edges between P and the leaves of T

Let $L = l_0, \dots, l_{k^2-2k+2}$ (resp. $Q = q_0, \dots, q_{k^2-2k+2}$) be the leaves of T (resp. of P)that are the end of an edge of F We assume without loss of generality that they appears in this order.

According to Proposition 4, there is a subsequence Q' of Q of length k such that the corresponding vertices L' of L appear in the same order. As one can easily see, this graph contains the yurt of order k and we are done. □

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