11th Annual European Symposium on Algorithms, ESA 2003, Springer Verlag, Lecture Notes in Computer Science, Vol. 2832, pp. 221 – 229, 2003.

Dominating sets and local treewidth*

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Abstract. It is known that the treewidth of a planar graph with a dominating set of size d is $O(\sqrt{d})$ and this fact is used as the basis for several fixed parameter algorithms on planar graphs. An interesting question motivating our study is if similar bounds can be obtained for larger minor closed graph families. We say that a graph family $\mathcal{F}$ has the domination-treewidth property if there is some function $f(d)$ such that every graph $G \in \mathcal{F}$ with dominating set of size $\leq d$ has treewidth $\leq f(d)$. We show that a minor-closed graph family $\mathcal{F}$ has the domination-treewidth property if and only if $\mathcal{F}$ has bounded local treewidth. This result has important algorithmic consequences.

1 Introduction

The last ten years has witnessed the of rapid development of a new branch of computational complexity: parameterized complexity (see the book of Downey & Fellows [9]). Roughly speaking, a parameterized problem with parameter k is *fixed parameter tractable* if it admits an algorithm with running time $f(k)|I|^{\beta}$. (Here f is a function depending only on k, $|I|$ is the length of the non parameterized part of the input and β is a constant.) Typically, $f(k) = c^k$ is an exponential function for some constant c.

A d-*dominating* set D of a graph G is a set of d vertices such that every vertex outside D is adjacent to a vertex of D. Fixed parameter version of the dominating set problem (the task is to compute, given a G and a positive integer d, a d-dominating set or to report that no such set exists) is one of the core problems in the Downey & Fellows theory. Dominating set is $W[2]$ complete and thus widely believed to be not fixed parameter tractable. However for planar graphs the situation is different and during the last five years a lot of work was done on fixed parameter algorithms for the dominating set problem on planar graphs and different generalizations of planar graphs. For planar graphs Downey

* The second author was supported by EC contract IST-1999-14186: Project ALCOM-FT (Algorithms and Complexity - Future Technologies and by the Spanish CICYT project TIC-2002-04498-C05-03 (TRACER).

and Fellows [9] suggested an algorithm with running time $O(11^d n)$. Later the running time was reduced to $O(8^d n)$ [2]. An algorithm with a sublinear exponent for the problem with running time $O(4^{6\sqrt{34d}}n)$ was given by Alber et al. [1]. Recently, Kanj & Perkovuć [16] improved the running time to $O(2^{27\sqrt{d}}n)$ and Fomin & Thilikos to $O(2^{15.13\sqrt{d}}d+n^3+d^4)$ [13]. The fixed parameter algorithms for extensions of planar graphs like bounded genus graphs and graphs excluding single-crossing graphs as minors are introduced in [10, 6].

The main technique to handle the dominating set problem which was exploited in several papers is that every graph G from a given graph family $\mathcal{F}$ with a domination set of size d has treewidth at most $f(d)$, where f is some function depending only on $\mathcal{F}$. With some work (sometimes very technical) a tree decomposition of width $O(f(d))$ is constructed and standard dynamic programming techniques on graphs of bounded treewidth are implemented. Of course this method can not be used for all graphs. For example, a complete graph K_n on n vertices has dominating set of size one and the treewidth of K_n is $n-1$. So the interesting question here is: Can this 'bounding treewidth method' be extended for larger minor-closed graph classes and what are the restrictions of these extensions?

In this paper we give a *complete* characterization of minor-closed graph families for which the 'bounding treewidth method' can be applied. More precisely, a minor-closed family $\mathcal{F}$ of graphs has the *domination-treewidth property* if there is some function $f(k)$ such that every graph $G \in \mathcal{F}$ with dominating set of size $\leq k$ has treewidth $\leq f(k)$. We prove that any minor-closed graph class has the domination-treewidth property if and only if it is of bounded local treewidth. Our proof is constructive and can be used for constructing fixed parameter algorithms for dominating set on minor-closed families of bounded local treewidth. The proof is based on Eppstein's characterization of minor-closed families of bounded local treewidth [11] and on a modification of the Robertson & Seymour excluded grid minor theorem due to Diestel et al.[8].

2 Definitions and preliminary results

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We let n denote the number of vertices of a graph when it is clear from context. For every nonempty $W \subseteq V(G)$, the subgraph of G induced by W is denoted by $G[W]$. We define the *r-neighborhood* of a vertex $v \in V(G)$, denoted by $N_G^r[v]$, to be the set of vertices of G at distance at most r from v. Notice that $v \in N_G^r[v]$. We put $N_G[v] = N_G^1[v]$. We also often say that a vertex v *dominates* subset $S \subset V(G)$ if $N_G[v] \supseteq S$.

Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a sequence of edge contractions is said to be a *contraction* of G. A graph H is a *minor* of a graph G if H is the subgraph of a contraction of G. We use the notation $H \preceq G$ (resp. $H \preceq_c G$) for H a minor (a contraction) of G.

The $m \times m$ grid is the graph on $\{1, 2, \ldots, m^2\}$ vertices $\{(i, j) \colon 1 \le i, j \le m\}$ with the edge set

$$\{(i, j)(i', j') \colon |i - i'| + |j - j'| = 1\}.$$

For $i \in \{1, 2, \ldots, m\}$ the vertex set (i, j), $j \in \{1, 2, \ldots, m\}$, is referred as the $i$throw and the vertex set (j, i), $j \in \{1, 2, \ldots, m\}$, is referred to as the ith column of the $m \times m$ grid.

The notion of treewidth was introduced by Robertson and Seymour [17]. A *tree decomposition* of a graph G is a pair $(\{X_i \mid i \in I\}, T = (I, F))$, with $\{X_i \mid i \in I\}$ a family of subsets of $V(G)$ and T a tree, such that

- $\bigcup_{i \in I} X_i = V(G)$.
- For all $\{v, w\} \in E(G)$, there is an $i \in I$ with $v, w \in X_i$.
- For all $i_0, i_1, i_2 \in I$: if i_1 is on the path from i_0 to i_2 in T, then $X_{i_0} \cap X_{i_2} \subseteq X_{i_1}$.

The *width* of the tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ is $\max_{i \in I} |X_i| - 1$. The treewidth $\mathbf{tw}(G)$ of a graph G is the minimum width of a tree decomposition of G.

We need the following facts about treewidth. The first fact is trivial.

- For any complete graph K_n on n vertices , $\mathbf{tw}(K_n) = n - 1$, and for any complete bipartite graph $K_{n,n}$, $\mathbf{tw}(K_{n,n}) = n$.

The second fact is well known but its proof is not trivial. (See e.g., [7].)

- The treewidth of the $m \times m$ grid is m.

A family of graphs $\mathcal{F}$ is minor-closed if $G \in \mathcal{F}$ implies that every minor of G is in $\mathcal{F}$. Graphs with the domination-treewidth property are the main issue of this paper. We say that a minor-closed family $\mathcal{F}$ of graphs has the *domination-treewidth property* if there is some function $f(d)$ such that every graph $G \in \mathcal{F}$ with dominating set of size $\le d$ has treewidth $\le f(d)$.

The next fact we need is the improved version of the Robertson & Seymour theorem on excluded grid minors [18] due to Diestel et al.[8]. (See also the textbook [7].)

Theorem 1 ([8]). *Let r, m be integers, and let G be a graph of treewidth at least $m^{4r^2(m+2)}$. Then G contains either K_r or the $m \times m$ grid as a minor.*

The notion of local treewidth was introduced by Eppstein [11] (see also [15]). The *local treewidth* of a graph G is

$$\mathbf{ltw}(G, r) = \max\{\mathbf{tw}(G[N_G^r[v]]) \colon v \in V(G)\}.$$

For a function $f \colon N \to N$ we define the minor closed class of graphs of bounded local treewidth

$$\mathcal{L}(f) = \{G \colon \forall H \preceq G \ \forall r \ge 0, \ \mathbf{ltw}(H, r) \le f(r)\}.$$

Also we say that a minor closed class of graphs $\mathcal{C}$ has bounded local treewidth if $\mathcal{C} \subseteq \mathcal{L}(f)$ for some function f.

Well known examples of minor closed classes of graphs of bounded local treewidth are planar graphs, graphs of bounded genus and graphs of bounded treewidth.

Many difficult graph problems can be solved efficiently when the input is restricted to graphs of bounded treewidth (see e.g., Bodlaender's survey [5]). Eppstein [11] made a step forward by proving that some problems like subgraph isomorphism and induced subgraph isomorphism can be solved in linear time on minor closed graphs of bounded local treewidth. Also the classical Baker's technique [4] for obtaining approximation schemes on planar graphs for different NP hard problems can be generalized to minor closed families of bounded local treewidth. (See [15] for a generalization of these techniques.)

An *apex graph* is a graph G such that for some vertex v (the *apex*), $G - v$ is planar. The following result is due to Eppstein [11].

Theorem 2 ([11]). *Let $\mathcal{F}$ be a minor-closed family of graphs. Then $\mathcal{F}$ is of bounded local treewidth if and only if $\mathcal{F}$ does not contain all apex graphs.*

3 Technical Lemma

In this section we prove the main technical lemma.

Lemma 1. *Let $G \in \mathcal{L}(f)$ be a graph containing the $m \times m$ grid H as a* subgraph*, $m > 2k^3$, where $k = 2f(2) + 2$. Then H contains the $(m/k^2 - 2k) \times (m - 2k)$ grid F as a subgraph such that for every vertex $v \in V(G)$, $|N_G[v] \cap V(F)| < k^2$, i.e. no vertex of G has $\geq k^2$ neighbors in F.*

Proof. We partition the grid H into k^2 subgraphs $H_1, H_2, \ldots, H_{k^2}$. Each subgraph H_i is the $m/k^2 \times m$ grid induced by columns $1 + (i-1)m/k^2, 2 + (i-1)m/k^2, \ldots, im/k^2$, $i \in \{1, 2, \ldots, k^2\}$. Every grid H_i contains *inner* and *outer* parts. Inner part $Inn(H_i)$ is the $(m/k^2 - 2k) \times (m - 2k)$ grid obtained from H_i by removing k outer rows and columns. (See Fig. 1.)

For the sake of contradiction, suppose that every grid $Inn(H_i)$ contains a set of vertices S_i of cardinality $\geq k^2$ dominated by some vertex of G. We claim that

$$\begin{aligned} &H \text{ contains as a contraction the } k \times k^2 \text{ grid } T \text{ such that in a graph} \\ &G_T \text{ obtained from } G \text{ by contracting } H \text{ to } T \text{ for every column } C \text{ of } T \\ &\text{there is a vertex } v \in V(G_T) \text{ such that } N_{G_T}[v] \supseteq C. \end{aligned} \quad (1)$$

Before proving (1) let us explain why this claim brings us to a contradiction. Let T be a grid satisfying (1). Suppose first that there is a vertex v of G_T that dominates (in G_T) all vertices of at least k columns of T. Then these columns are the columns of a $k \times k$ grid which is a contraction of T. Thus G_T can be contracted to a graph of diameter 2 containing the $k \times k$ grid as a subgraph. This contraction has treewidth $\geq k$.

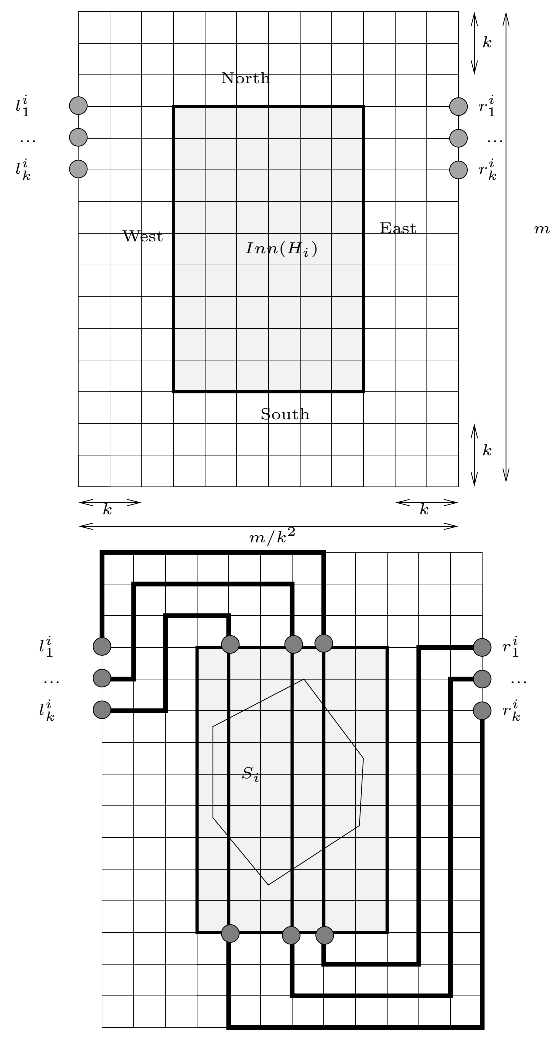

Fig. 1. Grid H_i and vertex disjoint paths connecting vertices $l_1^i, l_2^i, \ldots, l_k^i$ with $r_1^i, r_2^i, \ldots, r_k^i$.

If there is no such vertex v, then there is a set D of k vertices $v_1, v_2, \dots, v_k$ of G_T such that every vertex $v_i \in D$ dominates all vertices of some column of T. Let $v_1, v_2, \dots, v_l$, $l \leq k$, be the vertices of D that are in T. Then T contains as a subgraph the $k/2 \times k/2$ grid P such that at least $k - l/2 \geq k/2$ vertices of D are outside P. Let us call these vertices D'. Every vertex of D' is outside P and dominates some column of P. By contracting all columns of P into one column we obtain $k/2$ vertices and each of these $k/2$ vertices is adjacent to all vertices of D'. Thus G contains the complete bipartite graph $K_{k/2,k/2}$ as a minor. $K_{k/2,k/2}$ has diameter 2 and treewidth $k/2$. In both cases we have that G contains a minor of diameter ≤ 2 and of treewidth $\geq k/2 > f(2)$. Therefore $G \notin \mathcal{L}(f)$ which is a contradiction.

The remaining proof of the technical lemma is devoted to the proof of (1).

For every $i \in \{1, 2, \dots, k^2\}$, in the outer part of H_i we distinguish k vertices $l_1^i, l_2^i, \dots, l_k^i$ with coordinates $(k+1, 1), (k+2, 1), \dots, (2k, 1)$ and k vertices $r_1^i, r_2^i, \dots, r_k^i$ with coordinates $(k+1, m/k^2), (k+2, m/k^2), \dots, (2k, m/k^2)$. (See Fig. 1.)

We define *west (east) border* of $Inn(H_i)$ as the column of $Inn(H_i)$ which is the subcolumn of the $(k+1)$st $((m/k^2-k)$th) column of H_i. *North (south) border* of $Inn(H_i)$ is therow of $Inn(H_i)$ that is subrow of the $(k+1)$st $((m-k)$th)row in H_i

By assumption, every set S_i contains at least k^2 vertices in $Inn(H_i)$. Thus there are either k columns, or $k$rows of $Inn(H_i)$ such that each of these columns orrows has at least one vertex from S_i. This yields that there are k vertex disjoint paths either connecting north with south borders, or east with west borders and such that every path contains at least one vertex of S_i.

The subgraph of H_i induced by the first k columns and the first $k$rows is k-connected and by Menger's Theorem, for any k vertices of the west border of $Inn(H_i)$ (for any k vertices of the north border) there are k vertex disjoint paths connecting these vertices to the vertices $l_1^i, l_2^i, \dots, l_k^i$. By similar arguments any k vertices of the south border (east border) can be connected by k vertex disjoint paths with vertices $r_1^i, r_2^i, \dots, r_k^i$. (See Fig. 1.)

We conclude that for every $i \in \{1, 2, \dots, k^2\}$ there are k vertex disjoint paths in H_i with endpoints in $l_1^i, l_2^i, \dots, l_k^i$ and $r_1^i, r_2^i, \dots, r_k^i$ such that each path contains at least one vertex of S_i. Gluing these paths by adding edges (r_j^i, l_j^{i+1}), $i \in \{1, 2, \dots, k^2 - 1\}$, $j \in \{1, 2, \dots, k\}$, we construct k vertex disjoint paths $P_1, P_2, \dots, P_k$ in H such that for every $j \in \{1, 2, \dots, k\}$

- P_j contains vertices $l_j^1, r_j^1, l_j^2, r_j^2, \dots, l_j^{k^2}, r_j^{k^2}$,
- For every $i \in \{1, 2, \dots, k^2\}$ P_j contains a vertex from S_i.

The subgraph of G induced by the paths $P_1, P_2, \dots, P_k$ contains as a contraction a grid T satisfying (1). This grid can be obtained by contracting edges of P_j, $j \in \{1, 2, \dots, k\}$ in such way, that at least one vertex of S_i of the subpath of P_j between vertices l_j^i and r_j^i is mapped to l_j^i. This grid has k^2 columns and each of the k^2 columns of T is dominated by some vertex of G_T. This concludes the proof of (1) and the lemma follows.

Corollary 1. *Let $G \in \mathcal{L}(f)$ be a graph containing the $m \times m$, $m > 2k^3$, where $k = 2f(2) + 2$, grid H as a minor. Then every dominating set of G is of size $> \frac{m^2}{k^4}$.*

Proof. Assume that G has a dominating set of size d. G contains as a contraction a graph G' such that G' contains H as a subgraph. Notice that G' also has a dominating set of size d. By Lemma 1, H contains the $(m/k^2 - 2k) \times (m - 2k)$ grid F as a subgraph such that no vertex of G' has $\geq k^2$ neighbors in F. Thus

$$d \geq \frac{(m/k^2 - 2k) \times (m - 2k)}{k^2 + 1} > \frac{m^2}{k^4}.$$

4 Main theorem

Theorem 3. *Let $\mathcal{F}$ be a minor-closed family of graphs. Then $\mathcal{F}$ has the domination-treewidth property if and only if $\mathcal{F}$ is of bounded local treewidth.*

Proof. In one direction the proof follows from Theorem 2. The apex graphs A_i, $i = 1, 2, 3, \ldots$ obtained from the $i \times i$ grid by adding a vertex v adjacent to all vertices of the grid have a dominating set of size 1, diameter ≤ 2 and treewidth $\geq i$. So a minor closed family of graphs with domination-treewidth property cannot contain all apex graphs and hence it is of bounded local treewidth.

In the opposite direction the proof follows from the following claim

Claim. For any function $f\colon N \to N$ and any graph $G \in \mathcal{L}(f)$ with dominating set of size d, we have that $\mathbf{tw}(G) = 2^{O(\sqrt{d} \log d)}$.

Let $G \in \mathcal{L}(f)$ be a graph of treewidth $m^{4r^2(m+2)}$ and with dominating set of size d. Let $r = f(1) + 2$ and $k = 2f(2) + 2$. Then G has no complete graph K_r as a minor. By Theorem 1, G contains the $m \times m$ grid H as a minor and by Corollary 1 $d \geq \frac{m^2}{k^4}$. Since k and r are constants depending only on f, we conclude that $m = O(\sqrt{d})$ and the claim and thus the theorem follows.

5 Algorithmic consequences and concluding remarks

By general results of Frick & Grohe [14] the dominating set problem is fixed parameter tractable on minor-closed graph families of bounded local treewidth. However Frick & Grohe's proof is not constructive. It uses a transformation of first-order logic formulas into a 'local formula' according to Gaifman's theorem and even the complexity of this transformation is unknown.

Theorem 3 yields a constructive proof of the fact that the dominating set problem is fixed parameter tractable on minor-closed graph families of bounded local treewidth. It implies a fixed parameter algorithm that can be constructed as follows.

Let G be a graph from $\mathcal{L}(f)$. We want to check if G has a dominating set of size d. We put $r = f(1) + 2$ and $k = 2f(2) + 2$. First we check if the treewidth of

G is at most $(\sqrt{d}k^2)^{4r^2(\sqrt{d}k^2+2)}$. This step can be performed by Amir's algorithm [3], which for a given graph G and integer ω, either reports that the treewidth of G is at least ω, or produces a tree decomposition of width at most $3\frac{2}{3}\omega$ in time $O(2^{3.698\omega}n^3\omega^3\log^4 n)$. Thus by using Amir's algorithm we can either compute a tree decomposition of G of size $2^{O(\sqrt{d}\log d)}$ in time $2^{2^{O(\sqrt{d}\log d)}}n^{3+\epsilon}$, or conclude that the treewidth of G is more than $(\sqrt{d}k^2)^{4r^2(\sqrt{d}k^2+2)}$.

- If the algorithm reports that $\mathbf{tw}(G) > (\sqrt{d}k^2)^{4r^2(\sqrt{d}k^2+2)}$ then by Theorem 1 (G contains no K_r), G contains the $\sqrt{d}k^2 \times \sqrt{d}k^2$ grid as a minor. Then Corollary 1 implies that G has no dominating set of size d.
- Otherwise we perform a standard dynamic programming to compute dominating set. It is well known that the dominating set of a graph with a given tree decomposition of width at most ω can be computed in time $O(2^{2\omega}n)$ [1]. Thus this step can be implemented in time $2^{2^{O(\sqrt{d}\log d)}}n$.

We conclude with the following theorem.

Theorem 4. *There is an algorithm such that, for every minor-closed family $\mathcal{F}$ of bounded local treewidth and a graph $G \in \mathcal{F}$ on n vertices and an integer d, either computes a dominating set of size $\leq d$, or concludes that there is no such a dominating set. The running time of the algorithm is $2^{2^{O(\sqrt{d}\log d)}}n^{O(1)}$.*

Finally, some questions. For planar graphs and for some extensions it is known that for any graph G from this class with dominating set of size $\leq d$, we have $\mathbf{tw}(G) = O(\sqrt{d})$. It is tempting to ask if the same holds for all minor-closed families of bounded local treewidth. This will provide subexponential fixed parameter algorithms on graphs of bounded local treewidth for the dominating set problem.

Another interesting and prominent graph class is the class of graphs containing no minor isomorphic to some fixed graph H. Recently Flum & Grohe [12] showed that parameterized versions of the dominating set problem is fixed-parameter tractable when restricted to graph classes with an excluded minor. Our result shows that the technique based on the dominating-treewidth property can not be used for obtaining constructive algorithms for the dominating set problem on excluded minor graph families. So constructing fast fixed parameter algorithms for these graph classes requires fresh ideas and is an interesting challenge.

Addendum

Recently we were informed (personal communication) that a result similar to the one of this paper was also derived independently (with a different proof) by Erik Demaine and MohammadTaghi Hajiaghayi.

Acknowledgement

The last author is grateful to Maria Satratzemi for technically supporting his research at the Department of Applied Informatics of the University of Macedonia, Thessaloniki, Greece.

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