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Algorithms and obstructions for linear-width and related search parameters ☆

Dimitrios M. Thilikos

Departament de Llenguatges i Sistemes Informàtics, Universitat Politècnica de Catalunya, Campus Nord – Mòdul C5 – Desp. 211b, c/Jordi Girona Salgado, 1-3. E-08034, Barcelona, Spain

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Abstract

The *linear-width* of a graph G is defined to be the smallest integer k such that the edges of G can be arranged in a linear ordering $(e_1,\ldots,e_r)$ in such a way that for every $i=1,\ldots,r-1$, there are at most k vertices incident to edges that belong both to $\{e_1,\ldots,e_i\}$ and to $\{e_{i+1},\ldots,e_r\}$. In this paper, we give a set of 57 graphs and prove that it is the set of the minimal forbidden minors for the class of graphs with linear-width at most two. Our proof also gives a linear time algorithm that either reports that a given graph has linear-width more than two or outputs an edge ordering of minimum linear-width. We further prove a structural connection between linear-width and the mixed search number which enables us to determine, for any $k \geqslant 1$, the set of acyclic forbidden minors for the class of graphs with linear-width $\leqslant k$. Moreover, due to this connection, our algorithm can be transfered to two linear time algorithms that check whether a graph has mixed search or edge search number at most two and, if so, construct the corresponding sequences of search moves.

Keywords: Graph algorithm; Linear width; Graph minor; Obstruction set; Graph searching

1. Introduction

A *graph parameter* is a function which maps each graph to a positive integer. Given a graph parameter f and a positive integer k, we denote as $\mathscr{G}[f,k]$ the class of graphs for which the value of f does not exceed k.

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E-mail address: sedthilk@lsi.upc.es (D.M. Thilikos).

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Let $\mathscr{G}$ be a class of graphs. We say that $\mathscr{G}$ is *closed under taking of minors* if all the minors of graphs in $\mathscr{G}$ belong also in $\mathscr{G}$ (we say that a graph H is a minor of a graph G if it can be obtained from G after a number of vertex/edge removal and/or edge contractions — for the formal definitions, see Section 2.1). We also say that a graph parameter f is *closed under taking of minors* if, for every k, $\mathscr{G}[f,k]$ is closed under taking of minors.

The *obstruction set* of a graph class $\mathscr{G}$ — namely $\mathrm{ob}(\mathscr{G})$ — is defined to be the set of the minor minimal graphs that do not belong in $\mathscr{G}$. According to the result of Robertson and Seymour in their Graphs Minors series of papers (see [31] for a survey), the minor minimal elements of any graph class are finite. It follows that if a graph class $\mathscr{G}$ is closed under taking of minors then, for any graph G, $G \in \mathscr{G}$ iff none of the graphs in $\mathrm{ob}(\mathscr{G})$ is a minor of G. In the same series of papers, Robertson and Seymour prove that there exists an $\mathrm{O}(n^3)$ time algorithm checking if a given n-vertex graph G contains a fixed graph H as a minor [32–34]. A quite important consequence of that is that for any graph class that is closed under taking of minors there exists a polynomial time membership checking algorithm. Moreover, according to the result of Bodlaender in [5], this membership check can be done in linear time if some excluded minor is planar (see also [4,16]).

Many interesting graph classes/parameters have been proved to be closed under taking of minors. Unfortunately, the membership algorithm we mentioned above presumes the knowledge of the obstruction set. As there exists no general method to find the obstruction set of a graph class (see [17,18]), the research on this topic has been oriented to the specification of the obstruction set of individual graph classes (see [2,15,17,22,26,29]). Clearly, given a graph parameter f that is closed under taking of minors, each value of k corresponds to a different obstruction set, i.e. $\mathrm{ob}(\mathscr{G}[f,k])$. To our knowledge, obstruction sets have been found for the following graph parameters: treewidth, for $k \leqslant 3$ (see [1,20,35]), branchwidth, for $k \leqslant 3$ (see [8]), pathwidth, for $k \leqslant 2$ (see [22,23]), and mixed search number, for $k \leqslant 2$ (see [38]).

The *linear-width* of a graph G is defined to be the least integer k such that the edges of G can be arranged in a linear ordering $(e_1,\dots,e_r)$ in such a way that for every $i = 1,\dots,r-1$, there are at most k vertices incident to edges that belong both to $\{e_1,\dots,e_i\}$ and to $\{e_{i+1},\dots,e_r\}$. Linear-width was first mentioned by Thomas [40] and is strongly connected with the notion of *crusades* introduced by Bienstock and Seymour in [3]. In this paper we prove that several variants of problems appearing on graph searching can be reduced to the problem of computing linear-width.

In a graph searching game a graph represents a system of tunnels where an agile, fast, and invisible fugitive is resorting. We desire to capture this fugitive by applying a search strategy while using the fewest possible searchers. In short, the search number of a graph is the minimum number of searchers a searching strategy requires in order to capture the fugitive. Several variations on the way the fugitive can be captured during a search, define the parameters of the *edge*, *node*, and *mixed search number* of a graph (namely, $\mathrm{es}(G)$, $\mathrm{ns}(G)$, and $\mathrm{ms}(G)$). The first graph searching game was introduced by Breisch [9] and Parsons [30] and is the one of *edge searching*. *Node searching* appeared as a variant of edge searching and was introduced by Kirousis and Papadimitriou in [25]. Finally, *mixed searching* was introduced in [3,39] and is a natural generalization of the two previous variants (for the formal definitions see Section 5.1 — for analogues versions of the searching game without the agility requirement see [13,36]).

The problem of computing $\mathrm{es}(G), \mathrm{ns}(G), \mathrm{ms}(G)$, or linear-width$(G)$ is NP-complete (see [25,27,39] and Theorem 25(i) of this paper). On the other hand, since all of these parameters are closed under taking of minors, we know that there exists a linear-time algorithm checking membership in $\mathscr{G}[f,k]$ where f is ms, es, ns, or linear-width. Such a linear-time algorithm has been constructed for the node search number [5,6] (actually, the results in [5,6] concerns the parameter of pathwidth which is known to be equal to the node search number minus one – see [21,24,28]). Recently, a linear-time algorithm, checking if a graph belongs to $\mathscr{G}[\text{linear-width}, k]$, was found (see [7]). Moreover, the algorithm in [7] is constructive: for any fixed k, one can construct an optimal edge arrangement, if exists. On the other hand, the algorithm in [7] appears to be difficult to implement and rather impractical, even for small values of k, as the contribution of the fixed k on the "hidden" part of their linear-time complexity is heavily exponential.

In order to overcome the above problems one needs practical "tailor-made" algorithms for specific (usually small) values of k. Mainly, such kinds of algorithms are based on a complete structural characterization of the corresponding graph class. In this direction, an algorithm for the class of graphs with node search number $\leqslant 3$ has been given in [12] (actually the algorithm in [12] concerns graphs with pathwidth $\leqslant 2$ but can be easily transfered to the class of graphs with node search number $\leqslant 3$). However, no "tailor-made" algorithms for the linear-width, the mixed search number, or the edge search number are known.

In this paper we give a linear-time algorithm that checks if a graph has linear-width $\leqslant 2$ and, if so, outputs an edge ordering with optimal linear-width. Moreover, we prove a structural connection between linear-width and the three search parameters we mentioned before (this connection generalizes the one proved in [3]). According to this result, our algorithm can be directly modified to one that checks whether the mixed or the edge search number of a graph is at most 2 and, if so, outputs an optimal search.

Our algorithm is based on a complete structural characterization of the class of graphs with linear-width $\leqslant 2$. Using this characterization, we prove that $\mathrm{ob}(\mathscr{G}[\text{linear-width}, 2])$ consists of the 57 graphs depicted in the appendix (Fig. 13). Moreover, we prove that, for any k, there exists an injection from $\mathrm{ob}(\mathscr{G}[\mathrm{ms}, k])$ to $\mathrm{ob}(\mathscr{G}[\text{linear-width}, k])$. A direct consequence is that $\mathrm{ob}(\mathscr{G}[\mathrm{ms}, k])$ can be easily determined if we know $\mathrm{ob}(\mathscr{G}[\text{linear-width}, k])$. Applying this result for the case where $k=2$ we can determine $\mathrm{ob}(\mathscr{G}[\mathrm{ms}, 2])$ and, in that way, reproduce the result of [38].

Finally, for any k, we determine all the trees in $\mathrm{ob}(\mathscr{G}[\text{linear-width}, k])$. More specifically, we prove that, for any k, there exists a bijection between the trees in $\mathrm{ob}(\mathscr{G}[\text{linear-width}, k])$ and the trees in $\mathrm{ob}(\mathscr{G}[\mathrm{ms}, k])$. Our results indicate that, for $k > 2$, a complete structural characterization of the class of graphs with linear-width $\leqslant k$ is rather hard to find, even for small values of k.

The paper is organized as follows. In Section 2 we give some basic definitions and results concerning the structure of the graphs with linear-width $\leqslant 2$. In Section 3 we present the main algorithm of this paper. In Section 4 we prove the correctness of the algorithm and the obstruction set. Section 5 is devoted to the relation between linear-width and the three variants of the graph searching game. Finally, in Section 6, we end up with some conclusions and open problems.

2. Definitions and preliminary results

We consider finite undirected graphs without loops or multiple edges unless otherwise is mentioned.

Let G be a graph. If $S \subseteq V(G)$, we call the graph $(S, \{\{v,u\} \in E(G)\colon v,u \in S\})$ the *subgraph of G induced by S* and we denote it as $G[S]$. Given two graphs G_1, G_2 we set $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. Also, if $\mathscr{G} = \{G_1, \ldots, G_r\}$ is a set of graphs we define $\cup\mathscr{G} = G_1 \cup \cdots \cup G_r$. We denote as K_r the complete graph with r vertices and as $K_{q,r}$ the complete bipartite graph with parts consisting of q and r vertices each. Given a vertex $v \in V(G)$, we denote as $N_G(v)$ the vertices of G that are adjacent to v. We also define $G - v = G[V(G) - \{v\}]$. If $e \in E(G)$ we set $G - e = (V(G), E(G) - \{e\})$. A *contraction* of an edge $\{u,v\}$ of G to v is the operation that removes u and makes v adjacent to $N_G(u) - N_G(v) - \{v\}$. We denote the result of the contraction of e by $G \dot{-} e$. For any edge set $E \subseteq E(G)$ we denote by $V(E)$ the set of vertices that are incident to edges of H. We call $|N_G(v)|$ *degree* of a vertex v with respect to some graph G and we denote it by $d_G(v)$. We also denote by $A(G)$ the set of articulation vertices of G (i.e. $A(G) = \{v \in V(G) \mid G - v$ contains connected components than $G\}$).

We call a subgraph G' of G *pendant path* if $G' = (\{v_1, \ldots, v_r\}, \{\{v_1, v_2\}, \ldots, \{v_{r-1}, v_r\}\})$, $r \geqslant 2$, $d_G(v_1) \neq 2$, for $i = 2, \ldots, r-1$ $d_G(v_i) = 2$, and $d_G(v_r) = 1$. We call a vertex *pendant* if it has degree 1. We call an edge *pendant* if it contains a pendant vertex. We call a pendant vertex *fully pendant* if it is adjacent to a vertex of degree equal to 2, otherwise we call it *simply pendant*. We call a pendant edge *fully* (*simply*) *pendant* if one of its endpoints is a fully (simply) pendant vertex. We call a vertex *almost pendant* if it is adjacent to a fully pendant vertex. We call an edge *almost pendant* if it is not pendant and one of its endpoints is *almost pendant*. We denote by $A^*(G)$ the vertices of $A(G)$ that are not almost pendant vertices. Finally, we call $e = \{v,u\}$ *redundant* if $d_G(v) = d_G(u) = 2$. For an example of the given definitions see Fig. 1.

2.1. Minors: proper and rooted

We say that H is a *minor* of G (denoted by $H \preccurlyeq G$) if H can be obtained by a series of the following operations: vertex deletions, edge deletions, and edge contractions. We say that H is a proper *minor* of G (denoted by $H \prec G$) if $H \preccurlyeq G$ and H is not isomorphic to G. If $\mathscr{H}$ is a set of graphs containing at least one minor of G, then

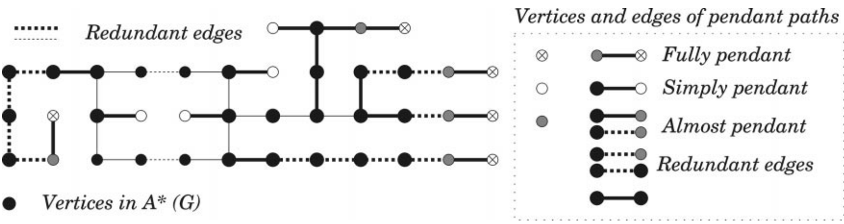

Fig. 1. An example for the notions of fully/simply/almost pendant vertices/edges.

we denote it by $\mathscr{H} \sqsubseteq G$. If no element of $\mathscr{H}$ is a minor of G then we denote it by $\mathscr{H} \not\sqsubseteq G$. Suppose that in some graph G we distinguish some vertex v. We call this graph v-rooted or, simply, v-graph (we also call v *root* of G). A pendant path rooted on its first vertex v_1 is called v_1-*pendant path*. Any v-graph H that can be obtained from a v-rooted graph G after a sequence of vertex/edge deletions and/or edge contractions that do not remove v is called v-*minor* of G and we denote it by $H \preccurlyeq_v G$ (from now on, whenever we mention a contraction in a rooted graph we will assume that the removed vertex is different than its root). Analogously to the non-rooted case, we define the relation "$\sqsubseteq_v$".

2.2. Linear-width

We define linear-width as follows. Let G be a graph and $l=(e_1,\ldots,e_{|E(G)|})$ be a linear ordering of $E(G)$. Let $\delta_l(e_i)=V(\{e_1,\ldots,e_i\})\cap V(\{e_{i+1},\ldots,e_{|E(G)|}\})$ (i.e. $\delta_l(e_i)$ is the set of vertices in $V(G)$ that are incident to an edge in $\{e_1,\ldots,e_i\}$ and also to an edge in $\{e_{i+1},\ldots,e_{|E(G)|}\}$. We set linear-width$(l)=\max_{1\leqslant i\leqslant |E(G)|-1}\{|\delta_l(e_i)|\}$. The linear-width of a graph is the minimum linear-width over all the orderings of $E(G)$ (if $|E(G)|\leqslant 1$ then linear-width(G)=0). If $l=(e_1,\ldots,e_{|E(G)|})$, we set $l^{-1}=(e_{|E(G)|},\ldots,e_1)$. Clearly, linear-width$(l)$ = linear-width(l^{-1}). If $\{E_i, i=1,\ldots,r\}$ is a partition of $E(G)$ and, for $i=1,\ldots,r$, $l_i=(e^i_1,\ldots,e^i_{|E_i|})$ is an edge ordering of E_i, we define $l_1\oplus l_2\oplus\cdots\oplus l_r=(e^1_1,\ldots,e^1_{|E_1|},e^2_1,\ldots,e^2_{|E_2|},\ldots,e^r_1,\ldots,e^r_{|E_r|})$, i.e. $l_1\oplus l_2\oplus\cdots\oplus l_r$ is the concatenation of $l_1,\ldots,l_r$.

Lemma 1. *The class of graphs with bounded* linear-width *is closed under taking of minors.*

Proof. Let G be a graph having an edge ordering l with linear-width equal to k. It is enough to prove that for any $v\in V(G), e\in E(G)$, graphs $G-v, G-e, G\dot{-}e$ have linear-width $\leqslant k$. Let l be an edge ordering of G. Clearly, the removal of an edge (or a set of edges) from l cannot increase its linear-width. Using this fact, it is straightforward to construct an edge ordering of $G-v$ or $G-e$ with linear-width $\leqslant k$. Suppose now that $G'=G\dot{-}e$. Let $e=\{v,u\}$ and assume that the contraction removes u. We now remove edge e from l and then replace u with v in any edge containing u (if during this operation an edge appears that is currently present in the ordering, then we remove it). It remains now to observe that the linear-width of the new ordering is no more than k. □

The following easy lemma will allow us to consider only connected graphs in the rest of this paper.

Lemma 2. *The* linear-width *of a graph is equal to the maximum* linear-width *of its connected components.*

Notice that a consequence of Lemma 2 is that, for every k, the graphs in ob($\mathscr{G}$[linear-width, k]) are all connected.

We denote by $\mathscr{L}_2$ the set consisting of the graphs depicted in the appendix (Fig. 13). The following lemma is a direct consequence of Lemma 1 and the fact that all the graphs in $\mathscr{L}_2$ have linear-width more than two.

Lemma 3. *Let G be a graph that $\mathscr{L}_2 \sqsubseteq G$. Then,* linear-width$(G) > 2$.

In the next two sections, we will prove that $\mathscr{L}_2$ is the minor minimal set of the graphs that do not belong in $\mathscr{G}$[linear-width, 2] and therefore $\mathscr{L}_2$ is the obstruction set for the class of graphs with linear-width $\leqslant 2$.

2.3. Reducing graphs to simpler ones

We will first prove a series of lemmata, enabling us to restrict our study to more simple graphs. Analogous lemmata for pathwidth and the mixed search number have been proved in [14,22,23] and [38], respectively.

Lemma 4. *Let H be a graph with* linear-width $\leqslant k$. *The following hold.*

(i) *Let v, v' be vertices such that $v \in V(H), d_H(v) \geqslant 2$, and $v' \notin V(H)$. If $H' = (V(H) \cup \{v'\}, E(H) \cup \{\{v, v'\}\})$, then* linear-width$(H') \leqslant k$ *(notice that v' is a simply pendant vertex of H').*

(ii) *Let v be a vertex that is adjacent only with vertices w and u in H and $u' \notin V(H)$. Let also $H' = (V(H) \cup \{u'\}, E(H) \cup \{\{u, u'\}, \{u', v\}\} - \{\{v, u\}\})$. Then,* linear-width$(H') \leqslant k$ *(notice that $\{u', v\}$ is a redundant edge of H').*

Proof. Let $l = (e_1, \ldots, e_r)$ be an edge ordering of H where linear-width$(l) = k$.

(i) Notice that, as $d_H(v) \geqslant 2$, l contains at least one edge e_i with $v \in \delta_l(e_i)$. It is now easy to see that $l' = (e_1, \ldots, e_i, \{v, v'\}, e_{i+1}, \ldots, e_r)$ is an edge ordering of G' with linear-width $\leqslant k$.

(ii) W.l.o.g. we assume that $\{v, u\}$ comes before $\{w, v\}$ in l (if not, we choose l^{-1}). Let also $e_i = \{v, u\}, 1 \leqslant i \leqslant r$. Now observe that $l' = (e_1, \ldots, e_{i-1}, \{u, u'\}, \{u', v\}, e_{i+1}, \ldots, e_r)$ is an edge ordering of G' with linear-width $\leqslant k$. □

Lemma 5. *Let G be a graph. Then, there exists a graph G' such that $G' \preccurlyeq G$, G' does not contain any redundant or simply pendant edges and* linear-width$(G) =$ linear-width(G'). *Moreover, if l' is an edge ordering of G' with* linear-width $\leqslant k$, *one can construct an edge ordering of G with* linear-width $\leqslant k$ *in* $O(|E(G)|)$ *time.*

Proof. Let G' be the graph that is obtained if we apply the following operation on G as long as this is possible:

- If e is a redundant or a simply pendant edge in G, then set $G \leftarrow G \dot{-} e$.

Clearly, if we have an edge ordering of G' with linear-width at most k, we can construct an edge ordering of G with linear-width at most k undoing the above sequence of contractions. Since we need $O(1)$ time for each contraction, the rebuilding process needs $O(|E(G)|)$ time. What remains is to prove that linear-width$(G) \leqslant k \Leftrightarrow$ linear-width$(G') \leqslant k$. We examine the nontrivial case where $|E(G)| \geqslant 2$. The "$\Rightarrow$" direction follows immediately from Lemma 1. The "$\Leftarrow$" direction follows if we apply inductively Lemma 4 on the number of the edges contracted. □

A graph G is *outerplanar* if it can be embedded in the plane such that all vertices are incident to one of its faces. It is known that a graph is outerplanar iff it does not contain K_4 or $K_{2,3}$ as a minor (e.g. see [11]). Moreover, for any outerplanar graph G, $|E(G)| = O(|V(G)|)$ (see [19, p. 107]). Using standard techniques, one can easily construct an algorithm that checks, in $O(|V(G)|)$ time, whether a graph G is outerplanar or not and, if so, outputs an outerplanar embedding of G (see also [10,19]). Clearly, as both K_4 and $K_{2,3}$ belong in ob($\mathscr{G}$[linear-width, 2]), every graph with linear-width $\leqslant 2$ must be outerplanar and this is a first (partial) characterization of the graphs in $\mathscr{G}$[linear-width,2]. We can now observe that if, linear-width$(G) \leqslant 2$, then $|E(G)| = O(|V(G)|)$.

We call the edges of the outer face of G *outer* edges and all the others *inner*. We denote the set of the outer (inner) edges of an outerplanar graph as out(G) (inn(G)). An outer edge $\{x, y\}$ is *weak* if none of its endpoints is an articulation vertex and there exists a vertex z such that if $E = \{\{z,x\}, \{z, y\}\}$ then inn$(G) \cap E \neq \emptyset$ and out$(G) \cap E = \emptyset$, i.e. E contains some inner edge of G and z is not adjacent to x or y through an outer edge (notice that E can contain an edge that is not necessarily an edge of G). As an example, notice that all the "fat" edges of the graphs depicted in Fig. 2 are weak.

Lemma 6. *Let e be a weak edge of an outerplanar graph G. Then* linear-width$(G \dot{-} e) \leqslant 2 \Rightarrow$ linear-width$(G) \leqslant 2$.

Proof. Let $e = \{x, y\}$ and suppose that $\{x,z\}$ is an inner edge of G. Let H be the result of the contraction of e to x. We observe that $\{x,z\}$ is an inner edge of H. Let B be the unique biconnected component of H that contains $\{x,z\}$ as an edge and set $e' = \{x,z\}$. Let C_1, C_2 be the two connected components of the graph occurring if we remove from B the endpoints of e'. Let $D_i = B[V(C_i) \cup \{x\} \cup \{z\}] - e', i = 1,2$ (see also Fig. 2). Suppose that $l = (e_1, \ldots, e_i, \ldots, e_r)$ is an edge ordering of H where

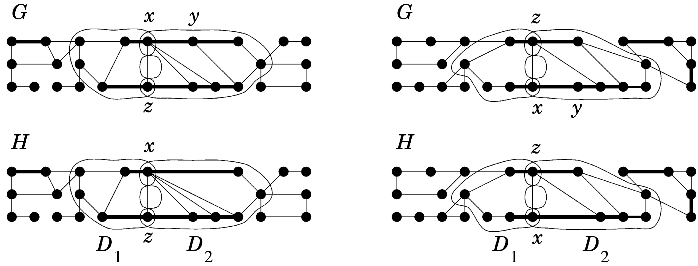

Fig. 2. Two examples of contractions of weak edges.

$e_i = e'$ and linear-width$(l) = 2$ (notice that linear-width$(G') < 2$) is impossible. Let e_j be the first edge of B appearing in l. Clearly $j < i$ because, otherwise, $|\delta_l(e_{j+1})| \geqslant 3$. W.l.o.g. we assume that $e_j \in E(D_1)$. Let e_h be the first edge of D_2 appearing in l. We claim that $h > i$. Suppose on the contrary that e_i comes after e_h in l. Notice that $\{e_j, \ldots, e_{h-1}\} \subseteq E(D_1)$, and thus $|\delta_l(e_{h-1})| \geqslant 2$. Moreover, as e_h is the first edge of D_2 in l we have that $\delta_l(e_h)$ will contain all the vertices in $\delta_l(e_{h-1})$ and at least one more from $V(C_2)$. Therefore, $|\delta_l(e_h)| \geqslant 3$, a contradiction. Now let e_k be the last edge in D_1 appearing in l. Applying the same arguments on l^{-1} (e_k is the first edge in D_1 appearing in l^{-1}), we can prove that $k < i$. We easily conclude that $k = i-1$, $h = i+1$ and thus $\delta_l(e_{i-1}) = \delta_l(e_i) = \{x, z\}$. We now set $l_1 = (e_1, \ldots, e_{i-1})$, and $l_3 = (e_{i+1}, \ldots, e_r)$. Moreover, if $\{y, z\} \in E(G)$, we set $l_2 = (\{x, z\}, \{x, y\}, \{y, z\})$, otherwise we set $l_2 = (\{x, z\}, \{x, y\})$. Observe that $l_1 \oplus l_2 \oplus l_3$ is an edge ordering of G with linear-width=2 and this completes the proof of the lemma. □

We point out that Lemma 6 holds even if we allow the endpoints of a weak edge to be articulation vertices. We choose to retain this requirement as this will facilitate the presentation of the next sections of this paper.

Lemma 7. *Let G be an outerplanar graph. Then, there exists a graph G' such that $G' \preccurlyeq G$, G' does not contain any weak edge, and* linear-width$(G) \leqslant 2 \Leftrightarrow$ linear-width$(G') \leqslant 2$. *Moreover, if l' is an edge ordering of G' with* linear-width $\leqslant 2$, *one can construct an edge ordering of G with* linear-width $\leqslant 2$ *in* $O(|V(E)|)$ *time.*

Proof. The proof is similar to the one of Lemma 5 with the difference that we now apply inductively Lemma 6 (in this case G' is constructed if we perform contractions of weak edges as long as this is possible). □

2.4. Bolbes, wings, and smooth graphs

Let G be an outerplanar graph. We will denote a face F of a planar embedding of G as the graph induced by the vertices that are incident to F (certainly, such a graph is always a cycle). For two vertices x, y, we say that $x \sim y$ if $\{x, y\} \in \text{out}(G)$. We

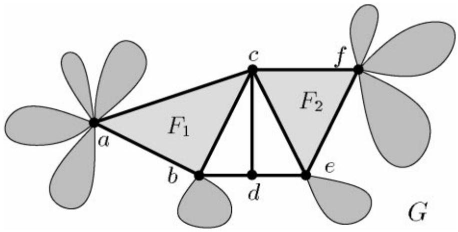

Fig. 3. $B = G[\{a,b,c,d,e,f\}]$ is a bolbe of an outerplanar graph G. The outer edges of B are $\{a,c\},\{c,f\},\{f,e\},\{e,d\},\{d,b\},\{b,a\}$. The inner edges of B are $\{b,c\},\{d,c\},\{e,c\}$. B contains two polar faces F_1,F_2 where $F_1=B[\{a,b,c\}]$ and $F_2=B[\{c,e,f\}]$. F_1 (F_2) contains only a (f) as polar vertex and the polar edges of F_1 (F_2) are $\{a,b\},\{a,c\}$ ($\{c,f\},\{e,f\}$). The critical vertices of F_1 (F_2) are b,c (c,e) and the critical edge of F_1 (F_2) is $\{b,c\}$ ($\{c,e\}$).

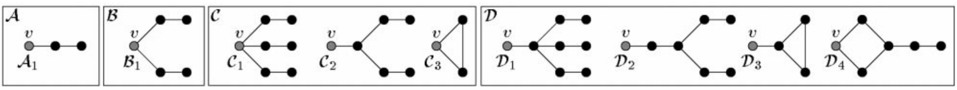

Fig. 4. The classes of rooted graphs $\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D}$.

call a face F *polar* if it contains at most one inner edge. Let F be a polar face. The edges F that belong to out(G) are called *polar*. The vertices of F that are not incident to the unique edge of F that is in inn(G) are called *polar*. If an edge (vertex) of F is not polar then we call it *critical*. (if G is a cycle or a single edge, all its edges are polar and outer). The set of polar faces of G is denoted by $\mathscr{S}(G)$. We say that a biconnected component of an outerplanar graph G is a *bolbe* if it does not consist of a pendant or an almost pendant edge. We denote as $\mathscr{B}(G)$ the set of all the bolbes of G. For an example of the given definitions see Fig. 3.

Lemma 8. *Let G be an outerplanar graph and B a bolbe of G containing more than two polar faces. Then,* $4K_3 \preccurlyeq G$.

Proof. Suppose now that G contains a bolbe B that has at least three polar faces F_1,F_2, and F_3. Then, if we first remove from G all the vertices in $V(G)-V(B)$, then contract all the edges not in $E(F_1)\cup E(F_2)\cup E(F_3)$, and, finally, contract all the redundant edges, we obtain $4K_3$. □

Lemma 8 is our second step towards describing the structure of the graphs with linear-width $\leqslant 2$. We now know that they are outerplanar and each of their bolbes contains at most two polar faces.

A v-graph G is called v-*wing* if $\mathscr{D} \sqsubseteq_v G$ (graphs in $\mathscr{D}$ are depicted in Fig. 4). Let G be a graph. If $\{C_1^v,\ldots,C_\rho^v\}$ is the set of the connected components of $G[V(G)-\{v\}]$, we set $\mathscr{X}(G,v)=\{X_1^v,\ldots,X_\rho^v\}$ where $X_i^v = G[V(C_i^v)\cup\{v\}], 1\leqslant i\leqslant \rho$. For any vertex $v\in V(G)$, we define $\alpha(G,v)$ as the number of v-wings in $\mathscr{X}(G,v)$.

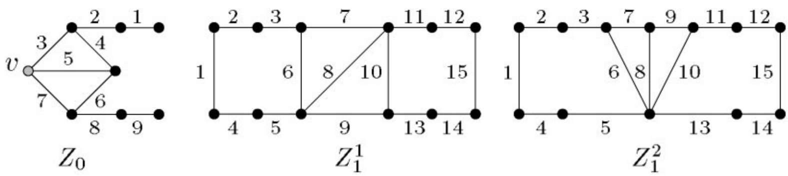

Fig. 5. The graphs Z_0, Z_1^1, Z_1^2.

Lemma 9. *Let G be a graph containing a vertex v such that $\alpha(G,v)>2$. Then, $\mathscr{L}_2^1 \sqsubseteq G$. (The graphs in $\mathscr{L}_2^1$ are depicted in Fig.* 13 *of the appendix.)*

Proof. Let $v \in V(G)$ such that $\mathscr{X}(G,v)$ contains at least three v-wings W_1, W_2, W_3. This means that $\mathscr{D} \sqsubseteq_v W_i, i=1,2,3$. It is now enough to observe that $W_1 \cup W_2 \cup W_3$ is a subgraph of G containing one of the graphs in $\mathscr{L}_2^1$ as a minor. □

Certainly, the above lemma gives further information about the structure of the graphs in $\mathscr{G}$[linear-width, 2]. According to the proof of Lemma 9, the graphs in $\mathscr{L}_2^1$ can be identified by taking into account all the possible ways the rooted graphs in $\mathscr{D}$ can be merged together.

We call a graph G *smooth* if each of the following conditions is satisfied.

(sm-i) G does not contain redundant or simply pendant edges

(sm-ii) G is outerplanar and does not contain weak edges

(sm-iii) For any bolbe B of G, $|\mathscr{S}(B)| \leqslant 2$, i.e. is contains at most two polar faces.

(sm-iv) For any vertex v, $\alpha(G,v) \leqslant 2$, i.e. there are at most two v-wings in $\mathscr{X}(G,v)$.

The proof of the following lemma is a direct consequence of Lemma 8 and 9.

Lemma 10. *Let G be a graph satisfying conditions* (sm-i) *and* (sm-ii) *above but not* (sm-iii) *or* (sm-iv). *Then* $\{4K_3\} \cup \mathscr{L}_2^1 \sqsubseteq G$.

Clearly, if G is a smooth graph, then any v-pendant path of G will contain exactly 3 vertices and, therefore, it will be isomorphic with $\mathscr{A}_1$ (graph $\mathscr{A}_1$ is depicted in Fig. 4).

Lemma 11. *Let G be a smooth v-graph such that $v \notin A(G)$. Then, either G is a v-wing or $G \preccurlyeq_v Z_0$ (Z_0 is depicted in* Fig. 5).

Proof. Suppose that $\mathscr{D} \not\sqsubseteq G$. We will prove that $G \preccurlyeq_v Z_0$. Clearly, we can assume that G is not isomorphic to $\mathscr{A}_1$ (graph $\mathscr{A}_1$ is depicted in Fig. 4). We distinguish the following cases. (The graphs $\mathscr{C}_i, i=1,\ldots,3$ and $\mathscr{D}_i, i=1,\ldots,4$ that are used in the case analysis below are depicted in Fig. 4.)

Case a. $d_G(v)=1$. Let u be the single neighbor of v. Notice that $d_G(u) \leqslant 3$, otherwise, $\mathscr{C}_1 \preccurlyeq_u G-v$ and thus $\mathscr{D}_1 \preccurlyeq_v G$. Also, $d_G(u) \geqslant 3$, otherwise, $\mathscr{C}_2 \preccurlyeq_u G-v$ and thus $\mathscr{D}_2 \preccurlyeq_v G$. Moreover, $G-v$ must be a tree, otherwise, $\mathscr{C}_3 \preccurlyeq_u G-v$ and thus $\mathscr{D}_3 \preccurlyeq_v G$. Let $\mathscr{X}(G-v,u)=\{G_1,G_2\}$. Clearly, both $G_i, i=1,2$ are u-pendant paths as, otherwise, $\{\mathscr{C}_2,\mathscr{C}_3\} \sqsubseteq_u G - v$ and thus $\{\mathscr{D}_2,\mathscr{D}_3\} \sqsubseteq_u G$. We conclude that G is isomorphic with $\mathscr{C}_2$ and we are done as $\mathscr{C}_2 \preccurlyeq_v Z_0$.

Case b. $d_G(v) \geqslant 2$. As G is outerplanar, one can easily see that G contains exactly one biconnected component B that is not a single edge and $v \in V(B)$ (otherwise $v \in A(G)$). From (sm-iii), B has two polar faces F_1, F_2. Notice now that if $M = A(G) \cap V(B)$ then $|M| \leqslant 2$ otherwise $\mathscr{D}_1 \preccurlyeq_v G$ (contract all the edges in $E(B)$). Now let v_1, v_2 be the two vertices of $V(B)$ such that $\{v,v_1\},\{v,v_2\} \in \mathrm{out}(B)$. Notice that if $x \in M$, then $x \in \{v_1,v_2\}$ otherwise, $\mathscr{D}_4 \preccurlyeq_v G$. Moreover, $|\mathscr{X}(G,x)| = 2$ (otherwise, $\mathscr{D}_2 \preccurlyeq_v G$) and the graph in $\mathscr{X}(G,x)$ not containing v as a vertex must be an x-pendant path (otherwise, $\{\mathscr{D}_2,\mathscr{D}_3\} \sqsubseteq_v G$. Observe now that v is incident to all the inner edges of B, otherwise, $\mathscr{D}_3 \preccurlyeq_v G$. Finally, notice that $|V(B) - \{v,v_1,v_2\}| \leqslant 1$ as G does not contain redundant or weak edges. Summing up all the previous observations we can easily see that $G \preccurlyeq_v Z_0$. □

Let $l = (e_1,\ldots,e_{|E(G)|})$ be an edge ordering of a v-rooted graph G. We say that l is a *v-simple* edge ordering of G if $\forall i, 1 \leqslant i \leqslant |E(G)|$ $|\delta_l(e_i) \cup \{v\}| \leqslant 2$.

Lemma 12. *Let G be a smooth graph where for some $v \in V(G)$ $\alpha(G,v) = 0$. Then there exists a v-simple edge ordering of G.*

Proof. From Lemma 11, any v-graph in $\mathscr{X}(G,v)$ is a v-minor of Z_0. The numbering depicted in Fig. 5 gives a v-simple edge ordering for Z_0. Using this, it is not hard to find a v-simple edge ordering for any of its minors. If now $l_1,\ldots,l_r$ are v-simple edge orderings for the graphs in $\mathscr{X}(G,v)$, then $l = l_1 \oplus \cdots \oplus l_r$ is a v-simple edge ordering of G. □

3. An algorithm for linear-width

It is easy to verify that ob($\mathscr{G}$[linear-width, 0])=$\{\mathscr{A}_1\}$ and that ob($\mathscr{G}$[linear-width, 1])= $\{\mathscr{C}_1,\mathscr{C}_3\}$ (graphs $\mathscr{A}_1$, $\mathscr{C}_1$ and $\mathscr{C}_3$ are depicted in Fig. 4). Using this fact, one can easily construct an algorithm that decides whether linear-width$(G) \leqslant 1$ and, if so, outputs an edge ordering of minimum linear-width. In this section we will present an algorithm, that, given a graph G, decides whether linear-width$(G) \leqslant 2$ and, if so, outputs an edge ordering of linear-width $\leqslant 2$. Before we present the algorithm we first need a series of definitions and lemmata about the structure of the graphs with linear-width $\leqslant 2$. The main structural lemma, supporting the correctness of the algorithm, is presented in the next section.

3.1. Doors and passages

Let B be a bolbe of a smooth graph G. We set $R(B) = A(G) \cap V(B)$, i.e. $R(B)$ contains the articulation vertices of B. For any $v \in R(B)$ we set $EX(B,v) = \{G_i^v \mid G_i^v \in$

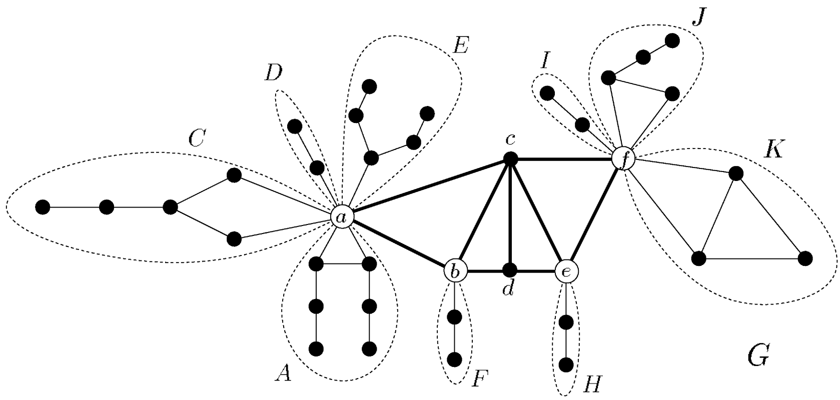

Fig. 6. An example of a smooth graph G containing a bolbe $B=G[\{a,b,c,d,e,f\}]$. The triples $(\emptyset,\mathcal{O},\mathcal{O})$, $(\{b\},F,\mathcal{O})$, $(\{f,e\},I,H)$, $(\{f,e\},K,H)$, $(\{f,e\},I\cup K,H)$, $(\{a,b\},F,D)$, $(\{a,b\},D,F)$, $(\{e,f\},H,I)$, and $(\{a,b\},A\cup C\cup D\cup E,F)$ are some of the doors of B.

$\mathscr{X}(G,v), V(G_i^v)\cap V(B)=\{v\}\}$, $EX(B)=\{H\mid \exists v\in R(B)$ such that $H\in EX(B,v)\}$, and $B_v=\bigcup_{G_i^v\in EX(B,v)} G_i^v$. For example, for the bolbe $B=G[\{a,b,c,d,e,f\}]$ depicted in Fig. 6 we have that $R(B)=\{a,b,e,f\}$, $EX(B,a)=\{A,C,D,E\}$, $EX(B,b)=\{F\}$, $EX(B,e)=\{H\}$, $EX(B,f)=\{I,J,K\}$, $EX(B)=\{A,C,D,E,F,H,I,J,K\}$, $B_a=A\cup C\cup D\cup E$, $B_b=F$, $B_e=H$, and $B_f=I\cup J\cup K$.

We denote the null graph as $\mathcal{O}$ (i.e. $\mathcal{O}=(\emptyset,\emptyset)$). Let $Q=(Y,H,I)$ be a triple consisting of a vertex set Y and two graphs H,I. We say that such a triple is a *door* of B if one of the following hold (for examples of doors, see Fig. 6).

(a) $Y=\emptyset$, $H=I=\mathcal{O}$. In such a case we call the door *empty*.

(b) $Y=\{v\}\subseteq R(B)$, $I=\mathcal{O}$, and $H=\bigcup_{G_i^v\in\mathscr{E}} G_i^v$ where $\mathscr{E}$ is a subset of $EX(B,v)$ that contains at most one v-wing. We call v a *passage* of the door.

(c) $Y=\{v,u\}\subseteq R(B)$, $v\sim u$, I is a u-pendant path in $EX(G,u)$ and $H=\bigcup_{G_i^v\in\mathscr{E}} G_i^v$ where $\mathscr{E}$ is a subset of $EX(B,v)$ that contains at most one v-wing. We call v a *passage* of the door.

Let B be a bolbe of a smooth graph G, let F_1,F_2 be the polar faces of B (if B has at most one polar face, we have $F_1=F_2=B$), and let $Q_i=(Y_i,H_i,I_i), i=1,2$ be two doors of B.

We say that the pair $\mathscr{P}=\{Q_1,Q_2\}$ *opens* B if each of the following conditions is satisfied:

(op-i). $G=B\cup H_1\cup I_1\cup H_2\cup I_2$.

(op-ii). $Y_i\subseteq F_i, i=1,2$,

(op-iii). $|V(H_1)\cap V(H_2)|\leqslant 1$ (i.e. if both $H_i, i=1,2$ are nonempty then they are different members of $EX(B)$).

(op-iv). $|V(I_1)\cap V(I_2)|\leqslant 1$ (i.e. if both $I_i, i=1,2$ are nonempty then they are different members of $EX(B)$).

(op-v). If $Q_i, i=1,2$ have the same passage v, then $\alpha(H_i,v)=1, i=1,2$.

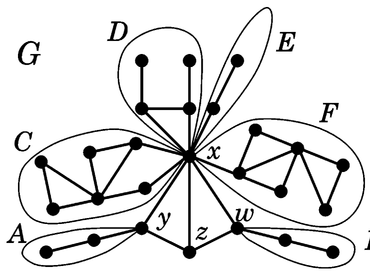

Fig. 7. G is a smooth graph and bolbe $B = G[\{w,x,y,z\}]$ can be opened by the following doors: $\{(\{x,y\},C,A),(\{x,w\},D\cup E\cup F,H)\}$, $\{(\{x,y\},C\cup D,A),(\{x,w\},E\cup F,H)\}$, $\{(\{x,y\},C\cup E,A),(\{x,w\},D\cup F,H)\}$, $\{(\{x,y\},C\cup D\cup E,A),(\{x,w\},F,H)\}$. Notice that, because of (op-v), if we replace F with a graph isomorphic to H or D, then there will exist only two doors that open B: $\{(\{x,y\},C\cup D\cup E\cup F,A),(\{w\},H,\emptyset)\}$ and $\{(\{y\},A,\emptyset),(\{x,w\},C\cup D\cup E\cup F,H)\}$.

(op-vi). If $E(B)>1$ and $Y_i, i=1,2$ induce edges in B, then these edges are different.
(op-vii). If $E(B)=1$ and $Q_i, i=1,2$ have different passages, then $|Y_1|=|Y_2|=1$.

We call $\mathscr{P}=\{(Y_i,H_i,I_i), i=1,2\}$ an *opening pair* of B. B is *open* when it is opened by some pair of doors. We call a vertex v a *passage* of an opening pair if v is a passage of its doors. For example, a pair opening the bolbe B depicted in Fig. 6 is $\mathscr{P}=\{(\{a,b\},A\cup C\cup D\cup E,F),(\{f,e\},I\cup J\cup K,H)\}$ and the corresponding passages are a and f. Notice that $\mathscr{P}$ is the unique pair opening B. Clearly, it is possible a bolbe to be opened by more than one pair (see e.g. Fig. 7). Notice that, if we know whether each rooted graph in $EX(B)$ is a wing, or a pendant path, we can assign to B an opening pair (if one exists) in $O(|EX(B)|)$ time. This observation will appear to be useful for proving the linearity of the algorithm $\mathsf{LW2}(G)$ that we will present in the proof of Theorem 18.

Lemma 13. *Any open bolbe B of a smooth graph G is a proper minor of one of the graphs Z_1^1, Z_1^2 depicted in Fig.* 5.

Proof. Let $Y_i, i=1,2$, be the vertex sets of an opening pair of B. The case where $\mathscr{S}(B)\leqslant 1$ is simple as in such a case G is either an edge or a cycle of at most 6 edges (notice that, since B is open, $|R(B)|\leqslant 4$). Suppose now that F_1,F_2 are the polar faces of B and assume w.l.o.g. that $Y_i\subseteq F_i, i=1,2$. Since $F_i, i=1,2$, does not contain redundant edges, it is a cycle containing at most 6 vertices. Taking now in mind that the critical edge of F_i has at most one vertex in common with Y_i, one can easily see that B is always a proper minor of Z_1^1 or Z_1^2. □

Lemma 14. *Let $\{(Y_i,H_i,I_i), i=1,2\}$ be a pair of doors opening a bolbe B of a smooth graph. Then, there exists an edge ordering of B with* linear-width $\leqslant 2$ *and with the property that Y_1 (Y_2) is a subset of its first* (*last*) *edge.*

Proof. From Lemma 13 we have that $B\prec Z_1^1$ or $B\prec Z_1^2$. Using now the orderings depicted in Fig. 5 for Z_1^1 and Z_1^2 as a starting point, one can easily construct a suitable edge ordering for any of their minors. □

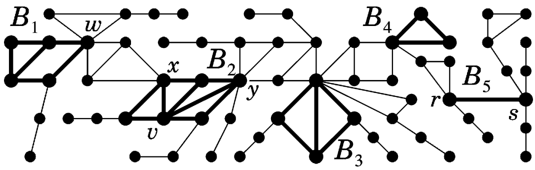

Fig. 8. All the bolbes of graph G are open. For $i=1,2$, B_i is non-marginal and there is only one pair opening it. The pair opening B_1 contains one empty door and one nonempty with w as a passage. The pair opening B_2 contains two doors with different passages x and y. For $i=3,4$, B_i is marginal. B_3 can be opened by more than one pairs and each of them contains doors with the same passage z. Moreover, there is only one pair opening B_4 and the common passage of its doors is u. Finally, B_5 is a non-marginal bolbe and the passages of the unique pair opening it are s and t (notice that, if we alter (op-vii), B_5 can be opened by more than one pairs).

Actually, the forms that open bolbes of smooth graphs can have are not many. Using Lemma 13 as a starting point, one can easily determine all of them. A back up of these graphs and the corresponding orderings (according to Lemma 14) can be useful for the implementation of the algorithm $\mathsf{LW2}(G)$ that we present in Theorem 18. The same remark holds for the graphs mentioned in Lemma 11.

3.2. *Finding a starting bolbe*

We plan to prove that any open smooth graph G has an edge ordering with linear-width $\leqslant 2$ (Theorem 17). In this direction, Lemma 12 and 14 show how to construct two different types of edge orderings for the bolbes that constitute G. What we now need is to merge all these orderings into an edge ordering of the whole graph. For this purpose we need to distinguish which parts of an open smooth graph require each type of ordering.

Given a v-graph G where v is not a pendant or an articulation vertex of G, we define the v-bolbe of G as the unique bolbe of G containing v as a vertex. For example, the v-bolbe of the graph of Fig. 8 is bolbe B_2. Let $\mathscr{P}$ be a pair opening a bolbe B. If the two doors in $\mathscr{P}$ are non-empty and have the same passage, then we call P *marginal.*

Notice that if a bolbe is opened by a (non)-marginal pair then all the pairs opening it are (non)-marginal. Using this remark, we define a bolbe B to be *marginal* if it is opened by some marginal pair. Otherwise, we call it non-marginal (for example, bolbe B in Fig. 6 is non-marginal and bolbe B in Fig. 7 is marginal). Finally, observe that if a bolbe is non-marginal then there is exactly one pair opening it. For an illustration of the distinction between marginal and non-marginal bolbes see Fig. 8

Lemma 15. *Let G be a smooth graph containing a marginal bolbe B. Let v be the unique passage of a marginal pair $\mathscr{P}$ opening B and G' be the graph in $\mathscr{X}(G,v)$ whose v-bolbe is B. Then the following hold.*

(i) *G' is not a v-wing,*

(ii) *$\mathscr{X}(G,v)$ contains two v-graphs whose v-bolbes are non-marginal.*

Proof. From (op-v) we have that $\mathscr{X}(G,v)$ contains two v-wings G_1, G_2 that are different from G'. We prove the two statements separately.

(i) Observe that from (sm-iv), $\alpha(G,v) \leqslant 2$ and therefore G' cannot be a v-wing.

(ii) Let now B_i be the v-bolbe of G_i, $i=1,2$. Then B_i is non-marginal, otherwise, applying Lemma 15(i) on B_i we have that G_i is not a v-wing, a contradiction. □

We call a smooth graph *open* if any bolbe in $\mathscr{B}(G)$ is open.

Lemma 16. *Let G be an open smooth graph. Then one of the following holds.*

(a) *There exists a vertex $v \in V(G)$ such that $\alpha(G,v)=0$.*

(b) *G has a non-marginal bolbe B opened by a non-marginal pair $\mathscr{P}$ where $\mathscr{P}$ has either an empty door or a passage $v \in V(B)$ such that $\alpha(G,v)=1$. We call such a bolbe* starting bolbe.

Proof. From (sm-iv) we have $\forall v \in V(G)$, $\alpha(G,v) \leqslant 2$. Assume now that (a) does not hold. Then, $\forall v \in A^*(G)$ $\alpha(G,v)=1$ or 2. We also assume that G contains at least one bolbe, otherwise, (a) holds. Finally, we can assume that for any bolbe of G the pairs opening it contain only non-empty doors as, otherwise, such a pair is clearly non-marginal and (b) holds. Let $\mathscr{W}$ be the set of the non-marginal bolbes of G. From Lemma 15(ii), we have that $\mathscr{W} \neq \emptyset$.

In what follows, we will prove that there exists a bolbe $B \in \mathscr{W}$ opened by a non-marginal pair $\mathscr{P}$ that contains a passage v where $\alpha(G,v)=1$. If $\mathscr{W}$ contains only one bolbe B, then it is trivial to see that, for any passage v of a pair opening B, $\alpha(G,v)=1$. We now assume that $|\mathscr{W}| \geqslant 2$. Suppose, towards a contradiction, that for any bolbe $B \in \mathscr{W}$ the unique pair opening it, contains two (different) passages v_1, v_2 such that $\alpha(G,v_1)=\alpha(G,v_2)=2$. We call these passages *passages* of B. Let A be the vertices of $A^*(G)$ that are also passages of bolbes in $\mathscr{W}$. Clearly, any bolbe of $\mathscr{W}$ contains two vertices of A as passages. Now let $v \in A$. Let also G_v^1, G_v^2 be the two v-wings in $\mathscr{X}(G,v)$ and B_1, B_2 be the v-bolbes of G_v^1 and G_v^2 respectively. Using Lemma 15(i), we have that $B_1, B_2 \in \mathscr{W}$ and therefore, each vertex in A is the common vertex of two different bolbes in $\mathscr{W}$. We construct now G' as follows: (a) remove from G all the vertices not belonging to graphs in $\mathscr{W}$, and (b) apply contractions that do not remove vertices in A as long as this is possible. It is not hard to see that $V(G')=A$. Moreover, notice that each vertex in G' has degree exactly 2 and therefore G' is a cycle, a contradiction as the vertices in A should be articulation vertices of G' as well. □

3.3. Constructing an edge ordering

In this subsection we present the way to merge the edge orderings of the trivial bolbes, the non-trivial bolbes, and the pendant paths of an open smooth graph. The non-trivial bolbes will form the main axis of the whole ordering.

Lemma 17. *Let G be an graph that is smooth and open. Then, there exists an edge ordering of G with* linear-width at most 2.

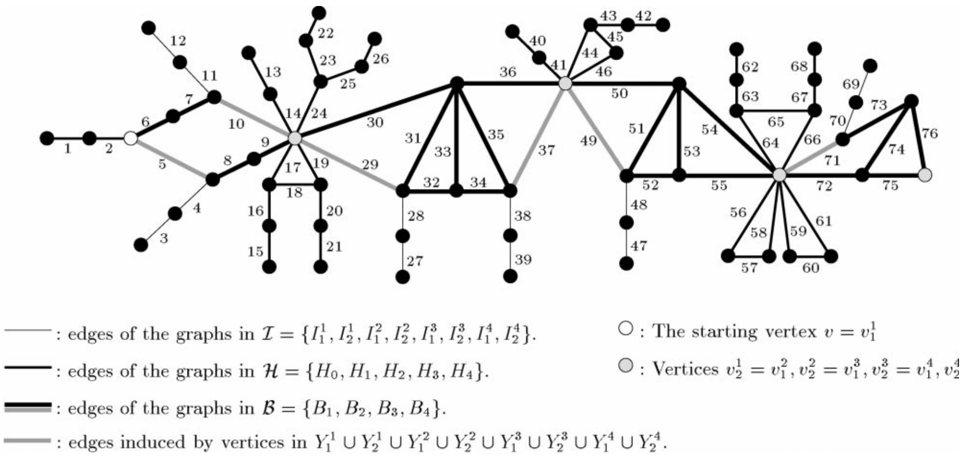

Fig. 9. An example of an edge ordering with linear-width $\leqslant 2$.

Proof. Using Lemma 16, we can assume that Lemma 16(b) holds as, otherwise, the result follows immediately from Lemma 12.

We apply the following procedure on G. For an example see Fig. 9.

1. Let B_1 be a starting bolbe of G and $\{Q_1, Q_2\} = \{(Y_j^1, H_j^1, I_j^1), j = 1,2\}$ be the non-marginal pair opening B_1.
   - If both Q_1, Q_2 are non-empty we can assume that their passages are v_1^1 and v_2^1 respectively and that $\alpha(G, v_1^1) = 1$. (Notice that, since (Q_1, Q_2) is non-marginal, $v_1^1 \neq v_2^1$.)
   - If only one, say Q_2, of Q_1, Q_2 has a passage, then we consider that v_2^1 is the passage of Q_2 and v_1^1 is some polar vertex of the polar face of B corresponding to Q_1 (if B is a cycle or a single edge, then we can choose as v_1^1 any vertex of B that is not v_2^1). We can also assume that $\alpha(G, v_1^1) = 1$, otherwise, the required edge ordering can be constructed according to Lemma 12.
   - If none of Q_1, Q_2 has a passage then $G = B_1$ and the required edge ordering can be constructed according to Lemma 14.
2. Let G_1 be the unique v-wing in $\mathscr{X}(G, v_1^1)$.
3. Set $H_0 = \{H_1^1\}$ (clearly, as $\alpha(G, v_1^1) = 1$, H_0 is not a v_1^1-wing).
4. Set $i = 1$.
5. If $Y_2^i = \emptyset$, then set $H_i = \mathcal{O}$, $\rho = i$, and **stop**.
6. If H_2^i is not a v_2^i-wing, then we set $H_i = H_2^i$, $\rho = i$ and **stop**.
7. If H_2^i is a v_2^i-wing, then let G_{i+1} be the unique member of $\mathscr{X}(H_2^i, v_2^i)$ that is a v_2^i-wing.
8. Set $H_i = \bigcup_{H \in \mathscr{X}(H_2^i, v_2^i) - \{G_{i+1}\}} H$ (i.e. H_i contains all the others). Notice also that $\alpha(H_i) = 0$.
9. Set $i = i + 1$.
10. Let B_i be the v_2^{i-1}-bolbe of G_i. Let also $\{Q_1^i, Q_2^i\} = \{(Y_j^i, H_j^i, I_j^i), j = 1,2\}$ be a pair opening B_i. Let also v_1^i, v_2^i be the corresponding passages. Clearly, v_2^{i-1} is one, say

v_1^i, of v_1^i, v_2^i. Recall that G_i is a v_1^i-wing. Therefore, using Lemma 15(i) we have that $\{Q_1^i, Q_2^i\}$ is a non-marginal pair and thus, $v_1^i \neq v_2^i$.

11. Goto to step 5.

Clearly, in each repetition of loop 5–10 the graph G_{i+1} produced has fewer vertices than G_i. Therefore, the procedure will stop after producing the sequence of graph sequences $\mathscr{H} = \{H_0, \ldots, H_\rho\}$ $\mathscr{I} = \{I_1^1, I_2^1, \ldots, I_1^\rho, I_2^\rho\}$ and $\mathscr{B} = \{B_1, \ldots, B_\rho\}$. As any member of $B_i \in \mathscr{B}$ is a bolbe, we can apply Lemma 14 to get, for any $i = 1, \ldots, \rho$, an edge ordering $l_i^{\mathscr{B}}$ of B_i with linear-width$(l) \leqslant 2$ and with the property that if e_1^i and $e_{|B_i|}^i$ are the first and last edges of f $l_i^{\mathscr{B}}$, then $Y_1^i \subseteq e_1^i$ and $Y_2^i \subseteq e_{|E(B_i)|}^i$. Notice now that any non-null member I_j^i of $\mathscr{I}$, $1 \leqslant i \leqslant \rho, j = 1,2$ is a u_j^i-pendant path $(\{u_j^i, x_j^i, y_j^i\}, \{\{u_j^i, x_j^i\}, \{x_j^i, y_j^i\}\})$ where u_j^i is the unique element of $Y_j^i - \{v_1^i\}$. For $i = 1, \ldots, \rho$ and $j = 1,2$ we define $l_{j,i}^{\mathscr{I}} = (\{u_j^i, x_j^i\}, \{x_j^i, y_j^i\})$. Let now H_i be a member of $\mathscr{H}, 0 \leqslant i \leqslant \rho$. Recall that H_i is not a v_2^i-wing. From Lemma 12, we have that there exists a v_2^i-simple edge ordering $l_i^{\mathscr{H}}$ of $E(H_i)$.

Notice now that $G = (\bigcup_{i=1,\ldots,\rho} B_i) \cup (\bigcup_{i=0,\ldots,\rho} H_i) \cup (\bigcup_{i=1,\ldots,\rho} I_1^i) \cup (\bigcup_{i=1,\ldots,\rho} I_2^i)$ and that if $l = l_0^{\mathscr{H}} \oplus (l_{1,1}^{\mathscr{I}})^{-1} \oplus l_1^{\mathscr{B}} \oplus l_{2,1}^{\mathscr{I}} \oplus l_1^{\mathscr{H}} \oplus (l_{1,2}^{\mathscr{I}})^{-1} \oplus l_2^{\mathscr{B}} \oplus l_{2,2}^{\mathscr{I}} \oplus l_2^{\mathscr{H}} \oplus \cdots \oplus (l_{1,\rho}^{\mathscr{I}})^{-1} \oplus l_\rho^{\mathscr{B}} \oplus l_{2,\rho}^{\mathscr{I}} \oplus l_\rho^{\mathscr{H}}$, then l is an edge ordering of G with linear-width $\leqslant 2$. □

3.4. The algorithm

We now present the main algorithm of this section.

Algorithm LW2(G)

Input: A graph G.

Output: If linear-width$(G) \leqslant 2$, the algorithm outputs an edge ordering of G with linear-width $\leqslant 2$. If not, the algorithm reports that "linear-width$(G) > 2$".

1. Let G^1 be a graph such that $G^1 \preccurlyeq G$, G^1 does not have redundant or simply pendant edges, and linear-width(G)=linear-width(G^1).
2. If G^1 is not outerplanar, then **return** "linear-width$(G) > 2$" and **stop**.
3. Let G^2 be a graph such that $G^2 \preccurlyeq G^1$, G^2 does not have weak edges, and linear-width$(G^1) \leqslant 2 \Leftrightarrow$ linear-width$(G^2) \leqslant 2$.
4. If $\exists B \in \mathscr{B}(G^2)$ such that $|\mathscr{S}(B)| \geqslant 3$, then **return** "linear-width$(G) > 2$" and **stop**.
5. If $\exists v \in A(G^2)$ such that $\alpha(G^2, v) \geqslant 3$, then **return** "linear-width$(G) > 2$" and **stop**. (Notice that if the algorithm does not stop here, then G is smooth.)
6. If $\exists B \in \mathscr{B}(G^2)$ such that B is not open, then **return** "linear-width$(G) > 2$" and **stop**. (Notice that if the algorithm does not stop here, then G^2 is smooth and open.)
7. If $\exists v \in A^*(G)$ such that $\alpha(G, v) = 0$, then construct an ordering l of G^2 according to Lemma 12 and goto step 10.
8. Find a starting bolbe of G^2 (this bolbe exists because of Lemma 16).
9. Construct an edge ordering l^2 of G^2 with linear-width $\leqslant 2$, using the procedure of the proof of Lemma 17.

10. Construct an edge ordering l^1 of G^1 with linear-width $\leqslant 2$.
11. Construct an edge ordering l of G with linear-width $\leqslant 2$.
12. **Return** l and **stop**.

Theorem 18. *Algorithm* LW2(G) *runs in* $O(|V(G)|)$ *time and outputs, if it exists, an edge ordering of G with* linear-width $\leqslant 2$.

Proof. We first prove that LW2(G) needs $O(|V(G)|)$ time. Steps 1 and 3 can be done in linear time because of Lemma 5 and 7 (take in mind that any outerplanar graph G has $O(|V(G)|)$ edges). Recall that step 2 can also be done in linear time. Moreover, it is possible in linear time to compute all the biconnected components of G^2 and, thus, step 4 needs $O(|V(G^2)|)$ time. Notice that it is possible to check in constant time whether a graph is v-minor of a graph with constant size. Therefore, according to Lemma 11, checking whether a graph is a v-wing or not requires constant time. Moreover, it is not hard to see that for any outerplanar graph $\sum_{v\in V(G)}|\mathscr{X}(G,v)| = O(|V(G)|)$ and $\sum_{B\in\mathscr{B}(G)}|EX(B)| = O(|V(G)|)$. Using the above observations and Lemma 13, one can easily verify that each of steps 5–9 can be done in $O(|V(G^2)|)$ time. Finally, the fact that steps 10 and 11 can be performed in linear time, follows directly from Lemma 7 and 5.

What remains now is to prove that algorithm LW2(G) is correct. Notice that if for some input G the algorithm enters step 7 then G^2 is smooth and open. Therefore, linear-width(G^2) $\leqslant 2$ and thus the required ordering can be correctly constructed according to Lemma 17. Suppose now that for some input G the algorithm never enters at step 7. We claim that, then, linear-width(G) > 2. In what follows we prove that $\mathscr{L}_2 \sqsubseteq G$ and the claim will be a direct consequence of Lemma 3.

Suppose first that LW2(G) stops at step 2. Then G^1 is not outerplanar and $\mathscr{L}_2 \sqsubseteq \{K_{2,3}, K_4\} \sqsubseteq G^1 \preccurlyeq G$ (recall that any non-outerplanar contains either $K_{2,3}$ or K_4 as a minor). If now LW2(G) stops at step 4 or 5, the result follows directly from the fact that $G^2 \preccurlyeq G^1 \preccurlyeq G$ and from Lemma 10. Finally, if the algorithm stops at step 6, this means that G is smooth and contains a bolbe B that is not open. The result now follows from the fact that $G^2 \preccurlyeq G^1 \preccurlyeq G$ and Lemma 9 (Lemma 9 will be presented in the next section). □

Concluding this section, we remark that the main algorithm of this section can be easily parallelized. We do not proceed with a detailed elaboration of the parallel case as it is easy and based on standard techniques.

4. Identifying the obstruction set

In this section we will prove the basic structural lemma of this paper. Moreover, we will examine the case where multiple edges are considered.

4.1. The main lemma

The proof of the main lemma is based in an exhaustive case analysis of all the possible ways the graphs in $EX(B)$ can be attached to a bolbe B. We will show that either an opening pair of B exists or some graph in $\mathscr{L}_2^3 \cup \cdots \cup \mathscr{L}_2^9$ is a minor of G.

Lemma 19. *Let B be a bolbe of a smooth graph G. Then, either B is open or $\mathscr{L}_2^3 \cup \cdots \cup \mathscr{L}_2^9 \sqsubseteq G$.*

Proof. We assume that $\mathscr{L}_2^3 \cup \cdots \cup \mathscr{L}_2^9 \not\sqsubseteq G$. We will construct a pair of doors $\mathscr{P} = \{(Y_i, H_i, I_i), i = 1, 2\}$ opening B. We examine first the case where there exists a vertex $v \in R(B)$ such that $\mathscr{X}(B_v, v)$ contains two v-wings G_1, G_2 (recall that the vertices in $R(B)$ are articulation vertices). Let G_3 be the v-graph in $\mathscr{X}(G, v)$ whose v-bolbe is B. Clearly, as $\alpha(G, v) \leqslant 2$ (recall that G is smooth), G_3 is not a v-wing and using Lemma 11, we have that G_3 is a v-minor of Z_0. Notice that $R(B) \leqslant 3$ and if $x \in R(G) - \{v\}$, then $x \sim v$. Let $H_1 = \cup(EX(B, v) - \{G_2\})$ and $H_2 = G_2$. If $R(B) - \{v\} = \emptyset$, then set $Y_i = \emptyset, I_i = \mathcal{O}, i = 1, 2$. If $R(B) - \{v\} = \{x\}$ and G_3 is isomorphic to $\mathscr{C}_2$ (graph $\mathscr{C}_2$ is depicted in Fig. 4) we set $Y_1 = Y_2 = \{v, x\}$, I_1 is one of the two x-pendant paths of $\mathscr{X}(G_3, x)$ and I_2 the other. If $R(B) - \{v\} = \{x\}$ and G_3 is not isomorphic to $\mathscr{C}_2$ we set $Y_1 = \{v, x\}, I_1 = B_x, Y_2 = \{v\}, I_2 = \mathcal{O}$. Finally, if $R(B) - \{v\} = \{x_1, x_2\}$, then set $Y_i = \{v, x_i\}, I_i = B_{x_i}, i = 1, 2$. It is now easy to observe that, in any case, pair $\{(Y_i, H_i, I_i), i = 1, 2\}$ opens B.

We assume now that $\forall v \in R(B)$, $EX(B, v)$ contains at most one v-wing. We define a function $\phi : R(B) \to \{0, 1, 2\}$ where for any $v \in R(B)$, $\phi(v) = 0$ if B_v is a v-pendant path, $\phi(v) = 1$ if B_v consists of two u-pendant paths (i.e. is isomorphic to graph $\mathscr{B}_1$ depicted in Fig. 4), and $\phi(v) = 2$ in any other case (i.e. contains some graph in $\mathscr{C}$ as a v-minor – $\mathscr{C}$ is depicted in Fig. 4). We call the value of $\phi(v)$ *strength* of v. Notice now the following.

(e-i) Any vertex in $R(B)$ belongs in some polar face, otherwise, $A_1^+ \preccurlyeq B$.
(e-ii) $|R(B)| \leqslant 4$ otherwise, $5A_1 \preccurlyeq G$.
(e-iii) $R(B)$ contains at most two vertices with strength 2, otherwise, $\mathscr{L}_2^3 \sqsubseteq G$.
(e-iv) If $|R(B)| = 3$ then $\exists v, u \in R(B) \sim u$, otherwise $3A_1^+ \preccurlyeq G$.
(e-v) If $|R(B)| = 4$ then $\exists v, u, w, x \in R(B)$ $v \sim u$ and $w \sim x$, otherwise $3A_1^+ \preccurlyeq G$.

Suppose now that $R(G) \leqslant 2$. If $R(B) = \emptyset$ then set $Y_i = \emptyset, H_i = I_i = \mathcal{O}, i = 1, 2$. If $R(B) = \{v\}$ then set $Y_1 = \{v\}, H_1 = B_v, Y_2 = \emptyset, H_2 = I_1 = I_2 = \mathcal{O}$. If $R(B) = \{v_1, v_2\}$ then set $Y_i = \{v_i\}, H_i = B_{v_i}, I_i = \emptyset, i = 1, 2$. Since in any of the above cases $\{(Y_i, H_i, I_i), i = 1, 2\}$ opens B, we may assume that $R(B) \geqslant 3$ (and thus $\mathscr{S}(B) \geqslant 1$). From the smoothness of G we have that $\mathscr{S}(G) \leqslant 2$. The proof proceeds with the following case analysis.

We examine first the case where $\mathscr{S}(B) = 1$ (Notice that, in this case B is a cycle).

(a) $R(B) = \{v, u, w\}$. From (e-iv) we can assume that $v \sim u$.

(a.I) For at least one, say u, of v, u, $\phi(u) = 0$. Then, $\mathscr{P} = \{(\{v, u\}, B_v, B_u), (\{w\}, B_w, \mathcal{O})\}$.

(a.II) $\phi(v)=\phi(u)=1$. Then, for one of v,u, say v, $v \sim w$ (otherwise $A_1 2B_1 \preccurlyeq G$). We set $\mathscr{P} = \{(\{u,v\},B_u,I_1),(\{w,v\},B_w,I_2)\}$ where I_1 is the one of the two v-pendant paths of B_v and I_2 is the other.

(a.III) $\phi(v)=2, \phi(u)=1$. If $\phi(w)=0$, then for one of v,u, say v, $v \sim w$ (otherwise $\{A_1 2B_1, A_1 B_1 C_3\} \sqsubseteq G$) and we set $\mathscr{P} = \{(\{u\},B_u,\emptyset),(\{v,w\}, B_v,B_w)\}$. If $\phi(w) \geqslant 1$, then $w \sim u$, otherwise, either $\mathscr{L}_2^8 \sqsubseteq G$ or $\mathscr{L}_2^9 \sqsubseteq G$. We set $\mathscr{P} = \{(\{w,u\},B_w,I_1),(\{v,u\},B_v,I_2)\}$ where I_1 is one of the two u-pendant paths of B_u and I_2 the other.

(a.IV.) $\phi(v) = \phi(u) = 2$. Then, $\phi(w) = 0$ and for one of v,u, say v, $w \sim v$, otherwise, either $\mathscr{L}_2^8 \sqsubseteq G$ or $\mathscr{L}_2^3 \sqsubseteq G$ or $\mathscr{L}_2^9 \sqsubseteq G$. We set $\mathscr{P} = \{(\{u\},B_u,\emptyset),(\{v,w\},B_v,B_w)\}$.

(b) $R(B)=\{v,u,w,x\}$. From (e-v) we assume that $v \sim u$ and $w \sim x$. Let N be the set of neighbors of v and u in B.

(b.I) $|N \cap \{w,x\}| \leqslant 1$. Notice that for at least one of v,u, say u, $\phi(u)=0$ (otherwise, $\mathscr{L}_2^8 \sqsubseteq G$) and for at least one of w,x, say x, $\phi(x)=0$ (otherwise, $\mathscr{L}_2^8 \sqsubseteq G$). We set $\mathscr{P} = \{(\{v,u\},B_v,B_u),(\{w,x\},B_w,B_x)\}$.

(b.II) $N \cap \{w,x\} = \{w,x\}$. If at least two vertices, say u,w, in $\{v,u,x,w\}$, have strength 0, then we set $\mathscr{P} = \{(\{v,u\},B_v,B_u),(\{v,w\},B_x,B_w)\}$. If at least 3 vertices in $\{v,u,x,w\}$ have strength $\geqslant 1$, then it is easy to see that, either $\mathscr{L}_2^6 \sqsubseteq G$ or $\mathscr{L}_2^3 \sqsubseteq G$.

It remains to examine the case where $\mathscr{S}(B)=2$. We set $\{F_1,F_2\} = \mathscr{S}(B)$. Clearly, $|F_1 \cap F_2| \leqslant 2$. We call a vertex *crucial* if it is a critical vertex of both polar faces of B (i.e. belongs in $F_1 \cap F_2$). We notice first the following fact.

(e-vi) Any face can contain at most 2 non-crucial vertices v,u that belong in $R(B)$ (otherwise, $2A_1 \preccurlyeq G$). Moreover $v \sim u$ (otherwise, $2A_1 \preccurlyeq G$) and for one of them, say u, $\phi(u)=0$ (otherwise, $\mathscr{L}_2^5 \sqsubseteq G$).

We distinguish the following three cases.

Case 1. $F_1 \cap F_2 = \emptyset$. Notice that all vertices in $R(B)$ are non-crucial.

(1.a) $R(B)=\{v,u,w\}$. From (e-vi) we may assume that $v,u \in F_1, w \in F_2, v \sim u, \phi(u)=0$ and we set $\mathscr{P} = \{(\{v,u\},B_v,B_u),(\{w\},B_w,\emptyset)\}$.

(1.b) $R(B)=\{v_1,u_1,v_2,u_2\}$. From (e-vi) we may assume that $v_i,u_i \in F_i, v_i \sim u_i, \phi(u_i)=0, i=1,2$ and we set $\mathscr{P} = \{(\{v_i,u_i\},B_{v_i},B_{u_i}), i=1,2\}$.

Case 2. $F_1 \cap F_2 = \{v\}$. We assume that $v \in R(B)$ (i.e. v is crucial) as, otherwise, Case 2 is reduced to Case 1.

(2.i.) $\phi(v)=0$.

(2.i.a.) $R(B)=\{v,u,w\}$. There are two cases.

(2.i.a.I) u,w belong to the same polar face, say F_1. As u,w are non-crucial, from (e-vi), we can assume that $u \sim w$ and $\phi(w)=0$. We set $\mathscr{P} = \{(\{v\},B_v,\emptyset),(\{u,w\},B_u,B_w)\}$.

(2.i.a.II) u,w belong to different polar faces. Then, for at least one of u,w, say u, we have that $u \sim v$ (otherwise $\{3A_1, 3A_1^+\} \sqsubseteq G$) and we set $\mathscr{P} = \{(\{u,v\},B_u,B_v),(\{w\},B_w,\emptyset)\}$.

(2.i.b) $R(B)=\{v,u,w,x\}$. As all the vertices u,w,x are non-crucial, from (e-vi), we can assume that $u,w\in F_1$, $x\in F_2$, $u\sim w$, and $\phi(u)=0$. Notice also that $x\sim v$, otherwise, $\{3A_1^+,3A_1\}\sqsubseteq G$. We set $\mathscr{P}=\{(\{w,u\},B_w,B_u),(\{x,v\},B_x,B_v)\}$.

(2.ii) $\phi(v)=1$.

(2.ii.a) $R(B)=\{v,u,w\}$. There are two cases.

(2.ii.a.I) u,w belong to the same polar face. Similar to Case 2.i.a.I.

(2.ii.a.II) u,w belong to different polar faces. Then, for at least one of u,w, say u, we have that $u\sim v$ (otherwise $\{3A_1,3A_1^+\}\sqsubseteq G$). If $\phi(u)=0$, we set $\mathscr{P}=\{(\{v,u\},B_v,B_u),(\{w\},B_w,\mathscr{O})\}$. If $\phi(u)>0$, then $w\sim v$ (otherwise $\{A_12B_1,A_1B_1C_3\}\sqsubseteq G$). We can now set $\mathscr{P}=\{(\{u,v\},B_u,I_1),(\{w,v\}, B_w,I_2)\}$ where I_1 is one of the two v-pendant paths of B_v and I_2 is the other.

(2.ii.b) $R(B)=\{v,u,w,x\}$. Using (e-vi), we can assume that $w,x\in F_1$, $y\in F_2$, $w\sim x$, and $\phi(x)=0$. We also notice that $y\sim v$ (otherwise, $3A_1^+G$) and $\phi(y)=0$ (otherwise, $\{A_12B_1,A_1B_1C_3\}\sqsubseteq G$). We set $\mathscr{P}=\{(\{w,x\},B_w,B_x),(\{v,y\},B_v,B_y)\}$.

(2.iii) $\phi(v)=2$.

(2.iii.a) $R(B)=\{v,u,w\}$. There are two cases.

(2.iii.a.I) u,w belong to the same polar face. Similar to Case 2(i.a.I).

(2.iii.a.II) u,w belong to different polar faces. Then, for at least one of u,w, say u, we have that $u\sim v$ (otherwise, $\{3A_1,3A_1^+\}\sqsubseteq G$). If $\phi(u)=0$, set $\mathscr{P}=\{(\{v,u\},B_v,B_u),(\{w\},B_w,\mathscr{O})\}$. If $\phi(u)>0$, then $w\sim v$ (otherwise, $\mathscr{L}_2^8\sqsubseteq G$) and $\phi(w)=0$ (otherwise $\mathscr{L}_2^9\sqsubseteq G$). We set $\mathscr{P}=\{(\{u\},B_u,\mathscr{O}),(\{v,w\},B_v,B_w)\}$.

(2.iii.b) $R(B)=\{v,u,w,x\}$. From (e-vi) we can assume that $u,w\in F_1, u\sim w$, and $\phi(w)=0$. We also notice that $x\sim v$ (otherwise, $3A_1^+G$) and that $\phi(x)=0$ (otherwise, $\mathscr{L}_2^8\sqsubseteq G$). We set $\mathscr{P}=\{(\{u,w\},B_u,B_w),(\{v,x\}, B_v, B_x)\}$.

Case 3. $F_1\cap F_2=\{v,u\}$. Notice that if $|R(B)\cap F_1\cap F_2|=0$ then Case 3 is reduced to Case 1. Also, if $|R(B)\cap F_1\cap F_2|=1$ then Case 3 is basically the same to Case 2 (the only difference is that set $\{3A_1,3A_1^+\}$ could be replaced by $\{3A_1^+\}$). We now assume that $|R(B)\cap F_1\cap F_2|=2$.

(3.i) $\phi(v)=0$ and $\phi(u)=0$.

(3.i.a) $R(B)=\{v,u,w\}$. From (e-iv), we can assume that $w\sim v$ and set $\mathscr{P}=\{(\{w,v\},B_w,B_v), (\{u\},B_u,\mathscr{O})\}$.

(3.i.b) $R(B)=\{v,u,w,x\}$. From (e-v), we can assume that $w\sim v$ and $x\sim u$. Also, w and x must belong into different polar faces, otherwise $4A_1G$. We can now set $\mathscr{P}=\{(\{w,v\},B_w,B_v),(\{x,u\},B_x,B_u)\}$.

(3.ii) $\phi(v)=0$ and $\phi(u)>0$.

(3.ii.a) $R(B)=\{v,u,w\}$. If $\phi(w)>0$, then $w\sim v$ (otherwise, $\{3A_1^+\}\cup\mathscr{L}_2^8\sqsubseteq G$) and we set $\mathscr{P}=\{(\{w,v\},B_w,B_v),(\{u\},B_u,\mathscr{O})\}$. If

$\phi(w)=0$ then either $w \sim v$ or $w \sim u$, otherwise, $3A_1^+G$. We may assume that $w \sim v$ and set $\mathscr{P}=\{(\{v,w\},B_v,B_w),(\{u\},B_u,\emptyset)\}$.

(3.ii.b) $R(B)=\{v,u,w,x\}$. We distinguish the following cases.

(3.ii.b.I) $\phi(w)=\phi(x)=0$. From (e-v) we may assume that $w \sim v$, and $x \sim u$. Also w,x must belong to different polar faces (otherwise $4A_1G$). We can now set $\mathscr{P}=\{(\{v,w\},B_v,B_w),(\{u,x\},B_u,B_x)\}$.

(3.ii.b.II) $\phi(w)=0$ and $\phi(x)>0$. In this case, $v \sim x$, $u \sim w$, and w and x must belong into different polar faces (in any other case $\{4A_1,3A_1^+\} \sqsubseteq G$). We can now set $\mathscr{P}=\{(\{x,v\},B_x,B_v),(\{u,w\},B_u,B_w)\}$.

(3.ii.b.III) If both $\phi(w),\phi(x) \geqslant 1$ then $\{4A_1\} \cup \mathscr{L}_2^6 \sqsubseteq G$.

(3.iii.) $\phi(v)>0$ and $\phi(u)>0$.

(3.iii.a) $R(B)=\{v,u,w\}$. Clearly $\phi(w)=0$, otherwise $\mathscr{L}_2^7 \sqsubseteq G$. Also, either $v \sim w$ or $w \sim u$ (otherwise $3A_1^+G$). We can assume that $v \sim w$ and set $\mathscr{P}=\{(\{v,w\},B_v,B_w),(\{u\},B_u,\emptyset)\}$.

(3.iii.b) $R(B)=\{v,u,w,x\}$. From (e-v) we may assume that $w \sim v$, $\phi(w)=0$, $x \sim u$, and $\phi(x)=0$. Also w and x belong to different polar faces (otherwise, $4A_1G$). We set $\mathscr{P}=\{(\{v,w\},B_v,B_w),(\{u,x\},B_u,B_x)\}$. □

Following the case analysis of the above proof one can easily enhance algorithm LW2 so that, in case linear-width$(G)>2$, it outputs the forbidden minor that G contains. Notice that LW2(G) is based only on the structural characterization of $\mathscr{G}$(linear-width, 2) given in Lemma 17 and does not involve at all the case analysis of the proof of Lemma 19 above.

After a detailed inspection, one can verify that any proper minor of a graph in $\mathscr{L}_2$ has linear-width $\leqslant 2$. Lemma 17 can accelerate this inspection as follows: Let G be a graph in $\mathscr{L}_2$. Let also H be any graph obtained after an edge removal/contraction on G. It is enough to observe that, if we apply the operation of the proof of Lemma 5 on H, the resulting graph H' is open and smooth. Clearly, linear-width(H)=linear-width(H') and applying Lemma 17 for H' we end up that linear-width $(H) \leqslant 2$.

Using now Lemma 3, we have the following.

Lemma 20. $\mathscr{L}_2 \subseteq \text{ob}(\mathscr{G}[\text{linear-width}, 2])$.

The next theorem gives a complete structural characterization of the class of graphs with linear-width $\leqslant 2$.

Theorem 21. *$\mathscr{L}_2$ is the obstruction set for the class of graphs with* linear-width $\leqslant 2$ *i.e.* $\mathscr{L}_2=\text{ob}(\mathscr{G}[\text{linear-width}, 2])$. (*The graphs in $\mathscr{L}_2$ are depicted in Fig.* 13 *of the appendix.*)

Proof. By Lemma 20, it is enough to prove that any graph with linear-width more than 2 contains at least one of the graphs in $\mathscr{L}_2$ as a minor. Suppose now that

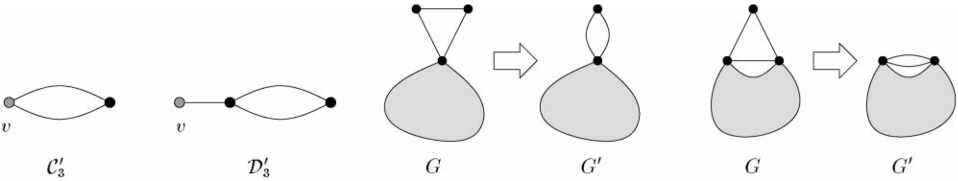

Fig. 10. The graphs $\mathscr{C}_3'$ and $\mathscr{D}_3'$ and the transformations for the case of multiple edges.

linear-width$(G) > 2$. Notice that, if G is not smooth, then $\mathscr{L}_2^1 \cup \mathscr{L}_2^2 \sqsubseteq G$ (use Lemma 10). If now G is smooth then it cannot be open, otherwise, from Lemma 17, linear-width $\leqslant 2$. Therefore, it contains a bolbe that is not open. From Lemma 19 we have that $\mathscr{L}_2^3 \cup \cdots \cup \mathscr{L}_2^9 \sqsubseteq G$. □

4.2. The case of multiple edges

During the presentation of the proof and the algorithm of Sections 3 and 4, we assumed that the graphs cannot contain loops or multiple edges. We have to mention that it is possible to obtain the same results without this restriction. The only essential difference is that graphs $\mathscr{C}_3$ and $\mathscr{D}_3$ should be replaced with graphs $\mathscr{C}_3'$ and $\mathscr{D}_3'$ depicted in Fig. 10. This would result in a different obstruction set. This obstruction set can be constructed from $\mathscr{L}_2$ if for any graph $G \in \mathscr{L}_2$ we apply the following two operations as long as this is possible (see Fig. 10):

- If G has a biconnected component that is a triangle, replace this triangle by $\mathscr{C}_3'$.
- If G has a polar face F containing only one polar vertex that is not an articulation vertex, remove this vertex (along with the two edges containing it) and introduce a new edge connecting the critical vertices of F.

We avoid examining the case of multiple edges in detail as it would be a tedious resumption of what we have already presented.

We conclude that the operation of adding copies of existing edges in a multigraph, can increase its linear-width. One can easily see that the same holds for the parameters of edge search number and mixed search number, defined in the next section. Interestingly, this is not the case for other relevant parameters like pathwidth, treewidth, or branchwidth where the obstruction set does not change if we consider multiple edges (see also [36]).

5. Linear-width and search parameters

In this section we give the definitions of edge searching, node searching, and mixed searching and we prove that the problem of computing the corresponding graph parameters can be reduced to the one of computing linear-width.

5.1. Mixed search and other variants

A *mixed searching game* is defined in terms of a graph representing a system of tunnels where an agile and omniscient fugitive with unbounded speed is hidden (alternatively, we can formulate the same problem considering that the tunnels are contaminated by some poisonous gas). The object of the game is to *clear* all edges, using one or more *searchers*. An edge of the graph is cleared if one of the following cases occurs.

(A) *both of its endpoints are occupied by a searcher*,

(B) *a searcher slides along it*, i.e., a searcher is moved from one endpoint of the edge to the other endpoint.

A search is a sequence containing some of the following moves. $a(v)$: placing a new searcher on v, $b(v)$: deleting a searcher from v, $c(v,u)$: sliding a searcher on v along $\{v,u\}$ and placing it on u.

The object of a mixed search is to clear all edges using a search. The search number of a search is the maximum number of searchers on the graph during any move. The mixed search number, $\mathrm{ms}(G)$, of a graph G is the minimum search number over all the possible searches of it. A move causes *recontamination* of an edge if it causes the appearance of a path from an uncleared edge to this edge not containing any searchers on its vertices or its edges. (Recontaminated edges must be cleared again.) A search without recontamination is called *monotone*.

The *node* (*edge*) *search number*, $\mathrm{ns}(G)$ ($\mathrm{es}(G)$) is defined similarly to the mixed search number with the difference that an edge can be cleared only if **A** (**B**) happens.

The following results were proved by Bienstock and Seymour in [3] (see also [39,36]).

Theorem 22. *For any graph G the following hold*:

(i) If $\mathrm{ms}(G) \leqslant k$ *then there exists a monotone mixed search in G using* $\leqslant k$ *searchers.*

(ii) $\text{linear-width}(G) \leqslant \mathrm{ms}(G)$.

(iii) *If G does not contain pendant vertices*, *then* $\text{linear-width}(G) = \mathrm{ms}(G)$.

(iv) *If* G^e *is the graph occurring from G after subdividing each of it edges*, *then* $\mathrm{es}(G) = \mathrm{ms}(G^e)$.

(v) *If* G^n *is the graph occurring if we replace every edge in G with two edges in parallel*, *then* $\mathrm{ns}(G) = \mathrm{ms}(G^n)$.

We mention that the mixed search number is equivalent with the parameter of proper-pathwidth defined by Takahashi et al. [37,39]. It is also known that the node search number is equivalent to the pathwidth, the interval thickness, and the vertex separation number (see [14,21,24,25,28]).

It is not hard to prove that the node search and the linear-width can differ by at most one (it appears as exercise in [40]). It is also easy to see that the same relation connects mixed search number and linear-width (see Theorem 25(iv)).

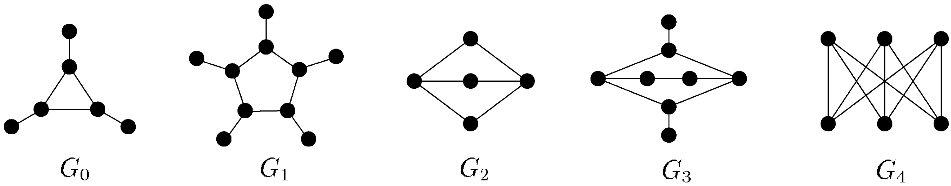

Fig. 11. An example of the variations of the values of linear-width, ms, es, and ns.

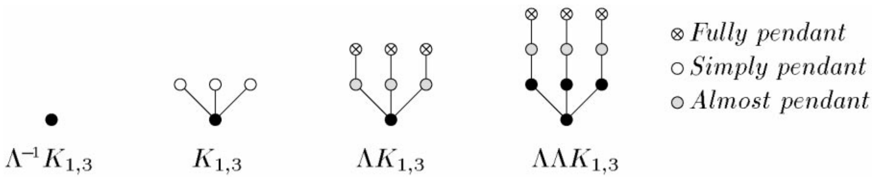

Fig. 12. The graphs $\Lambda^{-1}K_{1,3}, K_{1,3}, \Lambda K_{1,3}$ and $\Lambda\Lambda K_{1,3}$.

For an example of the values linear-width, ms, ns, and es can take, see the graphs of Fig. 11. We observe that

- linear-width$(G_0) = 2$, ms$(G_0) = 2$, es$(G_0) = 3$, ns$(G_0) = 3$,
- linear-width$(G_1) = 2$, ms$(G_1) = 3$, es$(G_1) = 3$, ns$(G_1) = 3$,
- linear-width$(G_2) = 3$, ms$(G_2) = 3$, es$(G_2) = 3$, ns$(G_2) = 3$,
- linear-width$(G_3) = 3$, ms$(G_3) = 3$, es$(G_3) = 3$, ns$(G_3) = 4$,
- linear-width$(G_4) = 4$, ms$(G_4) = 4$, es$(G_3) = 5$, ns$(G_4) = 4$.

Notice that, $G_0 \in \mathrm{ob}(\mathscr{G}[\mathrm{es}, 2])$, $G_1 \in \mathrm{ob}(\mathscr{G}[\mathrm{ms}, 2])$, $G_2 \in \mathrm{ob}(\mathscr{G}[\text{linear-width}, 2])$, $G_3 \in \mathrm{ob}(\mathscr{G}[\mathrm{ns}, 3])$, and $G_4 \in \mathrm{ob}(\mathscr{G}[\mathrm{es}, 4])$.

5.2. The relation between linear-width and mixed search

Let G be a graph. We denote by ΛG the graph obtained from G by introducing, for any pendant vertex, one new vertex and an edge connecting them (formally, if $P = \{p_1, \dots, p_r\}$, is the set of pendant vertices of G, then $\Lambda G = (V(G) \cup \{p'_1, \dots, p'_r\}, E(G) \cup \{\{p_1, p'_1\}, \dots, \{p_r, p'_r\}\})$ where $\{p'_1, \dots, p'_r\} \cap V(G) = \emptyset$. We denote by $\Lambda^{-1}G$ the graph obtained if we remove all the pendant vertices. Observe that if a graph does not contain simply pendant edges, the graphs G, $\Lambda\Lambda^{-1}G$, and $\Lambda^{-1}\Lambda G$ are isomorphic. For an example of operations Λ and Λ^{-1} see Fig. 12. Clearly, any pendant edge of G becomes almost pendant in ΛG and any almost pendant edge in G becomes pendant in $\Lambda^{-1}G$.

We will need the following easy result (for the proofs see e.g. [38]).

Lemma 23. *For any graph G the following hold.*
(i) *If v is a fully pendant vertex in G then* $\mathrm{ms}(G) = \mathrm{ms}(G - v)$.
(ii) *If e is a redundant edge in G then* $\mathrm{ms}(G) = \mathrm{ms}(G \dot{-} e)$.

Theorem 24. *Let G be a graph. Then*, ms(G)=linear-width(ΛG) *and* linear-width(G)= ms($\Lambda^{-1}G$).

Proof. Our first step is to prove that the first equality implies the second. We denote by G^s the graph obtained from G after removing, one by one, simply pendant vertices until this is not any more possible. Applying inductively Lemma 4(i) on the number of the simply pendant vertices of G that were removed, one can prove that linear-width(G)= linear-width(G^s). Since G^s has no simply pendant edges G^s is isomorphic with $\Lambda\Lambda^{-1}G^s$ and therefore, we have that linear-width(G^s) = linear-width($\Lambda\Lambda^{-1}G^s$). The first equality implies that linear-width($\Lambda\Lambda^{-1}G^s$) = ms($\Lambda^{-1}G^s$). Observe now that $\Lambda^{-1}G^s$ is isomorphic to $\Lambda^{-1}G$ and therefore linear-width(G) = ms($\Lambda^{-1}G$) as required. What now remains is to prove the first equality.

Let $E=\{g_1,\dots,g_n\}$ be the set of pendant edges of G and let $g_i=\{x_i,y_i\}, 1\leqslant i\leqslant n$ where $d_G(y_i)=1$, $1\leqslant i\leqslant n$. Also let $E'=(g_1,g_1',\dots,g_n,g_n')\subseteq E(\Lambda G)$ where $g_i=\{x_i,y_i\}$ is an almost pendant edge of ΛG and $g_i'=\{y_i,y_i'\}$ is a fully pendant edge of ΛG for $i=1,\dots,n$. Let $l=(e_1,\dots,e_r)$ be an edge ordering of ΛG with linear-width k.

For each $i=1,\dots,n$ we apply the following operation: let $g_i=e_j$ and $g_i'=e_h$ in l, w.l.o.g. we assume that $j<h$, and we replace l by the sequence $(e_1,\dots,e_{j-1},g_i,g_i',e_{j+1},\dots,e_{h-1},e_{h+1},\dots,e_r)$ (i.e., we remove g_i and g_i' and place first g_i and then g_i' in the position where the first of them appears). Notice that the above reordering operation does not increase the linear-width of the ordering. Therefore, we end up with an edge ordering $l^*=(f_1,\dots,f_r)$ of ΛG that has linear-width $\leqslant k$ and where every edge $\{y_i,y_i'\}$ appears immediately after $\{x_i,y_i\}$.

Let l' be the ordering of $E(G)$, obtained from l^* by replacing, for each $i, 1\leqslant i\leqslant n$, the pair of edges $\{x_i,y_i\}$ $\{y_i,y_i'\}$ by edge $\{x_i,y_i\}$. We claim that there exists a monotone mixed-search of G using $\leqslant k$ searchers, such that the edges of G are cleared in the order of l'. We prove the claim with induction. Suppose that there exists a sequence of search moves that clears the first i edges of l' (and not any other) in the order that they appear in l'. We denote this edge set as E_i. Let also f_j be the ith edge of l' (clearly, not all the edges of l^* are edges of l' and thus $j\geqslant i$). If f_{j+1} is missing from l' then set $h=j+1$ otherwise set $h=j$. Notice that no vertex in $\{y_1,\dots,y_n,y_1',\dots,y_n'\}$ belongs to $\delta_{l^*}(f_h)$. Using this fact, we have that any vertex $x\in\delta_{l^*}(f_h)$ is incident to an edge in $(f_1,\dots,f_h)-E'=E_i$ and to an edge in $(f_1,\dots,f_h)-E'=E(G)-E_i$. Notice now that all the vertices of $\delta_{l^*}(f_h)$ are occupied by a searcher in G as they are incident both to a clear edge (an edge in E_i) and to a contaminated edge (an edge in $E(G)-E_i$). Clearly, if we remove all the other searchers, no recontamination will occur. Let v,u be the endpoints of f_{h+1}. In case $|\{v,u\}\cup\delta_{l^*}(f_h)|\leqslant k$, we place new searchers on the endpoints of f_{h+1} and clear it. We can now assume that $|\{v,u\}\cup\delta_{l^*}(f_h)|>k$ and, as $\delta_{l^*}(f_h)\leqslant k$, at most one of the endpoints of f_{h+1}, is guarded (i.e. is occupied by some searcher). We now claim that exactly one of the endpoints of f_{h+1} is guarded. Indeed, using $\delta_{l^*}(f_h)\cap\{v,u\}=\emptyset$, we can prove that $\delta_{l^*}(f_{h+1})=\delta_{l^*}(f_h)\cup\{v,u\}$, and, as $|\delta_{l^*}(f_{h+1})|\leqslant k$, we conclude that $|\delta_{l^*}(f_h)|\leqslant k-2$, a contradiction to the assumption that $|\{v,u\}\cup\delta_{l^*}(f_h)|>k$.

W.l.o.g. we assume that v is the unguarded endpoint of f_{h+1}. As v is unguarded, either it is incident only to contaminated edges or only to clear edges in G. The second case is impossible as f_{h+1} is contaminated. If v is incident only to f_{h+1} in G, this means that v has degree 1 in G and therefore has degree 2 in ΛG. It is now clear that, in any case, v has degree $\geqslant 2$ in ΛG and therefore $v \in \delta_{l^*}(f_{h+1})$.

Notice that relations $|\{v,u\} \cup \delta_{l^*}(f_h)| > k$ and $v \in \delta_{l^*}(f_h)$ give that $|\delta_{l^*}(f_h)| \geqslant k$. Suppose now that u is incident to a contaminated edge different from f_{h+1}. This means that $\delta_{l^*}(f_h) \subseteq \delta_{l^*}(f_{h+1})$ and as, $v \notin \delta_{l^*}(f_h)$ and $v \in \delta_{l^*}(f_{h+1})$, we have that $|\delta_{l^*}(f_h)| < |\delta_{l^*}(f_{h+1})| \leqslant k$, a contradiction. Therefore, u is incident only with clear edges in $G - f_{h+1}$ and thus, we can clear f_{h+1} by sliding the searcher guarding u along f_{h+1} to v (i.e. applying $c(u,v)$), without causing any recontamination. This completes the proof of the fact that $\text{ms}(G) \leqslant \text{linear-width}(\Lambda G)$.

Suppose now that there exists a mixed search for G that uses k searchers. From Theorem 22(ii) we have that $\text{linear-width}(\Lambda G) \leqslant \text{ms}(\Lambda G)$. Therefore, it is enough to prove that $\text{ms}(\Lambda G) = \text{ms}(G)$. This fact follows from Lemma 23(i) by induction on the number of fully pendant vertices of ΛG. □

Notice that Theorem 24 is an extension of Theorem 22(iii). We summarize the consequences of Theorem 24 into the following theorem.

Theorem 25. *The following hold.*

(i) *The problem of computing* linear-width *is* NP-*complete*.

(ii) *There exists an algorithm that given a tree T computes* $\text{linear-width}(T)$ *in* $O(|V(T)|)$ *time*.

(iii) *One can construct a linear-time algorithm that, given a graph G, checks whether G has mixed (edge) search number at most 2 and, if so, outputs a mixed (edge) search strategy that uses the minimum number of searchers.*

(iv) *For any graph G*, $\text{linear-width}(G) \leqslant \text{ms}(G) \leqslant \text{linear-width}(G) + 1$.

Proof. (i) The NP-completeness of linear-width follows directly from Theorem 24 and the fact that computing $\text{ms}(G)$ is an NP-hard problem (see [39]).

(ii) The existence of an algorithm computing linear-width of trees is a consequence of Theorem 24 and the fact that there exists an algorithm that given a tree T computes $\text{ms}(T)$ in $O(|V(T)|)$ time (see [39]).

(iii) The result is trivial in case $\text{ms}(G) \leqslant 1$ ($\text{es}(G) \leqslant 1$). Using now Theorems 24 and 22(iv) we have that, in order to check whether $\text{ms}(G) \leqslant 2$ ($\text{es}(G) \leqslant 2$), it is enough to apply $\mathsf{LW2}(\Lambda G)$ ($\mathsf{LW2}(\Lambda G^e)$). If this is the case, $\mathsf{LW2}(\Lambda G)$ ($\mathsf{LW2}(\Lambda G^e)$) will output an edge ordering of ΛG (ΛG^e). It is not hard to see that, following the machinery of the proof of Theorem 24, this edge ordering can be transformed to a mixed (edge) search in linear time.

(iv) One can easily verify that $\text{linear-width}(\Lambda G) \leqslant \text{linear-width}(G)+1$. The required is now a direct consequence of this fact and Theorem 24. □

5.3. The acyclic minor minimal graphs with linear-width $> k$

It is easy to check that $\mathrm{ob}(\mathscr{G}[\mathrm{ms},1]) = \{K_3, K_{1,3}\}$. $\mathrm{ob}(\mathscr{G}[\mathrm{ms},2])$ has been determined by Takahashi et al. [38] and consists of 36 graphs. We also mention that Lemma 2 holds if we replace linear-width by the mixed search number and thus, for every k, $\mathrm{ob}(\mathscr{G}[\mathrm{ms},k])$ contains only connected graphs.

The following lemma is an immediate corollary of Lemma 4 and 23.

Lemma 26. *The following hold.*
(i) *No graph in* $\mathrm{ob}(\mathscr{G}[\mathrm{ms},k])$ *contains edges that are fully pendant or redundant.*
(ii) *No graph in* $\mathrm{ob}(\mathscr{G}[\text{linear-width},k])$ *contains edges that are simply pendant or redundant.*

Lemma 27. *For any* $k \geqslant 1$, $G \in \mathrm{ob}(\mathscr{G}[\mathrm{ms},k]) \Rightarrow \Lambda G \in \mathrm{ob}(\mathscr{G}[\text{linear-width},k])$.

Proof. Suppose that G is a graph in $\mathrm{ob}(\mathscr{G}[\mathrm{ms},k])$. Clearly $\mathrm{ms}(G) > k$ and, as G is minor minimal, $\forall_{H \prec G}\ \mathrm{ms}(H) \leqslant k$. From Theorem 24, linear-width$(\Lambda G) > k$. Suppose, towards a contradiction, that $\Lambda G \notin \mathrm{ob}(\mathscr{G}[\text{linear-width},k])$ and thus, there exists some edge $e \in E(\Lambda G)$ such that either linear-width$(\Lambda G - e) > k$ or linear-width$(\Lambda G \dot{-} e) > k$. In any case, we will find a proper minor of G that has linear-width $> k$. From Lemma 26(i) we have that ΛG does not contain any redundant edges. We examine three cases:

Case 1. $e = \{u, u'\}$ is a fully pendant edge of ΛG. We observe that one of u, u', say u' is a fully pendant vertex in G. Notice also that $\Lambda G - \{u, u'\}$ and $\Lambda G \dot{-} \{u, u'\}$ are isomorphic. For reasons of simplicity, we will denote both of them as $\Lambda G - \{u, u'\}$. Let v be the, unique, neighbor of u in $\Lambda G - \{u, u'\}$. As ΛG does not contain redundant edges, $\{v, u\}$ is the unique simply pendant edge in $\Lambda G - \{u, u'\}$ and from Lemma 4(i) we have that linear-width$((\Lambda G - \{u, u'\}) - \{v, u\}) > k$. Notice that, from Theorem 24, $\mathrm{ms}(\Lambda^{-1}((\Lambda G - \{u, u'\}) - \{v, u\})) > k$. As $\Lambda^{-1}((\Lambda G - \{u, u'\}) - \{v, u\}) \prec G$, we have a contradiction.

Case 2. $e = \{v, u\}$ is an almost pendant edge of ΛG. The case where $\mathrm{lw}(G \dot{-} \{v, u\}) > k$ is similar to Case 1. We assume that $\mathrm{lw}(\Lambda G - \{v, u\}) > k$. Let u be the almost pendant vertex of $\{v, u\}$ and u' be the fully pendant vertex of ΛG that is adjacent to u. Clearly, vertices u', u induce one, of the two connected components of $\Lambda G - \{v, u\}$. We denote the other as H. From Lemma 2, we have that linear-width$(H) > k$. Using now Theorem 24, we have that $\mathrm{ms}(\Lambda^{-1} H) > k$. As $\Lambda^{-1} H \prec G$, we have a contradiction.

Case 3. $e = \{x, y\}$ is not a fully or an almost pendant edge of ΛG. If linear-width $(\Lambda G - e) > k$ or linear-width$(\Lambda G \dot{-} e) > k$ then, from Theorem 24 we have that either $\mathrm{ms}(\Lambda^{-1}(\Lambda G - e)) > k$ or $\mathrm{ms}(\Lambda^{-1}(\Lambda G \dot{-} e)) > k$. Using now the fact that e is not a fully or an almost pendant edge of ΛG, one can easily see that $\Lambda^{-1}(\Lambda G - e)$ $(\Lambda^{-1}(\Lambda G \dot{-} e))$ is isomorphic to a proper minor of $G - e$ $(G \dot{-} e)$, a contradiction. □

According to Lemma 27, Λ is an injection from $\mathrm{ob}(\mathscr{G}[\mathrm{ms},k])$ to $\mathrm{ob}(\mathscr{G}[$linear-width, $k])$. Using this fact, it is easy to determine $\mathrm{ob}(\mathscr{G}[\mathrm{ms},k])$ if we know $\mathrm{ob}(\mathscr{G}[$linear-width, k]). Indeed, if we apply Λ^{-1} on all the graphs in ob($\mathscr{G}$[linear-width, k]), we will obtain a set $\mathscr{M}$ of graphs containing ob($\mathscr{G}$[ms, k]) as a subset. We can now obtain ob($\mathscr{G}$[ms, k]) from $\mathscr{M}$ by discarding all the graphs having proper minors in $\mathscr{M}$ (i.e. we keep only the minor minimal elements).

Using the above methodology, we can directly verify the result of Takahashi et al. [38]. One can easily see that ob($\mathscr{G}$[ms, k]) can be obtained if we apply Λ^{-1} on the 36 underlined graphs depicted in the appendix (Fig. 13).

We now denote by aob($\mathscr{G}$[linear-width, k]) (aob($\mathscr{G}$[ms, k])) the set consisting of the acyclic graphs in ob($\mathscr{G}$[linear-width, k]) (ob($\mathscr{G}$[ms, k])). Let $G_i, i=1,2,3$, be a triple of v_i-graphs ($i=1,2,3$) and let v be a vertex such that $v \notin V(G_1) \cup V(G_2) \cup V(G_3)$. We call the graph $G_1 \cup G_2 \cup G_3 \cup (\{v, v_1, v_2, v_3\}, \{\{v, v_1\}, \{v, v_2\}, \{v, v_3\}\})$ star-composition of $G_i, i=1,2,3$.

The following has been proved by Takahashi et al. [37].

Theorem 28. *Let $k \geqslant 2$. A tree T is in* aob($\mathscr{G}$[ms, k]) *iff T is a star decomposition of three* (*not necessarily distinct*) *graphs in* aob($\mathscr{G}$[ms, $k-1$]).

Notice that, as aob($\mathscr{G}$[ms, 1]) $=\{K_{1,3}\}$, Theorem 28 explicitly determines aob($\mathscr{G}$[ms, k]) for any $k \geqslant 1$. The following theorem shows that aob($\mathscr{G}$[ms, k]) and aob($\mathscr{G}$[linear-width, k]) are not very different.

Theorem 29. *Let T be tree and $k \geqslant 1$. Then, $T \in$* aob($\mathscr{G}$[ms, k])$\Lambda \Leftrightarrow \Lambda T \in$ aob ($\mathscr{G}$[linear-width, k]).

Proof. The "$\Rightarrow$" direction follows from Lemma 27. Now let $T \in$ aob($\mathscr{G}$[linear-width, k]). From Theorem 24 we have that $\text{ms}(\Lambda^{-1}T) > k$. Let $e \in E(\Lambda^{-1}T)$. We will prove that $\text{ms}(\Lambda^{-1}T - e) \leqslant k$ and $\text{ms}(\Lambda^{-1}T \dot{-} e) \leqslant k$. Suppose on the contrary, that for some edge $e \in E(\Lambda^{-1}T)$ either $\text{ms}(\Lambda^{-1}T - e) > k$ or $\text{ms}(\Lambda^{-1}T \dot{-} e) > k$. We examine first the case where $e = \{v, u\}$ is a pendant edge of $\Lambda^{-1}T$. W.l.o.g we assume that $d_{\Lambda^{-1}T}(u) = 1$. From Lemma 26(ii) we have that, in T, u is an almost pendant vertex adjacent to some pendant vertex u'. Moreover, from the same lemma, $\{v, u\}$ is a simply pendant edge in $\Lambda^{-1}(T)$ and the removal or the contraction of it does not result in the appearance of a new pendant edge. One can now see that $\Lambda(\Lambda^{-1}T - \{u, v\})$ is isomorphic to $(T - \{v, u\}) - \{u, u'\} \prec T$, which is a contradiction, as, from Theorem 24, linear-width($\Lambda(\Lambda^{-1}T - \{u, v\})) > k$ or linear-width($\Lambda(\Lambda^{-1}T \dot{-} \{u, v\})) > k$. Suppose now that $e = \{v, u\}$ is not a pendant edge of $\Lambda^{-1}T$. We examine two cases.

Case 1. $\text{ms}(\Lambda^{-1}T - e) > k$. We notice first that $\Lambda^{-1}T - e$ consists of two connected components T_1, T_2. Using Lemma 2, we may assume, w.l.o.g, that $\text{ms}(T_1) > k$. From Theorem 24, we have that linear-width($\Lambda T_1) > k$. It is now easy to see that that $\Lambda T_1 \prec T$, a contradiction.

Case 2. $\text{ms}(\Lambda^{-1}T \dot{-} e) > k$. We first claim that after the contraction of e no new pendant edge appears. Notice that the only case where the contraction of a non-pendant edge e results to the appearance of a new pendant edge is the case where exists a vertex adjacent with both of the endpoints of e. As T is a tree, this case must be excluded and the claim holds. Using now the claim, we can see that $\Lambda(\Lambda^{-1}T\dot{-}e)$ is isomorphic to $T\dot{-}e \prec T$. We now have a contradiction as, from Theorem 24, linear-width$(\Lambda(\Lambda^{-1}T\dot{-}e)) > k$. □

From Theorem 29, we have that Λ is an bijection from aob($\mathscr{G}$[ms, k]) to aob($\mathscr{G}$[linear-width, k]). Using this fact and Theorem 28, we can determine all the acyclic graphs in ob($\mathscr{G}$[linear-width, k]) for any $k \geqslant 1$. We can easily conclude the following result.

Theorem 30. *If* $T \in$ aob($\mathscr{G}$[linear-width, k]), *then* $|V(T)| = (3^{k+1} + 2 \cdot 3^k - 1)/2$. *Moreover*, $|$aob($\mathscr{G}$[linear-width, k])$| \geqslant (k!)^2$.

We mention that, according to [37], the cardinality of aob($\mathscr{G}$[linear-width, k]), for $k = 1, 2, 3$, and 4 is 1, 4, 1,330, and 2,875,919,312,080, respectively.

6. Conclusions — open problems

Theorem 30 suggests that a complete structural characterization of $\mathscr{G}$[linear-width, k] is not easy to find for $k > 2$. However, we believe that a more general version of the distinction between marginal and non-marginal bolbes, that we followed in this paper, can be applied in the more general cases.

Concluding, we mention that it is worth to investigate the following.

Conjecture. *For any* $k \geqslant 1$, *any graph in* ob($\mathscr{G}$[linear-width, k]) *with maximum number of vertices is a tree.*

A consequence of the above would be that no graph in ob[linear-width, k] has more than $f(k) = (3^{k+1} + 2 \cdot 3^k - 1)/2$ vertices. If this is true, we can enumerate all the graphs having at most $f(k)$ vertices, detect those that are minor minimal graphs that do not belong in $\mathscr{G}$[linear-width, k], and end up with ob($\mathscr{G}$[linear-width, k]) for any $k \geqslant 2$. Clearly, such a procedure would be rather impractical because of the immense number of graphs that have to be checked. Nevertheless, one could make it more efficient by applying further restrictions on the graphs enumerated (for example, graphs in ob($\mathscr{G}$[linear-width, k]) cannot have redundant or simply pendant edges). These restrictions can be based on some partial characterization of $\mathscr{G}$[linear-width, k] (for the case where $k = 2$, such a partial characterization could be the one of smoothness that we defined in Section 2.4).

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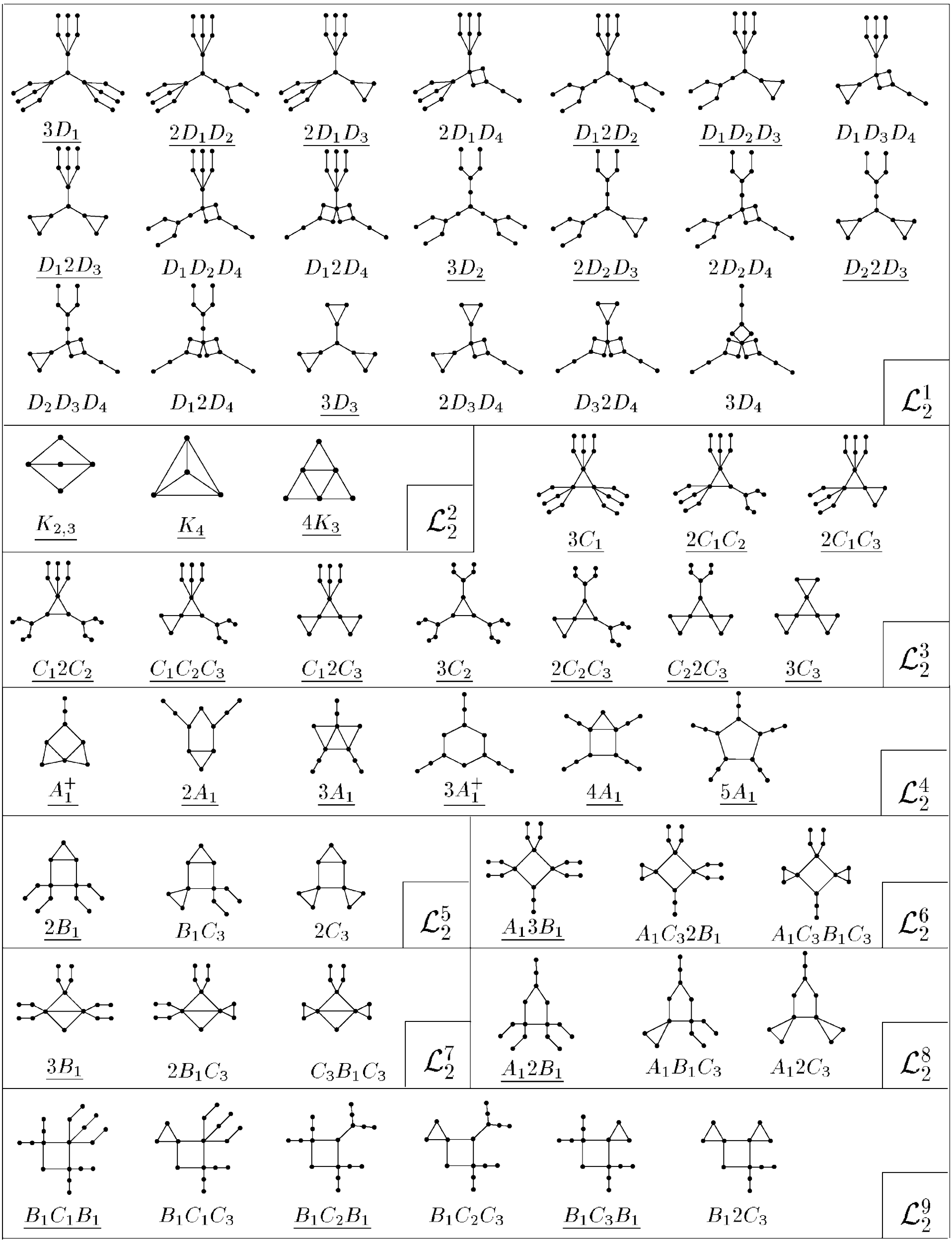

Fig. 13. The obstruction set $\mathscr{L}_2 = \mathscr{L}_2^1 \cup \mathscr{L}_2^2 \cup \mathscr{L}_2^3 \cup \mathscr{L}_2^4 \cup \mathscr{L}_2^5 \cup \mathscr{L}_2^6 \cup \mathscr{L}_2^7 \cup \mathscr{L}_2^8 \cup \mathscr{L}_2^9$.

Electronic Engineering of Tokyo Institute of Technology on February 1997, motivated the present research. Finally, I acknowledge the helpful and detailed comments of the referees that substantially improved the presentation of this paper.

Appendix

The obstruction set $\mathscr{L}_2 = \mathscr{L}_2^1 \cup \mathscr{L}_2^2 \cup \mathscr{L}_2^3 \cup \mathscr{L}_2^4 \cup \mathscr{L}_2^5 \cup \mathscr{L}_2^6 \cup \mathscr{L}_2^7 \cup \mathscr{L}_2^8 \cup \mathscr{L}_2^9$ for the class of graphs with linear-width $\leqslant 2$ is shown in Fig. 13.

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