



# LIFO-search: A min–max theorem and a searching game for cycle-rank and tree-depth<sup>☆</sup>

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## ABSTRACT

We introduce a variant of the classic node search game called LIFO-search where searchers are assigned different numbers. The additional rule is that a searcher can be removed only if no searchers of lower rank are in the graph at that moment. We show that all common variations of the game require the same number of searchers. We then introduce the notion of (directed) shelters in (di)graphs and prove a min–max theorem implying their equivalence to the cycle-rank/tree-depth parameter in (di)graphs. As (directed) shelters provide escape strategies for the fugitive, this implies that the LIFO-search game is monotone and that the LIFO-search parameter is equivalent to the one of cycle-rank/tree-depth in (di)graphs.

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## 1. Introduction

Graph searching games are increasingly becoming a popular way to characterize, and even define, practical graph parameters. There are many advantages to a characterization by graph searching games: it provides a useful intuition which can assist in constructing more general or more specific parameters; it gives insights into relations with other, similarly characterized parameters; and it is particularly useful from an algorithmic perspective as many parameters associated with such games are both structurally robust and efficiently computable.

One of the most common graph searching games is the node-search game. In this game several searchers and one fugitive occupy vertices of the graph and make simultaneous moves. The (omniscient) fugitive moves along searcher-free paths of arbitrary length whereas the searchers' movements are not constrained by the topology of the graph. The goal of the game is to minimize the number of searchers required to capture the fugitive by cornering him in some part of the graph and placing a searcher on the same vertex. This game has been extensively studied [7] and several important graph parameters such as treewidth [28] and pathwidth [18] can be characterized by natural variants of this game. One variation frequently used, indeed the one which separates treewidth and pathwidth, is whether the location of the fugitive is known or unknown to the searchers. Another common variation is whether the searchers use a monotone or a non-monotone searching strategy, that is, whether their strategy provides to the fugitive access to already searched areas (non-monotone strategy)

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or not (monotone strategy). Monotone search strategies lead to algorithmically useful decompositions, whereas non-monotone strategies are more robust under graph operations and hence reflect structural properties. Therefore, showing that monotone strategies require no more searchers than non-monotone strategies is an important and common question in the area. Whilst node-search games on undirected graphs tend to enjoy monotonicity [4,28,20], on digraphs the situation is much less clear [2,1,19].

Node-search games naturally extend to digraphs. However, in the translation another variation arises depending on how one views the constraints on the movement of the fugitive. One interpretation is that in the undirected case the fugitive moves along paths, so the natural translation would be to have the fugitive move along directed paths. Another view is that the fugitive moves to some other vertex in the same connected component, and here the natural translation would be to have the fugitive move within the same strongly connected component. Both interpretations have been studied in the literature, the former giving characterizations of parameters such as DAG-width [3,26] and directed pathwidth [2] and the latter giving a characterization of directed treewidth [16].

We define a variant of the node-search game in which only the most recently placed searchers may be removed; that is, the searchers must move in a last-in-first-out (LIFO) manner and we show that the minimum number of searchers required to capture a fugitive on a (di)graph with a LIFO-search is independent of:

- Whether the fugitive is invisible or visible,
- Whether the searchers use a monotone or non-monotone search, and
- Whether the fugitive is restricted to moving in searcher-free strongly connected components or along searcher-free directed paths.

This result is somewhat surprising: in the standard node-search game these options give rise to quite different parameters [2,3,19].

We show that on digraphs the LIFO-search game characterizes a pre-existing measure, cycle-rank—one of the possible generalizations of tree-depth to digraphs (though as the definition of cycle-rank predates tree-depth by several decades, it is perhaps more correct to say that tree-depth is an analogue of cycle-rank on undirected graphs). The cycle-rank of a digraph is an important parameter relating digraph complexity to other areas such as regular language complexity and asymmetric matrix factorization. It was defined by Eggan in [9], where it was shown to be a critical parameter for determining the star-height of regular languages. The success of tree-depth [10,14,12] rekindled interest in it as an important digraph parameter, especially from an algorithmic perspective.

It is well known that tree-depth can be characterized by a node-search game where a visible fugitive plays against searchers that are only placed and never moved [12]. In that paper, Ganian et al. considered one extension of this game to digraphs. Here we consider another natural extension, where the visible fugitive moves in strongly connected sets, and show that it also characterizes cycle-rank. From the above, we also obtain that the LIFO-search parameter is equivalent to the one of tree-depth.

Our final result uses these graph searching characterizations to define a dual parameter that characterizes structural obstructions for cycle-rank. We consider two kinds of obstructions. The first one is obtained from defining the notion of directed shelters. The second one is motivated by the havens of [16]. Both the directed shelters and LIFO-havens define simplified strategies for the fugitive. The game characterization then implies that these structural features are necessarily present when the cycle-rank of a graph is large. By showing that the aforementioned simplified strategies are also sufficient for the fugitive, we obtain a rare instance of an exact min–max theorem relating digraph parameters. This also implies that the notion of shelters when transferred to simple graphs characterizes structural obstructions for tree-depth.

The results of this paper can be summarized with the following characterizations of cycle-rank and tree-depth respectively.

**Theorem.** *Let  $G$  be a digraph, and  $k$  a positive integer. The following are equivalent:*

- (i)  $G$  has cycle-rank  $\leq k - 1$ ,
- (ii) On  $G$ ,  $k$  searchers can capture a fugitive with a LIFO-search strategy,
- (iii) On  $G$ ,  $k$  searchers can capture a visible fugitive restricted to moving in strongly connected sets with a searcher-stationary search strategy,
- (iv)  $G$  has no LIFO-haven of order  $> k$ , and
- (v)  $G$  has no directed shelter of thickness  $> k$ .

**Theorem.** *Let  $G$  be a non-empty graph and  $k$  be a positive integer. Then the following are equivalent.*

- (i)  $G$  has tree-depth at most  $k$ .
- (ii) there is a monotone LIFO-search strategy in  $G$  of cost at most  $k$  that captures an invisible and agile fugitive.
- (iii) there is a LIFO-search strategy in  $G$  of cost at most  $k$  that captures an invisible and agile fugitive.
- (iv) every shelter in  $G$  has thickness at most  $k$ .
- (v) there is a monotone LIFO-search strategy in  $G$  using  $k$  searchers against a visible and agile fugitive.

The paper is organized as follows. In Section 2 we recall the definitions and notation that we use throughout the paper. In Section 3 we define the LIFO-search and searcher-stationary games and show that they characterize cycle-rank. In Section 4 we prove the min–max theorem for cycle-rank. In Section 5 we consider simple graphs and argue that our results imply the existence of a min–max theorem for LIFO-search and that the LIFO-search parameter is equivalent to the one of tree-depth, and in Section 6 we conclude with a discussion on further research and open problems.

## 2. Preliminaries

All (di)graphs in this paper are finite, simple, (directed) and without self-loops, although the results readily extend to multigraphs with self-loops. For simplicity, we also assume that all (di)graphs contain at least one vertex unless explicitly mentioned. We use standard notation and terminology, in particular  $V(G)$  and  $E(G)$  denote the sets of vertices and edges respectively of a (di)graph  $G$  and between (di)graphs,  $\subseteq$  denotes the subgraph relation. We will often interchange an induced subgraph with the set of vertices which defines it, in particular strongly connected sets of vertices are sets of vertices that induce a strongly connected subgraph, and we will often view strongly connected components as sets of vertices. Given a (di)graph  $G$  and a set of vertices  $X \subseteq V(G)$ , we use  $G \setminus X$  to denote the subgraph of  $G$  induced by  $V(G) \setminus X$ .

Given a finite set  $V$ , we use  $\mathcal{P}(V)$  to denote its powerset,  $V^*$  to denote the set of finite words over  $V$ , and  $V^{<k}$  to denote the set of words over  $V$  of length  $<k$ . We use  $\epsilon$  to denote the empty word and  $\cdot$  or juxtaposition to denote concatenation. For  $X, Y \in V^*$  we write  $X \preceq Y$  if  $X$  is a prefix of  $Y$ , that is, if there exists a word  $Z \in V^*$  such that  $Y = X \cdot Z$ . For  $X = a_1 a_2 \cdots a_n \in V^*$ , we use  $|X|$  to denote the length of  $X$ , and  $\{|X|\}$  to denote the set  $\{a_1, a_2, \dots, a_n\}$ . Given two sets  $A$  and  $B$  we use  $A \Delta B$  to denote their symmetric difference, that is  $A \Delta B = (A \cup B) \setminus (A \cap B)$ . Given a set  $\mathcal{X} \subseteq \mathcal{P}(V)$  of subsets of  $V$ , a  $\subseteq$ -chain is a subset  $\{X_1, \dots, X_n\} \subseteq \mathcal{X}$  such that  $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n$ . If there is no  $Y \in \mathcal{X}$  such that  $Y \subset X_i, X_i \subset Y \subset X_{i+1}$  for some  $i$ , or  $X_n \subset Y$ , then  $\{X_1, \dots, X_n\}$  is a *maximal*  $\subseteq$ -chain.

**Definition 1** (Cycle-Rank). The *cycle-rank* of a digraph  $G$ ,  $\text{cr}(G)$ , is defined as follows:

- If  $G$  is acyclic then  $\text{cr}(G) = 0$ .
- If  $G$  is strongly connected then  $\text{cr}(G) = 1 + \min_{v \in V(G)} \text{cr}(G \setminus \{v\})$ .
- Otherwise  $\text{cr}(G) = \max_H \text{cr}(H)$  where the maximum is taken over all strongly connected components  $H$  of  $G$ .

We postpone the definition of tree-depth until Section 5.

## 3. Searching games for cycle-rank

We begin by formally defining the LIFO-search game, and its variants, for digraphs. Each variation of the LIFO-search game gives rise to a digraph parameter corresponding to the minimum number of searchers required to capture the fugitive under the given restrictions. The main result of this section is that for any digraph all these parameters are equal. Furthermore, we show they are all equal to one more than the cycle-rank of the digraph.

### 3.1. LIFO-search for digraphs

In summary, for the graph searching game in which we are interested the fugitive can run along searcher-free directed paths of any length, the searchers can move to any vertex in the graph, and the fugitive moves whilst the searchers are relocating. In this game, as in the classical node search game, the searchers first announce their move, then the fugitive moves taking into account this information and finally the searchers carry their already announced move. The searcher's strategy may apply two types of moves in each step: either placement of a searcher on a vertex or removal of a searcher from a vertex with the restriction that only the most recently placed searchers may be removed. If a searcher is placed on the fugitive then he/she is captured and the searchers win, otherwise the fugitive wins. The goal is to determine the minimum number of searchers required to capture the fugitive. The variants we are primarily interested in are whether the fugitive is visible or invisible, and whether or not the fugitive must stay within the same strongly connected component when he/she is moving. As our fundamental definitions are dependent on these two options, we define four *game variants*:  $i$ ,  $isc$ ,  $v$ ,  $vsc$ , corresponding to the visibility of the fugitive and whether he/she is constrained to moving within strongly connected components, that is,  $i$  and  $v$  correspond to an invisible and a visible fugitive respectively while  $isc$  and  $vsc$  correspond to an invisible and visible fugitive as above with the addition that the fugitives move inside the same strongly connected component. Then we parameterize our definitions by these variants.

Let us fix a digraph  $G$ . A position in a LIFO-search on  $G$  is a pair  $(X, R)$  where  $X \in V(G)^*$  and  $R$  is a (possibly empty) induced subgraph of  $G \setminus \{|X|\}$ . Intuitively  $X$  represents the position and ordered placement of the searchers and  $R$  represents the part of  $G$  that the fugitive can reach (in the visible case) or the set of vertices where he/she might possibly be located (in the invisible case). We say that a position  $(X, R)$  is

- an  *$i$ -position* if there are no edges in  $G$  from  $R$  to  $G \setminus R$ ,
- an  *$isc$ -position* if it is a union of strongly connected components of  $G \setminus \{|X|\}$ ,

- a *v*-position if there are no edges in  $G$  from  $R$  to  $G \setminus R$  and  $G[R]$  has a strongly connected component  $C$  with no edges from  $G \setminus C$  to  $C$ , and
- a *vsc*-position if  $R$  is a strongly connected component of  $G \setminus \{X\}$ .

To reflect how the game transitions to a new position during a round of the game we say, for  $\mathbf{gv} \in \{i, \text{isc}, v, \text{vsc}\}$ , a *gv*-position  $(X', R')$  is a *gv*-successor of  $(X, R)$  if either  $X \leq X'$  or  $X' \leq X$ , with  $|\{X\} \Delta \{X'\}| = 1$ , and

- (for  $\mathbf{gv} \in \{i, v\}$ ) For every  $v' \in V(R')$  there is a  $v \in V(R)$  and a directed path in  $G \setminus (\{X\} \cup \{X'\})$  from  $v$  to  $v'$ , or
- (for  $\mathbf{gv} \in \{\text{isc}, \text{vsc}\}$ ) For every  $v' \in V(R')$  there is a  $v \in V(R)$  such that  $v$  and  $v'$  are contained in the same strongly connected component of  $G \setminus (\{X\} \cup \{X'\})$ .

Ideally we would like to assume games start from  $(\epsilon, G)$ , however in the visible variants of the game this might not be a legitimate position. Thus, for  $\mathbf{gv} \in \{v, \text{vsc}\}$ , if  $(\epsilon, G)$  is not a *gv*-position we include it as a special case, and set as its *gv*-successors all *gv*-positions of the form  $(\epsilon, R)$ . We observe that in all variants, the successor relation is monotone in the sense that if  $(X, R)$  and  $(X, S)$  are positions with  $S \subseteq R$  and  $(X', S')$  is a successor of  $(X, S)$ , then there is a successor  $(X', R')$  of  $(X, R)$  with  $S' \subseteq R'$ .

For  $\mathbf{gv} \in \{i, \text{isc}, v, \text{vsc}\}$ , a (*gv*-LIFO)-search in a digraph  $G$  from *gv*-position  $(X, R)$  is a (finite or infinite) sequence of *gv*-positions  $(X, R) = (X_0, R_0), (X_1, R_1), \dots$  where for all  $i \geq 0$ ,  $(X_{i+1}, R_{i+1})$  is a *gv*-successor of  $(X_i, R_i)$ . A LIFO-search is *complete* if either  $R_n = \emptyset$  for some  $n$ , or it is infinite. We observe that if  $R_n = \emptyset$ , then  $R_{n'} = \emptyset$  for all  $n' \geq n$ .

We say that a complete LIFO-search is *winning for the searchers* if  $R_n = \emptyset$  for some  $n$ , otherwise it is winning for the fugitive. A complete LIFO-search from  $(\epsilon, G)$

- is *monotone* if  $R_{i+1} \subseteq R_i$  for all  $i$ , that is, the fugitive does not occupy positions of the graph from which he/she has already been banned,
- is *searcher-stationary* if  $X_i \leq X_{i+1}$  for all  $i$  where  $R_i \neq \emptyset$ , and
- *uses at most  $k$  searchers* if  $|X_i| \leq k$  for all  $i$ .

Whilst a complete LIFO-search from  $(\epsilon, G)$  describes a single run of the game, we are more interested in the cases where one of the players (particularly the searchers) can always force a win, no matter what the other player chooses to do. For this, we introduce the notion of a strategy. For  $\mathbf{gv} \in \{i, \text{isc}, v, \text{vsc}\}$ , a (*searcher*)-*gv* is a (partial<sup>1</sup>) function  $\sigma$  from the set of all *gv*-positions to  $V(G)^*$  such that for all  $(X, R)$ ,  $\sigma(X, R)$  is the first component of a *gv*-successor of  $(X, R)$ ; so with the possible exception of  $(X, R) = (\epsilon, G)$ , either  $\sigma(X, R) \leq X$  or  $X \leq \sigma(X, R)$ . A *gv*-LIFO-search  $(X_0, R_0), (X_1, R_1), \dots$  is *consistent* with a *gv*-strategy  $\sigma$  if  $X_{i+1} = \sigma(X_i, R_i)$  for all  $i \geq 0$ . A strategy  $\sigma$  is *winning* from  $(X, R)$  if all complete LIFO-searches from  $(X, R)$  consistent with  $\sigma$  are winning for the searchers. Likewise, a strategy is *monotone* (searcher-stationary, uses at most  $k$  searchers) if all consistent complete LIFO-searches from  $(\epsilon, G)$  are monotone (searcher-stationary, use at most  $k$  searchers respectively). We say  $k$  searchers can capture a fugitive on  $G$  in the *gv*-game with a (monotone) LIFO-search strategy if there is a (monotone) *gv*-strategy that uses at most  $k$  searchers and is winning for the searchers from  $(\epsilon, G)$ .

For  $\mathbf{gv} \in \{i, \text{isc}, v, \text{vsc}\}$ , we define the (monotone) *gv*-LIFO-search number of  $G$ ,  $\text{LIFO}^{\mathbf{gv}}(G)$  ( $\text{LIFO}^{\text{mgv}}(G)$ ), as the minimum  $k$  for which there is a (monotone) winning *gv*-strategy that uses at most  $k$  searchers. We also define the visible, strongly connected, searcher-stationary search number of  $G$ ,  $\text{SS}^{\text{vsc}}(G)$  as the minimum  $k$  for which there is a searcher-stationary winning *vsc*-strategy that uses at most  $k$  searchers.

In Section 4 we will also consider fugitive *gv*-strategies: a partial function  $\rho$  from  $V(G)^* \times \mathcal{P}(G) \times V(G)^*$  to induced subgraphs of  $G$ , defined for  $(X, R, X')$  if  $(X, R)$  is a *gv*-position and  $X'$  is the first component of a *gv*-successor of  $(X, R)$ . A LIFO-search  $(X_0, R_0), (X_1, R_1), \dots$  is *consistent* with a fugitive *gv*-strategy  $\rho$  if  $R_{i+1} = \rho(X_i, R_i, X_{i+1})$  for all  $i \geq 0$ , and a fugitive strategy is *winning* if all consistent complete LIFO-searches are winning for the fugitive. In this section, a strategy will always refer to a searcher strategy.

### 3.2. Relating the digraph searching parameters

We observe that in all game variants, a strategy that is winning from  $(X, R)$  can be used to define a strategy that is winning from  $(X, R')$  for any  $R' \subseteq R$ : the searchers can play as if the fugitive is located in the larger space; and from the monotonicity of the successor relation, the assumption that the actual set of locations of the fugitive is a subset of the assumed set of locations remains invariant. One consequence is that a winning strategy on  $G$  defines a winning strategy on any subgraph of  $G$ , so the search numbers we have defined are monotone with respect to the subgraph relation.

**Proposition 1.** *Let  $G$  be a digraph and  $G'$  a subgraph of  $G$ . Then:*

- $\text{SS}^{\text{vsc}}(G') \leq \text{SS}^{\text{vsc}}(G)$ , and
- $\text{LIFO}^{\mathbf{gv}}(G') \leq \text{LIFO}^{\mathbf{gv}}(G)$  for  $\mathbf{gv} \in \{i, \text{isc}, v, \text{vsc}, \text{mi}, \text{misc}, \text{mv}, \text{mvsc}\}$ .

<sup>1</sup> A strategy need only be defined for all positions  $(X, R)$  that can be reached from  $(\epsilon, G)$  in a LIFO-search consistent with the strategy. However, as this definition is somewhat circular, we assume strategies are total.

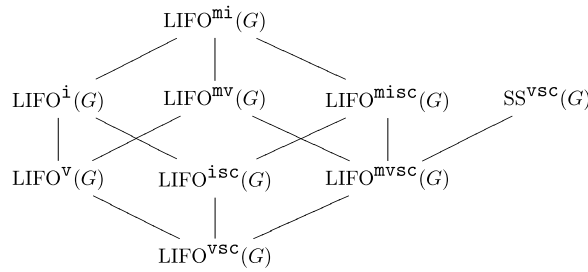


Fig. 1. Trivial relations between digraph searching parameters.

Another consequence is that a winning strategy in the invisible fugitive variant defines a winning strategy when the fugitive is visible; and a winning strategy when the fugitive is not constrained to moving within strongly connected components defines a winning strategy when he/she is. This corresponds to our intuition of the fugitive being more (or less) restricted. Also, in all game variants, a monotone winning strategy is clearly a winning strategy, and because a searcher-stationary LIFO-search is monotone, a winning searcher-stationary strategy is a monotone winning strategy. These observations yield several inequalities between the search numbers defined above. For example  $LIFO^{vsc}(G) \leq LIFO^{mi}(G)$  as any winning monotone *i*-strategy is also a winning *vsc*-strategy. The full set of these relationships is shown in a Hasse diagram in Fig. 1, with the larger measures towards the top.

The main result of this section is that all these digraph parameters are equal to one more than cycle-rank.

**Theorem 1.** For any digraph  $G$ :

$$\begin{aligned}
 1 + cr(G) &= LIFO^{mi}(G) = LIFO^i(G) = LIFO^{misc}(G) = LIFO^{isc}(G) \\
 &= LIFO^{mv}(G) = LIFO^v(G) = LIFO^{mvsc}(G) = LIFO^{vsc}(G) \\
 &= SS^{vsc}(G).
 \end{aligned}$$

**Proof.** From the above observations, to prove Theorem 1 it is sufficient to prove the following three inequalities:

- (1)  $LIFO^{vsc}(G) \geq SS^{vsc}(G)$ ,
- (2)  $SS^{vsc}(G) \geq 1 + cr(G)$ , and
- (3)  $1 + cr(G) \geq LIFO^{mi}(G)$ .  $\square$

These are established with the following series of lemmas.

**Lemma 1.** For any digraph  $G$ ,  $LIFO^{vsc}(G) \geq SS^{vsc}(G)$ .

**Proof.** We show that if a *vsc*-strategy is not searcher-stationary then it is not a winning strategy from  $(\epsilon, G)$ . The result then follows as this implies every winning *vsc*-strategy is searcher-stationary. Let  $\sigma$  be a *vsc*-strategy, and suppose  $(X_0, R_0), (X_1, R_1), \dots$  is a complete *vsc*-LIFO-search from  $(X_0, R_0) = (\epsilon, G)$  consistent with  $\sigma$  which is not searcher-stationary. Let  $j$  be the least index such that  $X_j \supseteq X_{j+1}$  and  $R_j \neq \emptyset$ . As  $X_0 = \epsilon$ , there exists  $i < j$  such that  $X_i = X_{j+1}$ . By the minimality of  $j$ , and the assumption that we only place or remove one searcher in each round,  $i = j - 1$ . As  $X_{j-1} \subseteq X_j, R_j \subseteq R_{j-1}$ , and as  $X_{j+1} \subseteq X_j, R_j \subseteq R_{j+1}$ . As  $R_j \neq \emptyset$ , it follows that  $R_{j-1}$  and  $R_{j+1}$  are the same strongly connected component of  $G \setminus \{|X_{j-1}|\}$ . Thus  $(X_{j-1}, R_{j-1})$  is a *vsc*-successor of  $(X_j, R_j)$ . As  $\sigma(X_j, R_j) = X_{j+1} = X_{j-1}$ , it follows that  $(X_0, R_0), (X_1, R_1), \dots, (X_{j-1}, R_{j-1}), (X_j, R_j), (X_{j-1}, R_{j-1}), (X_j, R_j), \dots$  is an infinite, and hence complete, *vsc*-LIFO-search (from  $(\epsilon, G)$ ) consistent with  $\sigma$ . As  $R_i \neq \emptyset$  for all  $i \geq 0$ , the LIFO-search is not winning for the searchers. Thus  $\sigma$  is not a winning strategy.  $\square$

**Lemma 2.** For any digraph  $G$ ,  $SS^{vsc}(G) \geq 1 + cr(G)$ .

**Proof.** We prove this by induction on  $|V(G)|$ .

If  $|V(G)| = 1$ , then  $SS^{vsc}(G) = 1 = 1 + cr(G)$ .

Now suppose  $SS^{vsc}(G') \geq 1 + cr(G')$  for all digraphs  $G'$  with  $|V(G')| < |V(G)|$ . We first consider the case when  $G$  is not strongly connected. From Proposition 1,  $SS^{vsc}(G) \geq \max_H SS^{vsc}(H)$  where the maximum is taken over all strongly connected components  $H$  of  $G$ . As  $G$  is not strongly connected,  $|V(H)| < |V(G)|$  for all strongly connected components  $H$  of  $G$ . Therefore, by the induction hypothesis

$$\begin{aligned}
 SS^{vsc}(G) &\geq \max_H SS^{vsc}(H) \\
 &\geq \max_H (1 + cr(H)) \\
 &= 1 + cr(G).
 \end{aligned}$$

Now suppose  $G$  is strongly connected. Let  $\sigma$  be a winning searcher-stationary vs $c$ -strategy which uses  $SS^{vs\mathit{c}}(G)$  searchers. As  $(\epsilon, G)$  is a legitimate vs $c$ -position, if  $(X, R)$  is a vs $c$ -successor of  $(\epsilon, G)$  then  $|X| = 1$ . Thus  $|\sigma(\epsilon, G)| = 1$ . Let  $\sigma(\epsilon, G) = v_0$ . As  $\sigma$  is a searcher-stationary strategy which uses the minimal number of searchers, it follows that  $SS^{vs\mathit{c}}(G \setminus \{v_0\}) = SS^{vs\mathit{c}}(G) - 1$ . Thus, by the induction hypothesis,

$$\begin{aligned} SS^{vs\mathit{c}}(G) &= SS^{vs\mathit{c}}(G \setminus \{v_0\}) + 1 \\ &\geq (1 + cr(G \setminus \{v_0\})) + 1 \\ &\geq \left(1 + \min_{v \in V(G)} cr(G \setminus \{v\})\right) + 1 \\ &= 1 + cr(G). \quad \square \end{aligned}$$

**Lemma 3.** For any digraph  $G$ ,  $1 + cr(G) \geq LIFO^{mi}(G)$ .

**Proof.** We also prove this by induction on  $|V(G)|$ .

If  $|V(G)| = 1$ , then  $1 + cr(G) = 1 = LIFO^{mi}(G)$ .

Now suppose  $1 + cr(G') \geq LIFO^{mi}(G')$  for all digraphs  $G'$  with  $|V(G')| < |V(G)|$ . First we consider the case when  $G$  is not strongly connected. As  $|V(G)| < |V(G)|$  for each strongly connected component  $H$ , by the inductive hypothesis, there is a monotone  $i$ -strategy,  $\sigma_H$ , which captures a fugitive using at most  $1 + cr(H)$  searchers. From the definition of cycle-rank, for each strongly connected component  $H$  of  $G$ ,  $cr(G) \geq cr(H)$ , thus  $\sigma_H$  uses at most  $1 + cr(G)$  searchers. We define a monotone  $i$ -strategy which captures a fugitive on  $G$  with at most  $1 + cr(G)$  searchers as follows. Intuitively, we search the strongly connected components of  $G$  in topological order using the monotone strategies  $\sigma_H$ . More precisely, let  $H_1, H_2, \dots, H_n$  be an ordering of the strongly connected components of  $G$  such that if there is an edge from  $H_i$  to  $H_j$  then  $i < j$ . We define  $\sigma$  as follows.

- $\sigma(\epsilon, G) = \sigma_{H_1}(\epsilon, H_1)$ ,
- For  $1 \leq i$ , if  $\{|X|\} \subseteq H_i$  and  $R = R' \cup \bigcup_{j=i+1}^n H_j$  where  $\emptyset \neq R' \subseteq H_i$ ,  $\sigma(X, R) = \sigma_{H_i}(X, R')$ ,
- For  $1 \leq i < n$ , if  $\emptyset \neq \{|X|\} \subseteq H_i$  and  $R = \bigcup_{j=i+1}^n H_j$  then  $\sigma(X, R) = X'$  where  $X'$  is the maximal proper prefix of  $X$ .

From the definition of  $i$ -successors and the ordering of the strongly connected components if  $(X_0, R_0), \dots, (X_n, R_n)$  is an  $i$ -LIFO-search on  $G$  where  $\{|X_n|\} \subseteq H_i$  and  $\bigcup_{j>i} H_j \subseteq R_{n-1} \subseteq \bigcup_{j \geq i} H_j$ , then  $\bigcup_{j>i} H_j \subseteq R_n \subseteq \bigcup_{j \geq i} H_j$ . It follows (by induction on the length of a LIFO-search) that every LIFO-search from  $(\epsilon, G)$  consistent with  $\sigma$  can be divided into a sequence of LIFO-searches  $\lambda_1, \lambda_2, \dots, \lambda_n$ , where  $\lambda_i$  can be viewed as a LIFO-search consistent with  $\sigma_{H_i}$  with  $\bigcup_{j>i} H_j$  added to the second component of every position. Thus if each  $\sigma_{H_i}$  is monotone, winning and uses at most  $1 + cr(G)$  searchers, then  $\sigma$  is also monotone, winning and uses at most  $1 + cr(G)$  searchers.

Now suppose  $G$  is strongly connected. Let  $v_0$  be the vertex which minimizes  $f(v) = cr(G \setminus \{v\})$ . Let  $G' = G \setminus \{v_0\}$ , so  $cr(G) = 1 + cr(G')$ . By the induction hypothesis, there exists a winning monotone  $i$ -strategy  $\sigma'$  which uses at most  $1 + cr(G')$  searchers to capture a fugitive on  $G'$ . We define an  $i$ -strategy  $\sigma$  on  $G$  which uses at most  $2 + cr(G') = 1 + cr(G)$  searchers as follows. Initially, place (and keep) a searcher on  $v_0$ , then play the strategy  $\sigma'$  on  $G \setminus \{v_0\}$ . More precisely,  $\sigma(\epsilon, G) = v_0$  and  $\sigma(v_0X, R) = v_0 \cdot \sigma'(X, R)$ . Clearly any LIFO-search consistent with  $\sigma$  can be viewed as a LIFO-search consistent with  $\sigma'$  prepended with the position  $(\epsilon, G)$  and where the first component of every position is prepended with  $v_0$ . Thus if  $\sigma'$  is monotone, then  $\sigma$  is monotone, and if  $\sigma'$  is winning then  $\sigma$  is winning. Thus  $\sigma$  is a monotone winning  $i$ -strategy which uses at most  $1 + cr(G)$  searchers.  $\square$

### 3.3. Relation with other graph parameters

With a characterization of cycle-rank in terms of several graph searching games we can compare it with other digraph measures defined by similar games. In particular, the directed pathwidth of a digraph,  $dpw(G)$ , which can be characterized by an invisible-fugitive graph searching game [2], and the DAG-depth,  $dd(G)$  which can be characterized by a visible-fugitive, searcher-stationary searching game [12]. Whilst the relationships we present here are known [14,12], using the game characterizations we obtain a more simple and more intuitive proof.

**Corollary 1.** For any digraph  $G$ ,  $dpw(G) \leq cr(G) \leq dd(G) - 1$ .

## 4. Obstructions for cycle-rank

In this section we consider the dual parameter arising from considering the graph searching games from the fugitive's perspective. We show that it can be characterized by two types of structural features, akin to the havens and brambles used to dually characterize treewidth [28]. To do so we first define the notion of the directed shelter of a digraph, a structural obstruction which we show to be dual to cycle-rank.

**Definition 2.** A *directed shelter* of a digraph  $G$  is a collection  $\mathcal{S}$  of non-empty strongly connected sets of vertices such that for any non-minimal  $S \in \mathcal{S}$

$$\bigcap \{S' : S' \in M_{\mathcal{S}}(S)\} = \emptyset,$$

where  $M_{\mathcal{S}}(S)$  is the  $\subseteq$ -maximal elements of  $\{S' \in \mathcal{S} : S' \subset S\}$ . The *thickness* of a shelter  $\mathcal{S}$  is the minimal length of a maximal  $\subseteq$ -chain.

The second structural obstruction we consider is motivated by the definition of a haven in [16], a structural feature dual to directed treewidth.

**Definition 3.** A *LIFO-haven of order  $k$*  of a digraph  $G$  is a function  $h$  from  $V(G)^{<k}$  to induced subgraphs of  $G$  such that:

- (H1)  $h(X)$  is a non-empty strongly connected component of  $G \setminus \{|X|\}$ , and
- (H2) If  $X \preceq Y$  and  $|Y| < k$  then  $h(Y) \subseteq h(X)$ .

Whilst Adler [1] has shown that the havens of [16] do not give an exact min–max characterization of directed treewidth and Safari [27] has shown that directed versions of havens and brambles give rise to distinct parameters, we show that LIFO-havens and directed shelters both give a tight min–max characterization of cycle-rank.

**Theorem 2 (Min–Max Theorem for Cycle-Rank).** Let  $G$  be a digraph and  $k$  a positive integer. The following are equivalent:

- (i)  $G$  has cycle-rank less than  $k$ ,
- (ii)  $G$  has no LIFO-haven of order greater than  $k$ , and
- (iii)  $G$  has no directed shelter of thickness greater than  $k$ .

**Proof.** (i)  $\Rightarrow$  (ii). Assume that it is not the case that  $G$  has no LIFO-haven of order greater than  $k$ , that is,  $G$  has a LIFO-haven  $h$  of order at least  $k+1$ . We show that the fugitive has a winning strategy against  $k$  searchers, so by Theorem 1,  $\text{cr}(G) \geq k$ . Define a vsc-strategy  $\rho$  for the fugitive (against  $k$  searchers) by defining  $\rho(X, R, X') = h(X')$  for all suitable triples  $(X, R, X')$ . From (H1),  $(X', \rho(X, R, X'))$  is a valid vsc-position. Furthermore, (H2) implies that if  $(X, R)$  is a vsc-position such that  $R = h(X)$ , then  $(X', \rho(X, R, X'))$  is a vsc-successor of  $(X, R)$ , so  $\rho$  is a vsc-strategy (defined for all LIFO-searches that use at most  $k$  searchers). Also, if  $(X_0, R_0), (X_1, R_1) \dots$  is a complete LIFO-search consistent with  $\rho$  then  $R_i = h(X_i)$  for all  $i > 0$ . As  $h(X) \neq \emptyset$  when  $|X| \leq k$ , it follows that all consistent complete LIFO-searches that use at most  $k$  searchers are winning for the fugitive. Thus  $\rho$  is a winning strategy for the fugitive, so  $\text{LIFO}^{\text{vsc}}(G) > k$ . By Theorem 1,  $\text{cr}(G) \geq k$ .

(ii)  $\Rightarrow$  (iii). We show that a directed shelter  $\mathcal{S}$  of thickness at least  $k$  can be used to define a haven of order at least  $k$ . For each  $X \in V(G)^{<k}$  we define  $S_X \in \mathcal{S}$  inductively as follows. For  $X = \epsilon$ , let  $S_\epsilon$  be any  $\subseteq$ -maximal element of  $\mathcal{S}$ . Note that  $\{S \in \mathcal{S} : S \subset S_\epsilon\}$  is a directed shelter of thickness at least  $k - 1$ . Now suppose  $X = X'v, S_{X'}$  is defined,  $S_{X'} \cap \{|X'|\} = \emptyset$ , and  $\mathcal{S}_{X'} = \{S \in \mathcal{S} : S \subset S_{X'}\}$  is a directed shelter of thickness at least  $k - 1 - |X'|$ . From the definition of a directed shelter, there exists a  $\subseteq$ -maximal element of  $\mathcal{S}_{X'}$  that does not contain  $v$ , as otherwise  $v \in S$  for all  $S \in M_{\mathcal{S}}(S_{X'})$ . Let  $S_X$  be that element. As  $S_{X'} \cap \{|X'|\} = \emptyset$  and  $v \notin S_X$ , it follows that  $S_X \cap \{|X|\} = \emptyset$ . Further,  $\{S \in \mathcal{S} : S \subset S_X\}$  is a directed shelter of thickness at least  $(k - 1 - |X'|) - 1 = k - 1 - |X|$ , satisfying the assumptions necessary for the next stage of the induction. Now for all  $X \in V(G)^{<k}$ ,  $S_X$  is a non-empty strongly connected set such that  $S_X \cap \{|X|\} = \emptyset$ . Thus there is a unique strongly connected component of  $G \setminus \{|X|\}$  that contains  $S_X$ . Defining  $h(X)$  to be that component we see that  $h$  satisfies (H1). For (H2), from the definition of  $S_X$ , if  $X \preceq Y$  and  $|Y| < k$ , then  $S_X \supseteq S_Y$ , so  $h(X) \supseteq h(Y)$ . Therefore  $h$  is a haven of order at least  $k$ .

(iii)  $\Rightarrow$  (i). Again, we prove the contrapositive. Suppose  $\text{cr}(G) \geq k$ . Let  $G'$  be a strongly connected component of  $G$  which has cycle-rank  $\geq k$ . We prove by induction on  $k$  that  $G'$ , and hence  $G$ , has a directed shelter of thickness at least  $k + 1$ . Every digraph with  $|V(G)| \geq 1$  has a directed shelter of thickness 1: take  $\mathcal{S} = \{\{v\}\}$  for some  $v \in V(G)$ . Thus for  $k = 0$ , the result is trivial. Now suppose for  $k' < k$  every digraph of cycle-rank  $\geq k'$  contains a directed shelter of thickness at least  $k' + 1$ . For  $v \in V(G')$ , let  $G'_v = G' \setminus \{v\}$ . From the definition of cycle-rank,  $\text{cr}(G'_v) \geq k - 1$  for all  $v \in V(G')$ . Thus, by the induction hypothesis,  $G'_v$  contains a directed shelter,  $\mathcal{S}_v$ , of thickness at least  $(k - 1) + 1$ . As  $v \notin S$  for all  $S \in \mathcal{S}_v$ , it follows that  $\mathcal{S} = \{G'\} \cup \bigcup_{v \in V(G')} \mathcal{S}_v$  is a directed shelter of  $G$ . As  $\mathcal{S}_v$  has thickness at least  $k$  for all  $v \in V(G')$ ,  $\mathcal{S}$  has thickness at least  $k + 1$ .  $\square$

Combining Theorems 1 and 2 we obtain the following.

**Theorem 3.** Let  $G$  be a digraph, and  $k$  a positive integer. The following are equivalent:

- (i)  $G$  has cycle-rank at most  $k - 1$ ,
- (ii) On  $G$ ,  $k$  searchers can capture a fugitive with a LIFO-search strategy,
- (iii) On  $G$ ,  $k$  searchers can capture a visible fugitive restricted to moving in strongly connected sets with a searcher-stationary search strategy,
- (iv)  $G$  has no LIFO-haven of order greater than  $k$ , and
- (v)  $G$  has no directed shelter of thickness greater than  $k$ .

## 5. LIFO-search in simple graphs

In this section we consider the consequences of our results to simple graphs. In order to do so, we first give the definition of tree-depth and then restrict the notions of directed shelters on simple graphs.

**Definition 4** (*Tree-Depth*). The *tree-depth* of a graph  $G$ ,  $\text{td}(G)$ , is defined as follows:

- If  $|V(G)| = 1$  then  $\text{td}(G) = 1$ .
- If  $|V(G)| > 1$  and  $G$  is connected then  $\text{td}(G) = 1 + \min_{v \in V(G)} \text{td}(G \setminus \{v\})$ .
- Otherwise  $\text{td}(G) = \max_H \text{td}(H)$  where the maximum is taken over all connected components  $H$  of  $G$ .

Tree-depth is also known as the vertex ranking problem [5], the ordered colouring problem [17], or the minimum-height of an elimination tree of a graph [6,8,23] and has received much attention, mostly because of the theory of graph classes of bounded expansion, developed by Nešetřil and Ossona de Mendez in [23,21,24,25,22].

Given a graph  $G$  we define the digraph  $G_d$  where  $V(G_d) = V(G)$  and  $E(G_d) = \{(x, y) \mid \{x, y\} \in E(G)\}$ , that is,  $G_d$  is obtained from  $G$  after replacing every edge  $\{x, y\} \in E(G)$  with the arcs  $(x, y)$  and  $(y, x)$ . From Definitions 1 and 4 we get that.

**Observation 1.** For every graph  $G$ ,  $\text{td}(G) = \text{cr}(G_d) + 1$ .

We now give the definition of shelters when restricted to simple graphs.

**Definition 5** (*Shelter*). A *shelter* of  $G$  is a collection  $\mathcal{S}$  of non-empty connected sets in  $G$  such that for every non-minimal set  $S \in \mathcal{S}$  no vertex belongs to all its children, in other words,

$$\bigcap \{S' \mid S' \in M_{\mathcal{S}}(S)\} = \emptyset,$$

where  $M_{\mathcal{S}}(S)$  is the  $\subseteq$ -maximal elements of  $\{S' \in \mathcal{S} : S' \subset S\}$ . The *thickness* of a shelter  $\mathcal{S}$  is the minimal length of a maximal  $\subseteq$ -chain.

Observation 1 ensures that we may restate Theorem 3 for simple graphs in the following way.

**Theorem 4.** Let  $G$  be a non-empty graph and  $k$  be a positive integer. Then the following are equivalent.

- (i)  $G$  has tree-depth at most  $k$ .
- (ii) there is a monotone LIFO-search strategy in  $G$  of cost at most  $k$  that captures an invisible and agile fugitive.
- (iii) there is a LIFO-search strategy in  $G$  of cost at most  $k$  that captures an invisible and agile fugitive.
- (iv) every shelter in  $G$  has thickness at most  $k$ .
- (v) there is a monotone LIFO-search strategy in  $G$  using  $k$  searchers against a visible and agile fugitive.

## 6. Conclusions and further work

To conclude, this multiple characterization of cycle-rank gives a new perspective on the measure which can be useful for further investigation. For example, whilst it is known that computing the cycle-rank is NP-complete [14], it holds that, for any fixed  $k$ , deciding whether a  $n$ -vertex digraph has cycle-rank at most  $k$  is decidable in  $O(n^k)$  steps (this follows from its definition and the fact that strongly connected components can be computed in linear time). From the parameterized complexity perspective, this means that this problem, parameterized by  $k$ , belongs in the class XP. It is an open question whether the same problem belongs in the class FPT, that is it can be solved by an algorithm in  $f(k) \cdot n^{O(1)}$  steps. Techniques based on separators have shown that parameterized problems corresponding to related measures such as directed treewidth belong in FPT. Whether the visible, strongly connected game characterizations of cycle-rank can improve the known complexity from XP to FPT is part of ongoing research.

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