Forbidding Kuratowski Graphs as Immersions

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Abstract: Immersion is a containment relation on graphs that is weaker than topological minor. (Every topological minor of a graph is also its immersion.) The graphs that do not contain any of the Kuratowski graphs (K_5 and $K_{3,3}$) as topological minors are exactly planar graphs. We give a structural characterization of graphs that exclude the Kuratowski graphs as immersions. We prove that they can be constructed from planar graphs that are subcubic or of branch-width at most 10 by repetitively applying i-edge-sums, for $i \in \{1, 2, 3\}$. We also use this result to give a structural characterization of graphs that exclude $K_{3,3}$ as an immersion. J. Graph Theory 78: 43–60, 2015

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1. INTRODUCTION

A famous graph-theoretic result is the theorem of Kuratowski that states that a graph G is planar if and only if it does not contain K_5 and $K_{3,3}$ (also known as the Kuratowski graphs) as a topological minor, that is, if K_5 and $K_{3,3}$ cannot be obtained from the graph by applying vertex and edge removals and vertex dissolutions. It is well known that the topological minor relation defines a (partial) ordering on the class of graphs.

In a similar way, the immersion and the minor orderings can be defined in graphs if instead of vertex dissolutions we ask for edge lifts and edge contractions, respectively. (For detailed definitions see Section 2.) Notice that the topological minor ordering is stronger than the minor and the immersion orderings, in the sense, that if a graph G contains a graph H as a topological minor then it also contains it as an immersion and as a minor but the inverse direction does not always hold.

In the celebrated theory of Graph Minors, developed by Robertson and Seymour, it was proven that both the immersion and minor orderings are well-quasi-ordered, that is, there are no infinite sets of mutually noncomparable graphs [17, 18] according to these orderings. This result has as a consequence the complete characterization of the graph classes that are closed under taking immersions or minors in terms of forbidden graphs. (A graph class is closed under taking immersions, respectively minors, if for any graph that belongs to the graph class all of its immersions, respectively minors, also belong to the graph class.) For example, by an extension of the Kuratowski theorem (also known as Wagner's theorem), it is also known that a graph is planar if and only if it does not contain K_5 and $K_{3,3}$ as a minor.

Thus, a question that naturally arises is about the characterization of the structure of a graph G that excludes some fixed graph H as an immersion or as a minor. While this subject has been extensively studied for the minor ordering (see [2, 3, 6, 9, 13, 15, 16, 19–21]), the immersion ordering only recently attracted the attention of the research community [1, 4, 8, 10, 12]. DeVos et al. [4] proved that for every positive integer t, every simple graph of minimum degree at least $200t$ contains the complete graph on t vertices as a (strong) immersion and Ferrara et al., given a graph H, provide a lower bound on the minimum degree of any graph G in order to ensure that H is contained in G as an immersion [7]. More recently, Wollan [23] proved a structure theorem for graphs excluding complete graphs as immersions.

In terms of graph colorings, Abu-Khzam and Langston [1] provided evidence supporting the immersion ordering analog of Hadwiger's Conjecture, that is, the conjecture stating that if the chromatic number of a graph G is at least t, then G contains the complete graph on t vertices as an immersion, and proved it for $t \leq 4$. This conjecture is proven for $t = 5, 6$ and $t \leq 7$ by Lescure and Meyniel [14] and by DeVos et al. [5] independently. The most recent result on colorings is an approximation of the list coloring number on graphs excluding the complete graph as immersion [12].

Finally, in terms of algorithms, Grohe et al. gave a cubic time algorithm that decides whether a fixed graph H is contained in the input graph G as immersion [10] and Giannopoulou et al. provided sufficient conditions which, when given, make the computation of the minimal graphs not belonging to a graph class closed under immersions effective [8].

Going back to the subject of the structural characterization of the graphs that exclude some fixed graph H as an immersion we notice that it is straightforward to find such characterizations in the cases where $H = K_{i,j}$, where $i \in [2]$ and $j \in [3]$. In particular, the graphs that exclude $K_{1,1}$ are exactly all edgeless graphs, and the graphs that exclude $K_{1,2}$ are disjoint unions of (possibly multiple) edges. It is also easy to verify, that the connected graphs that exclude $K_{1,3}$ are the subgraphs of the following graphs; the cycle of length 3, where some of its edges may be multiple, the cycles of length at least four that have no multiple edges and the paths on at least two vertices where only the edges that are incident to the endpoints of the path may appear multiple times (for this notice that every vertex of a graph that excludes $K_{1,3}$ as an immersion has at most two neighbors). Note here that the tree-width of graphs that exclude $K_{1,j}$ as an immersion is upper bounded by $j - 1$.

Similarly, one can show that the the tree-width of the graphs that exclude $K_{2,2}$ as an immersion is upper bounded by 2. The reason for this is that if a graph excludes $K_{2,2}$ as an immersion then it also has to exclude $K_{2,2}$, and thus K_4, as a (topological) minor. Furthermore, after careful inspection one can show that these graphs have a much more specific form; their biconnected components are either single edges (edges that have multiplicity exactly 1), triangles that may have multiple edges, or edges of multiplicity at least 2 and also no triangle shares a common vertex with a biconnected component unless it is an edge of multiplicity 1, and no edge of multiplicity at least 2 shares a vertex with more than one other edge of multiplicity at least 2.

Finally, for the case where a graph G excludes $K_{2,3}$ as an immersion it is also easy to see that the tree-width of G is upper bounded by 3. Indeed, first notice that the following four graphs contain $K_{2,3}$ as a minor; K_5, the pentagonal prism, the octahedron, and Wagner's graph. This implies that if a graph G contains either one of those graphs as a minor, then it also contains $K_{2,3}$ as a minor. As the maximum degree of $K_{2,3}$ is at most 3, a folklore result ensures that the graph G also contains $K_{2,3}$ as a topological minor (and therefore as an immersion). Hence, any graph G that excludes $K_{2,3}$ as an immersion, also excludes the four graphs mentioned above as minors. From a well-known result [2] it follows that the tree-width of G is upper bounded by 3.

In this note we characterize the structure of the graphs that do not contain K_5 and $K_{3,3}$ as immersions. As these graphs already exclude Kuratowski graphs as topological minors, they are planar. Additionally, we show that they have a more special structure: they can be constructed by repetitively joining together simpler graphs, starting from either graphs of small decomposability or by planar graphs with maximum degree at most 3. In particular, we prove that a graph G that contains neither K_5 nor $K_{3,3}$ as immersions can be constructed by applying consecutive i-edge-sums, for $i \in [3]$, to graphs that are planar subcubic or of branch-width at most 10.

Furthermore, we show that our main result can be employed to obtain a structural characterization for the graphs that only exclude $K_{3,3}$ as an immersion.

2. DEFINITIONS

For every integer n, we let $[n] = \{1, 2, \ldots, n\}$. All graphs we consider are finite, undirected, and loopless but may have multiple edges. Given a graph G we denote by $V(G)$ and $E(G)$ its *vertex set* and *edge set,* respectively. Given a set $F \subseteq E(G)$ (resp. $S \subseteq V(G)$), we denote by $G \setminus F$ (resp. $G \setminus S$) the graph obtained from G if we remove the edges in F (resp. the vertices in S along with their incident edges). We denote by $\mathcal{C}(G)$ the set of the *connected components* of G. Given a vertex $v \in V(G)$, we also use the notation $G \setminus v = G \setminus \{v\}$. The *neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of edges in G that are adjacent to v. We denote by $E_G(v)$ the set of the edges of G that are incident with v. The *degree* of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is the number of edges that are incident with it, i.e. $\deg_G(v) = |E_G(v)|$. Notice that, as we are dealing with multigraphs, $|N_G(v)| \leq \deg_G(v)$. The minimum degree of a graph G, denoted by $\delta(G)$, is the minimum of the degrees of the vertices of G, that is, $\delta(G) = \min_{v \in V(G)} \deg_G(v)$. A graph is called *subcubic* if all its vertices have degree at most 3. We also denote by K_r the *complete graph* on r vertices and by $K_{r,q}$ the *complete bipartite graph* with r vertices in its one part and q in the other. Let P be a path and $v, u \in V(P)$. We denote by $P[v, u]$ the subpath of P with end-vertices v and u.

We say that a graph H is a *subgraph* of a graph G, denoted by $H \subseteq G$, if H can be obtained from G by removing edges or vertices. An *edge cut* in a graph G is a nonempty set F of edges that belong to the same connected component of G and such that $G \setminus F$ has more connected components than G. If $G \setminus F$ has one more connected component than G then we say that F is a *minimal* edge cut. Let F be an edge cut of a graph G and let G' be the connected component of G containing the edges of F. We say that F is an *internal edge cut* if it is minimal and each of the two connected components of $G' \setminus F$ contains at least two vertices. An edge cut is also called *i-edge-cut* if it has cardinality $\leq i$.

In this article, we mostly deal with planar graphs, that is, graphs that are embedded in the sphere $\mathbb{S}_0$. We call such a graph, along with its embedding, *Σ_0-embeddable graph*. Let C_1, C_2 be two disjoint cycles in a Σ_0-embeddable graph G. Let also Δ_i be the open disk of $\mathbb{S}_0 \setminus C_i$ that does not contain points of C_{3-i}, $i \in [2]$. The *annulus between* C_1 and C_2 is the set $\mathbb{S}_0 \setminus (\Delta_1 \cup \Delta_2)$ and we denote it by $A[C_1, C_2]$. Notice that $A[C_1, C_2]$ is a closed set. Let $\mathcal{A} = \{C_1, \ldots, C_r\}$ be a collection of cycles of a $\mathbb{S}_0$-embeddable graph G. We say that $\mathcal{A}$ is *nested* if for every $i \in [r-2]$, $A[C_i, C_{i+1}] \cup A[C_{i+1}, C_{i+2}] = A[C_i, C_{i+2}]$.

Contractions and minors. The *contraction of an edge* $e = \{x, y\}$ from G is the removal from G of all edges incident with x or y and the insertion of a new vertex v_e that is made adjacent to all the vertices of $(N_G(x) \setminus \{y\}) \cup (N_G(y) \setminus \{x\})$ such that edges corresponding to the vertices in $(N_G(x) \setminus \{y\}) \cap (N_G(y) \setminus \{x\})$ increase their multiplicity, that is, if there was a vertex $v \in (N_G(x) \setminus \{y\}) \cap (N_G(y) \setminus \{x\})$, k edges joining v and x and, l edges joining v and y then in the resulting graph there will be $k + l$ edges joining v with v_e. Finally, remove any loops resulting from this operation. Given two graphs H and G, we

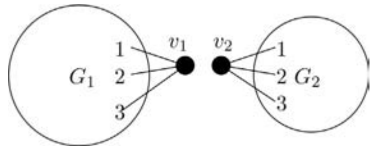

FIGURE 1. The graphs G_1 and G_2 before the edge sum.

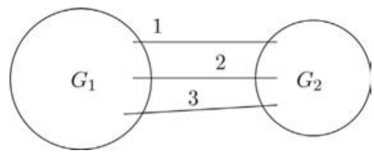

FIGURE 2. The graph obtained after the edge sum.

say that H is a *contraction* of G, denoted by $H \leq_c G$, if H can be obtained from G after a (possibly empty) series of edge contractions. Moreover, H is a *minor* of G if H is a contraction of some subgraph of G.

Topological minors. A *subdivision* of a graph H is any graph obtained after replacing some of its edges by paths between the same endpoints. A graph H is a *topological minor* of G (denoted by $H \leq_t G$) if G contains as a subgraph some subdivision of H.

Immersions. The *lift* of two edges $e_1 = \{x, y\}$ and $e_2 = \{x, z\}$ to an edge e is the operation of removing e_1 and e_2 from G and then adding the edge $e = \{y, z\}$ in the resulting graph. We say that a graph H can be *(weakly) immersed* in a graph G (or is an *immersion* of G), denoted by $H \leq_{im} G$, if H can be obtained from a subgraph of G after a (possibly empty) sequence of edge lifts. Equivalently, we say that H is an immersion of G if there is an injective mapping $f : V(H) \to V(G)$ such that, for every edge $\{u, v\}$ of H, there is a path from $f(u)$ to $f(v)$ in G and for any two distinct edges of H the corresponding paths in G are *edge-disjoint*, that is, they do not share common edges. Additionally, if these paths are internally disjoint from $f(V(H))$, then we say that H is *strongly immersed* in G (or is a *strong immersion of* G). The injective mapping f together with the edge-disjoint paths is called *a model of H in G defined by f.*

Edge sums. Let G_1 and G_2 be graphs, let v_1, v_2 be vertices of $V(G_1)$ and $V(G_2)$ respectively such that $\deg_G(v_1) = \deg_G(v_2)$, and consider a bijection $\sigma : E_{G_1}(v_1) \to E_{G_2}(v_2)$, where $E_{G_1}(v_1) = \{e_1^i \mid i \in [k]\}$. We define the *k-edge sum* of G_1 and G_2 on v_1 and v_2 as the graph G obtained if we take the disjoint union of G_1 and G_2, identify v_1 with v_2, and then, for each $i \in \{1, \ldots, k\}$, lift e_1^i and $\sigma(e_1^i)$ to a new edge e^i and remove the vertex v_1. (See Figs. 1 and 2.)

Let G be a graph, let F be a minimal i-edge cut in G, and let G' be the connected component of G that contains F. Let also C_1 and C_2 be the two connected components of $G' \setminus F$. We denote by C_i' the graph obtained from G' after contracting all edges of C_{3-i}' to a single vertex v_i, $i \in [2]$. We say that the graph consisting of the disjoint union of the graphs in $\mathcal{C}(G) \setminus \{C_1, C_2\} \cup \{C_1', C_2'\}$ is the *F-split* of G and we denote it by $G|_F$. Notice that if G is connected and F is a minimal i-edge cut in G, then G is the result of the i-edge sum of the two connected components G_1 and G_2 of $\mathcal{C}(G|_F)$ on the vertices v_1 and v_2. From Menger's Theorem we obtain the following.

Observation 2.1. *Let k be a positive integer. If G is a connected graph that does not contain an internal i-edge cut, for some $i \in [k-1]$ and $v, v_1, \ldots, v_i \in V(G)$ are distinct vertices such that $\deg_G(v) \geq i$ then there exist i edge-disjoint paths from v to $v_1, v_2, \ldots, v_i$.*

Lemma 2.2. *If G is a $\{K_5, K_{3,3}\}$-immersion free connected graph and F is a minimal internal i-edge cut in G, for $i \in [3]$, then both connected components of $G|_F$ are $\{K_5, K_{3,3}\}$-immersion free.*

Proof. For contradiction assume that G is a $\{K_5, K_{3,3}\}$-immersion free connected graph and one of the connected components of $G|_F$, say C_1', contains K_5 or $K_{3,3}$ as an immersion, where F is a minimal internal i-edge cut in G, $i \in [3]$. Assume that $H \in \{K_5, K_{3,3}\}$ is immersed in C_1' and let $f : V(H) \to V(C_1')$ be a model of H in C_1'. Let also v_1 be the newly introduced vertex of C_1'. Notice that if $v_1 \notin f(V(H))$ and v_1 is not an internal vertex of any of the edge-disjoint paths between the vertices in $f(V(H))$, then f is a model of H in C_1. As $C_1 \subseteq G$, f is a model of H in G, a contradiction to the hypothesis. Thus, we may assume that either $v_1 \in f(V(H))$ or v_1 is an internal vertex in at least one of the edge-disjoint paths between the vertices in $V(H)$. Note that, as neither K_5 nor $K_{3,3}$ contain vertices of degree 1, $\deg_{C_1'}(v_1) = 2$ or $\deg_{C_1'}(v_1) = 3$.

We first exclude the case where $v_1 \notin f(V(H))$, that is, v_1 only appears as an internal vertex on the edge-disjoint paths. Observe that, as $\deg_{C_1'}(v_1) \leq 3$, v_1 belongs to exactly one path P in the model defined by f. Let v_1^1 and v_1^2 be the neighbors of v_1 in P. Recall that, by the definition of an internal F-split, there are vertices v_2^1 and v_2^2 in C_2 such that $\{v_1^1, v_2^1\}, \{v_2^1, v_2^2\} \in E(G)$. Furthermore, as C_2 is connected, there exists a (v_2^1, v_2^2)-path P' in C_2. Therefore, be substituting the subpath $P[v_1^1, v_1^2]$ by the path defined by the union of the edges $\{v_1^1, v_2^1\}, \{v_2^1, v_2^2\} \in E(G)$ and the path P' in C_2 we obtain a model of H in G defined by f, a contradiction to the hypothesis.

Thus, the only possible case is that $v_1 \in f(V(H))$. As $\delta(K_5) = 4$ and $\deg_{C_1'}(v_1) \leq 3$, f defines a model of $K_{3,3}$ in C_1'. Let v_1^1, v_1^2 and v_1^3 be the neighbors of v_1 in C_1'. We claim that there is a vertex v in C_2 and edge-disjoint paths from v to v_1^1, v_1^2, v_1^3 in G, thus proving that there exists a model of $K_{3,3}$ in G as well, a contradiction to the hypothesis. By the definition of an internal F-split, there are vertices v_2^1, v_2^2 , and v_2^3 in C_2 such that $\{v_1^i, v_2^i\} \in E(G)$, $i \in [3]$. Recall that C_2 is connected. Therefore, if for every vertex $v \in C_2$, $\deg_{C_2}(v) \leq 2$, C_2 contains a path whose endpoints, say u and u' belong to $\{v_2^1, v_2^2, v_2^3\}$ and internally contains the vertex in $\{v_2^1, v_2^2, v_2^3\} \setminus \{u, u'\}$, say u''. Then it is easy to verify that u'' satisfies the conditions of the claim. Assume then that there is a vertex $v \in C_2$ of degree at least 3. Let G' be the graph obtained from G after removing all vertices in $V(C_1) \setminus \{v_1^1, v_1^2, v_1^3\}$ and adding a new vertex that we make it adjacent to the vertices in $\{v_1^1, v_1^2, v_1^3\}$. As G does not contain an internal i-edge cut, $i \in [2]$, G' does not contain an internal i-edge cut, $i \in [2]$. Therefore, from Observation 2.1 and the fact that $v \notin \{v_1^1, v_1^2, v_1^3\}$, we obtain that there exist three edge-disjoint paths from v to v_1^1, v_1^2, v_1^3 in G' and thus in G. This completes the proof of the claim and the lemma follows. ■

Let r and q be integers such that $r \geq 3$ and $q \geq 1$. A *(r, q)-cylinder*, denoted by $C_{r,q}$, is the Cartesian product of a cycle on r vertices and a path on q vertices. (See, for

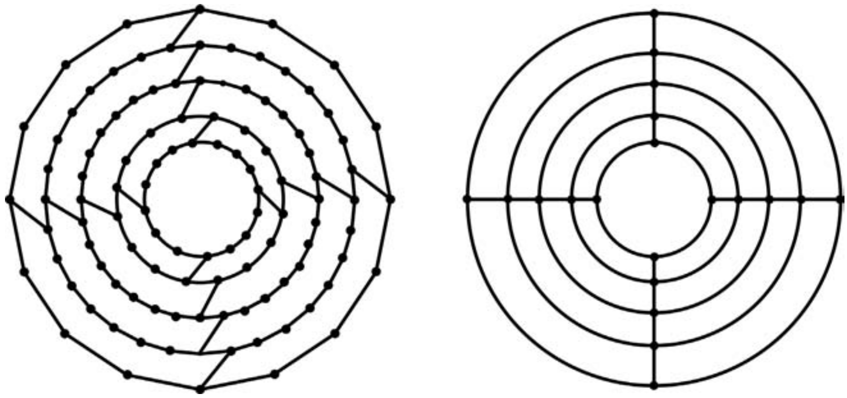

FIGURE 3. A (4,5)-railed annulus and a (4,5)-cylinder.

example, Fig. 3.) A (r, q)-*railed annulus* in a graph G is a pair $(\mathcal{A}, \mathcal{W})$ such that $\mathcal{A}$ is a collection of r nested cycles $C_1, C_2, \ldots, C_r$ that are all met by a collection $\mathcal{W}$ of q paths $P_1, P_2, \ldots, P_q$ (called *rails*) in such a way that the intersection of a rail and a path is always connected, that is, it is a (possibly trivial, that is, consisting of only one vertex) path. (See, e.g. Fig. 3.) Notice that given a graph G embedded in the sphere and a (k, h)-*cylinder* ((r, q)-*railed annulus*, respectively) of G, then any two cycles of the (k, h)-*cylinder* ((r, q)-*railed annulus*, respectively) define an annulus between them.

Branch decompositions. A *branch decomposition of a graph G* is a pair $\mathcal{B} = (T, \tau)$, where T is a ternary tree and $\tau : E(G) \to \mathcal{L}(T)$ is a bijection of the edges of G to the leaves of T, denoted by $\mathcal{L}(T)$. Given a branch decomposition $\mathcal{B}$, we define $\sigma_{\mathcal{B}} : E(T) \to \mathbb{N}$ as follows.

Given an edge $e \in E(T)$ let T_1 and T_2 be the trees in $T \setminus \{e\}$. Then $\sigma_{\mathcal{B}}(e) = |\{v \mid$ there exist $e_i \in \tau^{-1}(\mathcal{L}(T_i))$, $i \in [2]$, such that $e_1 \cap e_2 = \{v\}\}|$. The *width of a branch decomposition* $\mathcal{B}$ is $\max_{e \in E(T)} \sigma_{\mathcal{B}}(e)$ and the *branch-width of a graph G*, denoted by $\mathbf{bw}(G)$, is the minimum width over all branch decompositions of G. When $|V(T)| \leq 1$ the width of the branch decomposition is defined to be 0.

Theorem 2.3 ([11]). *If G is a planar graph and* k, h *are integers with* $k \geq 3$ *and* $h \geq 1$ *then G either contains the* (k, h)*-cylinder as a minor or has branch-width at most* $k + 2h - 2$.

We now prove the following.

Lemma 2.4. *If G is a planar graph of branch-width at least 11, then G contains a (4,4)-railed annulus as a subgraph.*

Proof. Let G be a planar graph of branch-width at least 11. Then by Theorem 2.3, G contains $(4, 4)$-cylinder as a minor. By the definition of the minor relation, G contains a $(4, 4)$-railed annulus as a subgraph. ■

Confluent paths. Let G be a graph embedded in some surface Σ and let $x \in V(G)$. A *disk around x is an open disk* Δ_x with the property that each point in $\Delta_x \cap G$ is either x or belongs to the edges incident with x. Let P_1 and P_2 be two edge-disjoint paths in G and x be a vertex of $V(P_1) \cap V(P_2)$ that is not an endpoint of P_1 or P_2. From now on, we restrict the disks Δ_x to be such that $\Delta_x \setminus P_1$ and $\Delta_x \setminus P_2$ have exactly two connected components each. We say that P_1 and P_2 are *confluent* if for every $x \in V(P_1) \cap V(P_2)$, that is not an endpoint of P_1 or P_2, and for every disk Δ_x around x, one of the connected components of the set $\Delta_x \setminus P_1$ does not contain any point of P_2. We also say that a collection of paths is *confluent* if the paths in it are pairwise confluent.

Moreover, given two edge-disjoint paths P_1 and P_2 in G we say that a vertex $x \in V(P_1) \cap V(P_2)$ that is not an endpoint of P_1 or P_2 is an *overlapping vertex of* P_1 *and* P_2 if there exists a Δ_x around x such that both connected components of $\Delta_x \setminus P_1$ contain points of P_2. For a family of paths $\mathcal{P}$, a vertex v of a path $P \in \mathcal{P}$ is called an *overlapping vertex of* P if there exists a path $P' \in \mathcal{P}$ such that v is an overlapping vertex of P and P'.

Finally, given two paths P_1 and P_2 that share a common endpoint v, we say that they are *well-arranged* if their common vertices appear in the same order in both paths.

3. PRELIMINARY RESULTS ON THE CONFLUENCE OF PATHS

Lemma 3.1. *Let G be a graph and $v, v_1, v_2 \in V(G)$ such that there exist edge-disjoint paths P_1 and P_2 from v to v_1 and v_2 , respectively. If the paths P_1 and P_2 are not well arranged then there exist edge-disjoint paths P_1' and P_2' from v to v_1 and v_2 respectively such that $E(P_1') \cup E(P_2') \subsetneq E(P_1) \cup E(P_2)$.*

Proof. Let $U = V(P_1) \cap V(P_2) = \{v, u_1, u_2, \dots, u_k\}$, where $(v, u_1, u_2, \dots, u_k)$ is the order that the vertices in U appear in P_1 and, $(v, u_{i_1}, u_{i_2}, \dots, u_{i_k})$ is the order that they appear in P_2. As the paths are not well arranged there exists $\lambda \in [k]$ such that $u_\lambda \neq u_{i_\lambda}$. Without loss of generality assume that λ is the smallest such integer. Also, without loss of generality assume that $u_\lambda < u_{i_\lambda}$. We define

$$P_1' = P_1[v, u_{\lambda-1}] \cup P_2[u_{\lambda-1}, u_{i_\lambda}] \cup P_1[u_{i_\lambda}, v_1]$$
$$P_2' = P_2[v, u_{\lambda-1}] \cup P_1[u_{\lambda-1}, u_\lambda] \cup P_2[u_\lambda, v_2].$$

and observe that P_1' and P_2' satisfy the desired properties. (For an example, see Fig. 4.) ■

Before proceeding to the statement and proof of the next proposition we need the following definition. Given a collection of paths $\mathcal{P}$ in a graph G, we define the function $f_{\mathcal{P}} : \bigcup_{P \in \mathcal{P}} V(P) \to \mathbb{N}$ such that $f_{\mathcal{P}}(x)$ is the number of pairs of paths $P, P' \in \mathcal{P}$ for which x is an overlapping vertex. Let

$$g(\mathcal{P}) = \sum_{x \in \bigcup_{P \in \mathcal{P}} V(P)} f_{\mathcal{P}}(x).$$

Notice that $f_{\mathcal{P}}(x) \geq 0$ for every $x \in \bigcup_{P \in \mathcal{P}} V(P)$ and thus $g(\mathcal{P}) \geq 0$. Observe also that $g(\mathcal{P}) = 0$ if and only if $\mathcal{P}$ is a confluent collection of paths.

Lemma 3.1 allows us to prove the main result of this section. We state the result for general surfaces as the proof for this more general setting does not have any essential difference than the case where Σ is the sphere $\mathbb{S}_0$.

Proposition 3.2. *Let r be a positive integer. If G is a graph embedded in a surface Σ, $v, v_1, v_2, \dots, v_r \in V(G)$ and $\mathcal{P}$ is a collection of r edge-disjoint paths from v to*

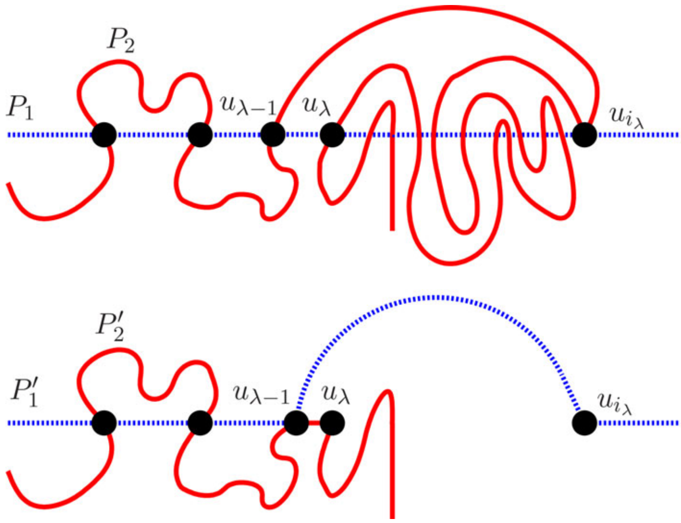

FIGURE 4. An example of the procedure in Lemma 3.1.

$v_1, v_2, \ldots, v_r$ in G, then G contains a confluent collection $\mathcal{P}'$ of r well-arranged edge-disjoint paths from v to $v_1, v_2, \ldots, v_r$ where $|\mathcal{P}'| = |\mathcal{P}|$ and such that $E(\bigcup_{P\in\mathcal{P}'} P) \subseteq E(\bigcup_{P\in\mathcal{P}} P)$.

Proof. Let $\hat{G}$ be the spanning subgraph of G induced by the edges of the paths in $\mathcal{P}$ and let G' be a minimal spanning subgraph of $\hat{G}$ that contains a collection of r edge-disjoint paths from v to $v_1, v_2, \ldots, v_r$. Let also $\mathcal{P}'$ be the collection of r edge-disjoint paths from v to $v_1, v_2, \ldots, v_r$ in G' for which $g(\mathcal{P}')$ is minimum. It is enough to prove that $g(\mathcal{P}') = 0$.

For a contradiction, we assume that $g(\mathcal{P}') > 0$ and we prove that there exists a collection $\tilde{\mathcal{P}}$ of r edge-disjoint paths from v to $v_1, v_2, \ldots, v_r$ in G' such that $g(\tilde{P}) < g(\mathcal{P}')$. As $g(\mathcal{P}') > 0$, then there exists a path, say $P_1 \in \mathcal{P}'$, that contains an overlapping vertex u. Let z_1 be the endpoint of P_1 which is different from v. Without loss of generality we may assume that u is the overlapping vertex of P_1 that is closer to z_1 in P_1, that is, there is no other overlapping vertex vertex of P_1 that is also a vertex of $P_1[u, z_1]$. Then there is a (v, z_2)-path $P_2 \in \mathcal{P}'$ such that u is an overlapping vertex of P_1 and P_2. Let $\tilde{P}_i = P_{3-i}[v, u] \cup P_i[u, z_i]$, $i \in [2]$ and $\tilde{P} = P$ for every $P \in \mathcal{P}' \setminus \{P_1, P_2\}$. As Lemma 3.1 and the edge minimality of G' imply that the paths P_1 and P_2 are well arranged, we obtain that $\tilde{P}_i$ is a path from v to v_i, $i \in [2]$. Let $\tilde{\mathcal{P}}$ be $\{\tilde{P} \mid P \in \mathcal{P}'\}$. It is easy to verify that $\tilde{\mathcal{P}}$ is a collection of r edge-disjoint paths from v to $v_1, v_2, \ldots, v_r$. We will now prove that $g(\tilde{\mathcal{P}}) < g(\mathcal{P}')$.

First notice that if $x \neq u$, then $f_{\tilde{\mathcal{P}}}(x) = f_{\mathcal{P}'}(x)$. Thus, it is enough to prove that $f_{\tilde{\mathcal{P}}}(u) < f_{\mathcal{P}'}(u)$. Observe that if $\{P, P'\} \subseteq \mathcal{P}' \setminus \{P_1, P_2\}$ and u is an overlapping vertex of P and P' then u is also an overlapping vertex of $\tilde{P}$ and P'. Furthermore, while u is an overlapping vertex in the case where $\{P, P'\} = \{P_1, P_2\}$, it is not an overlapping vertex of $\tilde{P}_1$ and $\tilde{P}_2$. It remains to examine the case where $|\{P, P'\} \cap \{P_1, P_2\}| = 1$. In other words, we examine the case where one of the paths P and P', say P', is P_1 or P_2, and $P \in \mathcal{P}' \setminus \{P_1, P_2\}$. Let

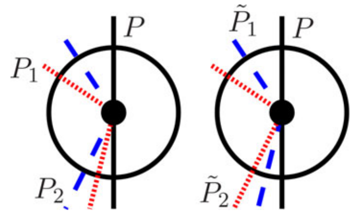

FIGURE 5. The paths P (black), P_1 (red - dotted) and P_2 (blue - dashed) and the paths $\tilde{P}_1$ (blue - dashed) and $\tilde{P}_2$ (red - dotted).

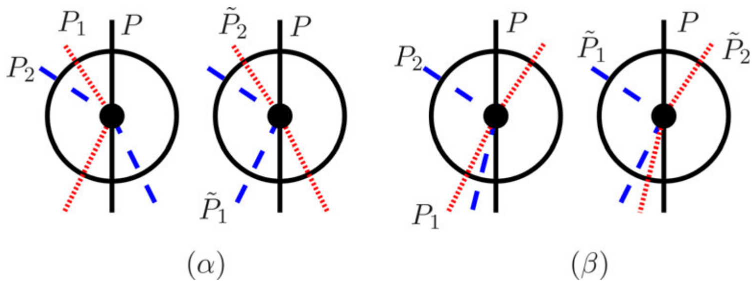

FIGURE 6. The paths P (black), P_1 (red - dotted) and P_2 (blue - dashed) and the paths $\tilde{P}_1$ (blue - dashed) and $\tilde{P}_2$ (red - dotted).

Δ_u be a disk around u and Δ_1, Δ_2 be the two distinct disks contained in the interior of Δ_u after removing P. We distinguish the following cases.

Case 1. u is neither an overlapping vertex of P_1 and P, nor of P_2 and P (see Fig. 5).

Then it is easy to see that the same holds for the pairs of paths $\tilde{P}_1$ and P and, $\tilde{P}_2$ and P. Indeed, notice that for every $i \in [2]$, P_i intersects exactly one of Δ_1 and Δ_2. Furthermore, as u is an overlapping vertex of P_1 and P_2, both paths intersect the same disk. From the observation that $P_1 \cup P_2 = \tilde{P}_1 \cup \tilde{P}_2$, we obtain that u is neither an overlapping vertex of $\tilde{P}_1$ and P nor of $\tilde{P}_2$ and P.

Case 2. u is an overlapping vertex of P_i and P but not of P_{3-i} and P, $i \in [2]$ (see Fig. 6).

Notice that exactly one of the following holds.

- $P_i[v, u] \cup P_{3-i}[v, u]$ intersects exactly one of the disks Δ_1 or Δ_2, say Δ_1. Then $P_i[u, z_i]$ intersects Δ_2 and $P_{3-i}[u, z_{3-i}]$ intersects Δ_1. Therefore, it is easy to see that, u is not an overlapping vertex of P_i and P anymore but becomes an overlapping vertex of $\tilde{P}_{3-i}$ and P.
- $P_i[u, z_i] \cup P_{3-i}[u, z_{3-i}]$ intersects exactly one of the disks Δ_1 or Δ_2, say Δ_1. Then $P_i[v, u]$ intersects Δ_2 and $P_{3-i}[v, u]$ intersects Δ_1. Therefore, it is easy to see that u remains an overlapping vertex of $\tilde{P}_i$ and P and does not become an overlapping vertex of P_{3-i} and P.

Case 3. u is an overlapping vertex of both P_1 and P and, P_2 and P (see Fig. 7).

As above, exactly one of the following holds.

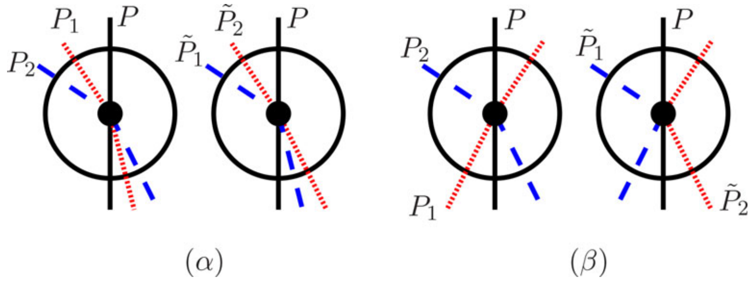

FIGURE 7. The paths P (black), P_1 (red - dotted) and P_2 (blue - dashed) and the paths $\tilde{P}_1$ (blue - dashed) and $\tilde{P}_2$ (red - dotted).

- $P_1[v, u] \cup P_2[v, u]$ intersects exactly one of the disks Δ_1 or Δ_2, say Δ_1. Then $P_1[u, z_1] \cup P_2[u, z_2]$ intersects Δ_2. It follows that u is an overlapping vertex of both $\tilde{P}_1$ and P and, $\tilde{P}_2$ and P.
- $P_1[v, u] \cup P_2[u, z_2]$ intersects exactly one of the disks Δ_1 or Δ_2, say Δ_1. Then $P_1[u, z_1] \cup P_2[v, u]$ intersects Δ_2. It follows that u is neither an overlapping vertex of $\tilde{P}_1$ and P nor of $\tilde{P}_2$ and P.

From the above cases we obtain that $f_{\tilde{\mathcal{P}}}(u) < f_{\mathcal{P}'}(u)$ and therefore $g(\tilde{\mathcal{P}}) < g(\mathcal{P}')$, contradicting the choice of $\mathcal{P}'$. This completes the proof of the proposition. ■

4. A DECOMPOSITION THEOREM

In this section, we give a decomposition theorem for $(K_5, K_{3,3})$-immersion free graphs and use it to obtain as a corollary a decomposition theorem for $K_{3,3}$-immersion free graphs.

A. The structure of $(K_5, K_{3,3})$-immersion free graphs

We first prove the following decomposition theorem for $(K_5, K_{3,3})$-immersion free graphs.

Theorem 4.1. *If G is a graph not containing K_5 or $K_{3,3}$ as an immersion, then G can be constructed by applying consecutive i-edge sums, for $i \in [3]$, to graphs that are planar and are subcubic or have branch-width at most 10.*

Proof. Observe first that a $(K_5, K_{3,3})$-immersion-free graph is also $(K_5, K_{3,3})$-topological-minor-free, therefore, from Kuratowski's theorem, G is planar. Applying Lemma 2.2, we may assume that G is a $(K_5, K_{3,3})$-immersion-free graph G without any internal i-edge cut, $i \in [3]$. It is now enough to prove that G is either subcubic or has branch-width at most 10. For a contradiction, we assume that $\mathbf{bw}(G) \geq 11$ and that G contains some vertex v of degree at least 4. Our aim is to prove that G contains $K_{3,3}$ as an immersion. First, let G^s be the graph obtained from G after subdividing all of its edges once. Notice that G^s contains $K_{3,3}$ as an immersion if and only if G contains $K_{3,3}$ as an immersion. Hence, from now on, we want to find $K_{3,3}$ in G^s as an immersion.

From Lemma 2.4, G and thus G^s, contains a $(4, 4)$-railed annulus as a subgraph. Observe then that G^s also contains as a subgraph a $(2, 4)$-railed annulus such that the vertex v of degree at least 4 does not belong to the annulus between its cycles. (Fig. 8

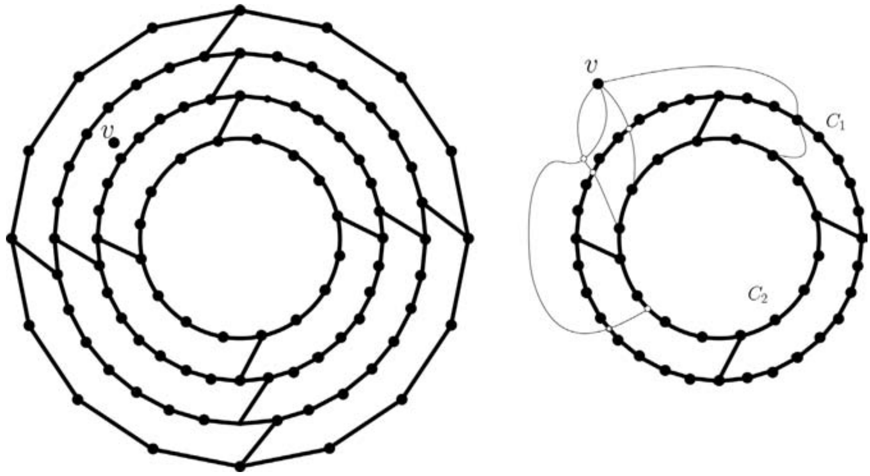

FIGURE 8. The (4, 4)-railed annulus and the vertex v.

depicts the case where v is inside the annulus between the second and the third cycle.) We denote by C_1 and C_2 the nested cycles and by R_1, R_2, R_3, and R_4 the rails of the aforementioned (2, 4)-railed annulus. Let A be the annulus between C_1 and C_2. Without loss of generality we may assume that C_1 separates v from C_2 and that A is edge-minimal, that is, there is no other annulus A' such that $|E(A')| < |E(A)|$ and $A' \subseteq A$.

Let now $G_1, G_2, \ldots, G_p$ be the connected components of $A \setminus (C_1 \cup C_2)$.

Claim 1. *For every $i \in [p]$ and every $j \in [2]$, $|N_{G^s}(V(G_i)) \cap V(C_j)| \leq 1$.*

Proof of Claim 1. Assume the contrary. Then there is a cycle C'_j such that C'_j and $C_{j \bmod 2+1}$ define an annulus A' with $A' \subseteq A$ and $|E(A')| < |E(A)|$; a contradiction to the edge-minimality of the annulus A. ■

For every $l \in [p]$, we denote by u_1^l and u_2^l the unique neighbor of G_k in C_1 and C_2, respectively (whenever they exist). We call the connected components of $A \setminus (C_1 \cup C_2)$ that have both a neighbor in C_1 and a neighbor in C_2 *substantial*. Let

$$\mathcal{C} = \left\{\widehat{G}_i = G\left[V(G_i) \cup \left\{u_1^i, u_2^i\right\}\right] \middle| G_i \text{ is a substantial connected component}\right\}.$$

That is, $\mathcal{C}$ is the set of graphs induced by the substantial connected components and their neighbors in the cycles C_1 and C_2. Note that every edge of G has been subdivided in G^s and thus every edge $e \in G$ for which $e \cap C_1 \neq \emptyset$ and $e \cap C_2 \neq \emptyset$ corresponds to a substantial connected component in $\mathcal{C}$.

We now claim that there exist four confluent edge-disjoint paths $P_1, P_2, P_3,$ and P_4 from v to C_2 in G^s. Indeed, recall first that G, and hence G^s, does not contain an internal i-edge cut, $i \in [3]$. Moreover, C_2 contains at least four vertices, say w_i, $i \in [4]$. Then, as $\deg_{G^s}(v) \geq 4$, Observation 2.1 yields that there exist four edge-disjoint paths P_1, P_2, P_3, and P_4 from v to w_1, w_2, w_3, and w_4. Finally, from Proposition 3.2, we may assume that P_1, P_2, P_3, and P_4 are confluent.

Let P'_i be the subpath $P_i[v, z_i]$ of P_i, where z_i is the vertex in $V(P_i) \cap V(C_2)$ whose distance from v in P_i is minimum, $i \in [4]$. Recall that all edges of G have been subdivided in G^s. This implies that there exist four (possibly not disjoint) graphs in $\mathcal{C}$, say $\widehat{G}_1$, $\widehat{G}_2$, $\widehat{G}_3$, and $\widehat{G}_4$ such that $z_i = u_2^i$, $i \in [4]$. We distinguish two cases.

Case 1. The graphs $\widehat{G}_1$, $\widehat{G}_2$, $\widehat{G}_3$, and $\widehat{G}_4$ are vertex disjoint.

This implies that the endpoints of P'_1, P'_2, P'_3, and P'_4 in C_2 are disjoint. Let G' be the graph induced by the cycles C_1, C_2 , and the paths P'_1, P'_2, P'_3, P'_4 and let $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$, and $\widehat{P}_4$ be confluent edge-disjoint paths from v to u_2^1, u_2^2, u_2^3, and u_2^4 in G' such that

(i) $\sum\{e \mid e \in \bigcup_{i\in[4]} E(\widehat{P}_i) \setminus E(A)\}$ is minimum, that is, the number of the edges of the paths that are outside of A is minimum, and
(ii) subject to (i), $\sum\{e \mid e \in \bigcup_{i\in[4]} E(\widehat{P}_i)\}$ is minimum.

Let also $\widehat{G}$ be the graph induced by C_1, C_2, $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$, and $\widehat{P}_4$. From now on we work toward showing that $\widehat{G}$ contains $K_{3,3}$ as an immersion. For every $i \in [4]$ we call a connected component of $\widehat{P}_i \cap C_1$ nontrivial if it contains at least an edge.

Claim 2. *For every $i \in [4]$, $\widehat{P}_i \cap C_1$ contains at most one nontrivial connected component Q_i and u_1^i is an endpoint of Q_i.*

Proof of Claim 2. First, notice that any path from v to z_i in $\widehat{G}$ contains u_1^i and thus, $u_1^i \in V(\widehat{P}_i)$. Observe now that $\widehat{P}_i[u_1^i, u_2^i]$ is a subpath of $\widehat{P}_i$ whose internal vertices do not belong to C_1, thus if u_1^i belongs to a nontrivial connected component Q_i of $\widehat{P}_i \cap C_1$, then u_1^i is an endpoint of Q_i. We will now prove that any nontrivial connected component of $\widehat{P}_i \cap C_1$ contains u_1^i. Assume to the contrary that there exists a nontrivial connected component P of $\widehat{P}_i \cap C_1$ that does not contain u_1^i. Let u be the endpoint of P for which $\mathbf{dist}_{\widehat{P}_i}(u, u_1^i)$ is minimum. Let also u' be the vertex in $V(\widehat{P}_i[u, u_1^i] \cap C_1) \setminus \{u\}$ such that $\mathbf{dist}_{\widehat{P}_i}(u, u')$ is minimum. Let P' be the subpath of C_1 with endpoints u, u' such that $\widehat{P}_i[u, u'] \cup P'$ is a cycle C with $C \cap P = \{u\}$. We further assume that the interior of $\widehat{P}_i[u, u'] \cup P'$ is the open disk that does not contain any vertices of $\widehat{P}_i$. We will prove that for every path $\widehat{P}_j$, $j \in [4]$, $\widehat{P}_j \cap P' \subseteq \{u, u'\}$. As this trivially holds for $j = i$ we will assume that $j \neq i$. Observe that, for every $j \in [4]$, $\widehat{P}_j[v, u_1^j] \cap A \subseteq C_1$ as for every connected component H of $A \setminus (C_1 \cup C_2)$ it holds that $|N_{G^s}(V(H)) \cap V(C_j)| \leq 1$. Furthermore, observe that $\widehat{P}_i[u, u'] \cup P'$ is a separator in $\widehat{G}$. This implies that v does not belong to the interior of $\widehat{P}_i[u, u'] \cup P'$. Thus, if there is a vertex z such that $z \in \widehat{P}_j \cap (P' \setminus \{u, u'\})$, $j \neq i$, there is a vertex $z' \in \widehat{P}_j \cap \widehat{P}_i[u, u']$, a contradiction to the confluence of the paths. We may then replace $\widehat{P}_i[u, u']$ by P', a contradiction to (i). ■

Let us denote by v_i the endpoint of Q_i that is different from u_1^i if Q_i is a nontrivial connected component of $\widehat{P}_i \cap C_1$, $i \in [4]$. Observe that $\widehat{P}_i = \widehat{P}_i[v, v_i] \cup Q_i \cup \widehat{P}_i[u_1^i, u_2^i]$, where we let $Q_i = \emptyset$ in the case where $\widehat{P}_i \cap C_1$ is edgeless, $i \in [4]$. We denote by T_i the subpath of C_1 with endpoints u_1^i and $u_1^{i \bmod 4+1}$ such that $T_i \cap \{\{u_1^1, u_1^2, u_1^3, u_1^4\} \setminus \{u_1^i, u_1^{i \bmod 4+1}\}\} = \emptyset$, $i \in [4]$. From the confluence of the paths $\widehat{P}_i$ and the fact that u_1^i is an endpoint of Q_i it follows that $Q_i \subseteq T_i$ or $Q_i \subseteq T_{i-1}$, $i \in [4]$ where $T_{i-1} = T_{3+i \bmod 4}$ if $i - 1 \notin [4]$.

Claim 3. *There exists an $i_0 \in [4]$ such that $T_{i_0} \cap (Q_{i_0}, Q_{i_0 \bmod 4+1}) \neq T_{i_0}$.*

Proof of Claim 3. Toward a contradiction assume that for every $i \in [4]$, it holds that $T_i \cap (Q_i, Q_{i \bmod 4+1}) = T_i$. It follows that either $Q_i = T_i = \widehat{P}_i[v_i, u_1^i]$, $i \in [4]$, or $Q_{i \bmod 4+1} = T_i$, $i \in [4]$. Notice then that either $v_i = u_1^{i \bmod 4+1}$, $i \in$

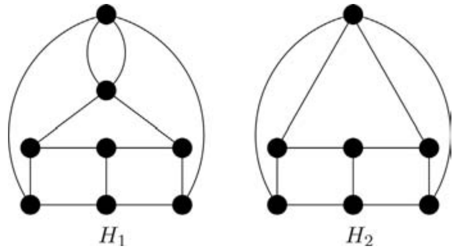

FIGURE 9. The graphs H_1 and H_2.

[4], or $v_{i \bmod 4+1} = u_1^i$, $i \in [4]$, respectively. Then, we let $\tilde{P}_{i \bmod 4+1} = \widehat{P}_i[v, v_i] \cup \widehat{P}_{i \bmod 4+1}[u_1^{i \bmod 4+1}, u_2^{i \bmod 4+1}]$, or $\tilde{P}_i = \widehat{P}_{i \bmod 4+1}[v, v_{i \bmod 4+1}] \cup \widehat{P}_i[u_1^i, u_2^i]$, $i \in [4]$, respectively. Notice that the paths $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$, and $\tilde{P}_4$ are confluent edge-disjoint paths from v to u_2^1, u_2^2, u_2^3, and u_2^4 such that $\cup_{i\in[4]} \tilde{P}_i$ is a proper subgraph of $\cup_{i\in[4]} \widehat{P}_i$. Therefore, we have that $\sum\{e \mid e \in \bigcup_{i\in[4]} E(\tilde{P}_i)\} < \sum\{e \mid e \in \bigcup_{i\in[4]} E(\widehat{P}_i)\}$, a contradiction to (ii). ■

It is now easy to see that $\widehat{G}$, and thus G, contains $K_{3,3}$ as an immersion. Indeed, first remove all edges of $C_1 \setminus T_{i_0}$ that do not belong to any path $\widehat{P}_i$, $i \in [4]$. Then lift the paths $\widehat{P}_i$ to a single edge where $i \neq i_0, i_0 \mod 4+1$. Now let u_{i_0} ($u_{i_0 \bmod 4+1}$, respectively) be the vertex of T_{i_0} that belongs to $\widehat{P}_{i_0}$ ($\widehat{P}_{i_0 \bmod 4+1}$, respectively) whose distance from v in $\widehat{P}_{i_0}$ ($\widehat{P}_{i_0 \bmod 4+1}$, respectively) is minimum and lift the paths $\widehat{P}_{i_0}[v, u_{i_0}]$ and $\widehat{P}_{i_0 \bmod 4+1}[v, u_{i_0 \bmod 4+1}]$ to single edges. Notice now that $\widehat{G}$ contains the graph H_2 depicted in Fig. 9 as an immersion. Thus, we get that $\widehat{G}$ contains $K_{3,3}$ as an immersion.

Case 2. There exist $i_1, i_2 \in [4]$ such that $\widehat{G}_{i_1}$ and $\widehat{G}_{i_2}$ are not vertex disjoint.

Let G^μ be the graph induced by the cycles C_1 and C_2 and the graphs in $\mathcal{C}'$. We will show that G^μ contains $K_{3,3}$ as an immersion. First recall that the common vertices of $\widehat{G}_{i_1}$ and $\widehat{G}_{i_2}$ lie in at least one of the cycles C_1 and C_2. Without loss of generality assume that they have a common vertex in C_1. Recall that, as every edge of G has been subdivided in G^s, there does not exist an edge $e \in G^s$ such that $e \cap C_j \neq \emptyset$, $j \in [2]$. This observation and the fact that there exist four rails between C_1 and C_2 imply that there exist at least four graphs in $\mathcal{C}'$ that are vertex disjoint. It follows that there exist three vertex-disjoint graphs, say $\widehat{G}_{i_3}, \widehat{G}_{i_4}, \widehat{G}_{i_5}$, in $\mathcal{C}'$ with the additional properties that $\widehat{G}_{i_{2+r}} \cap \widehat{G}_{i_1} \cap C_1 = \emptyset$, $r \in [3]$, and that at most one of the $\widehat{G}_{i_3}, \widehat{G}_{i_4}, \widehat{G}_{i_5}$ has a common vertex with one of the $\widehat{G}_{i_1}, \widehat{G}_{i_2}$. Note here that none of the $\widehat{G}_{i_3}, \widehat{G}_{i_4}, \widehat{G}_{i_5}$ can have a common vertex with one of the $\widehat{G}_{i_1}, \widehat{G}_{i_2}$ in C_2, in the case where $\widehat{G}_{i_1} \cap \widehat{G}_{i_2} \cap C_2 \neq \emptyset$. It is now easy to see that G^μ contains H_1 or (H_2, respectively) depicted in Figure 9 as a topological minor when $\widehat{G}_{i_1} \cap \widehat{G}_{i_2} \cap C_2 \neq \emptyset$ ($\widehat{G}_{i_1} \cap \widehat{G}_{i_2} \cap C_2 = \emptyset$, respectively). Observe now that H_1 contains H_2 as an immersion. Moreover, notice that H_2 contains $K_{3,3}$ as an immersion. Thus G^μ, and therefore G^s and G, contain $K_{3,3}$ as an immersion, a contradiction.

B. A decomposition theorem for $K_{3,3}$-immersion free graphs

In this subsection, we show how we can obtain a decomposition theorem for $K_{3,3}$-immersion free graphs from the decomposition theorem of the previous subsection. We first need the following definition.

Clique-sums. Let G_1 and G_2 be two graphs with disjoint vertex sets, let $k \geq 0$ be an integer, and let $X_i \subseteq V(G)$ be a set of pairwise adjacent vertices in G_i of size k, $i \in [2]$. Let G'_i be the graph obtained from G_i after deleting a (possibly empty) set of edges whose both endpoints belong to X_i. If $f : X_1 \to X_2$ is a bijection, the graph $G = G'_1 \oplus_k G'_2$ obtained from the union of G'_1 and G'_2 by identifying x with $f(x)$, $x \in X_1$, is called a k-clique-sum of G_1 and G_2.

Theorem 4.2 ([22]). *A graph G does not contain $K_{3,3}$ as a minor if and only if it can be constructed from planar graphs and K_5 by applying i-clique-sums, $i \in \{0\} \cup [2]$.*

Let us note here that the analog of Lemma 2.2 for $K_{3,3}$-immersion free graphs and i-clique-sums, $i \in \{0\} \cup [2]$, also holds.

Lemma 4.3. *If G is a $K_{3,3}$-immersion free graph such that there exist G'_1 and G'_2 with $G = G'_1 \oplus_i G'_2$, $i \in [2]$, that is, if G can be obtained from G'_1 and G'_2 by applying an i-clique-sum, $i \in \{0\} \cup [2]$, then there also exist $K_{3,3}$-immersion free graphs G_1 and G_2 such that $G = G_1 \oplus_i G_2$, $i \in [2]$.*

Proof. Notice first that the graph G is not 3-connected as the vertices occurring in the i-clique-sum, $i \in \{0\} \cup [2]$ form a separator of G. Notice also that the lemma trivially holds in the case where G is not connected as it can be considered as the 0-clique-sum of the graph induced by exactly one of its connected components and the graph induced by the rest of its connected components. Thus, notice that it is enough to prove the lemma for the case where G is either 1-connected or biconnected. We assume first that G is 1-connected and x is a cut-vertex of G. Let $C_1, C_2, \ldots, C_l$, be the connected components of $G \setminus \{x\}$ and notice that $G = G_1 \oplus_1 G_2$, where $G_1 = G[V(C_1) \cup \{x\}]$ and $G_2 = G[\cup_{i=2}^{l} V(C_i) \cup \{x\}]$. We claim that G_1 and G_2 satisfy the requirements of the lemma. Indeed, if $K_{3,3}$ is an immersion of G_1 or G_2 then, as G_1 and G_2 are subgraphs of G, $K_{3,3}$ is also an immersion of G.

Finally, let us consider the case where the graph G is biconnected. We denote by x and y the vertices of a separator of G of minimum size. Let $C_1, C_2, \ldots, C_l$ be the connected components of $G \setminus \{x, y\}$. Let also G_1 be the graph induced by $V(C_1) \cup \{x, y\}$ and containing the edge $\{x, y\}$ (if $\{x, y\} \notin E(G)$) and G_2 be the graph induced by $\cup_{i=2}^{l} V(C_i) \cup \{x, y\}$ again containing the edge $\{x, y\}$ (if $\{x, y\} \notin E(G)$). We claim that the lemma holds for the graphs G_1 and G_2. Indeed, to the contrary, let us assume that G_1 contains $K_{3,3}$ as an immersion. Observe that if there is a model h of $K_{3,3}$ in G_1^*, where G_1^* is the graph obtained from G_1 after the removal of the edge $\{x, y\}$, then as G_1^* is a subgraph of G, h is also a model of $K_{3,3}$ in G, a contradiction to the hypothesis that G does not contain $K_{3,3}$ as an immersion. Therefore, every model h of $K_{3,3}$ in G_1 uses the edge $\{x, y\}$. This implies that $\{x, y\} \notin E(G)$ as otherwise G_1 is a subgraph of G and therefore, $K_{3,3}$ is also an immersion of G. However, as $\{x, y\}$ is a minimal separator of G, there exists an (x, y)-path P in $G[V(C_2) \cup \{x, y\}]$. It follows that by replacing $\{x, y\}$ with the path P in h we obtain a model h' of $K_{3,3}$ in G, a contradiction to the hypothesis. Similarly,

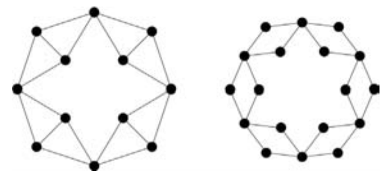

FIGURE 10. Simple nonsub-cubic graphs of branch-width 3 without K_5 or $K_{3,3}$ as immersions.

G_2 does not contain $K_{3,3}$ as an immersion. This completes the proof of the claim and the lemma. ■

Theorem 4.4. *If G is a $K_{3,3}$-immersion free graph then it can be constructed by applying i-edge-sums, $i \in [3]$, and j-clique-sums, $j \in \{0\} \cup [2]$, to a (possibly empty) set of disjoint copies of K_5 and planar graphs that are sub-cubic or have branch-width at most 10, with the further restriction that no 2-clique-sum is applied on two edges that belong to two disjoint copies of K_5.*

Proof. Let G be a graph that does not contain $K_{3,3}$ as an immersion. Then, it also does not contain $K_{3,3}$ as a topological minor. Furthermore, as the maximum degree of $K_{3,3}$ is upper bounded by 3, from a folklore result, it follows that G does not contain $K_{3,3}$ as a minor as well. Combining Lemmata 4.2 and 4.3 we may also assume that G is either isomorphic to K_5 or planar. Applying Lemma 2.2, we may further assume that G does not contain any internal i-edge-cut, $i \in [3]$. Therefore, G is internally 4-edge-connected and is either isomorphic to K_5 or planar. Hence, by following along the lines of the proof of Theorem 4.1, we obtain that G is either K_5 or it is a planar graph which is either subcubic or has branch-width at most 10. Finally, in order to see that no 2-clique-sums are applied on edges of two disjoint copies of K_5, it is enough to observe that $K_5 \oplus_2 K_5$ contains $K_{3,3}$ as an immersion. ■

Remark 4.5. It is easy to verify that our results hold for both the weak and strong immersion relations.

We believe that the upper bound on the branch-width of the building blocks of Theorem 4.1 can be further reduced, especially if we restrict ourselves to simple graphs. There is an infinite family of graphs that are not subcubic and have branch-width 3; two of them are depicted in Figure 10. However, we have not been able to find any simple nonsubcubic graph of branch-width greater than 3 that does not contain K_5 or $K_{3,3}$ as an immersion.

Finally, let us mention here that finding an exact structural characterization of the graphs that do not contain K_5 as a topological minor (which would also imply a structural characterization of the graphs that exclude K_5 as an immersion) is a long-standing open problem.

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