Increasing the minimum degree of a graph by contractions☆

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ABSTRACT

The Degree Contractibility problem is to test whether a given graph G can be modified to a graph of minimum degree at least d by using at most k contractions. We prove the following three results. First, Degree Contractibility is NP-complete even when $d = 14$. Second, it is fixed-parameter tractable when parameterized by k and d. Third, it is W[1]-hard when parameterized by k. We also study its variant where the input graph is weighted, i.e., has some edge weighting and the contractions preserve these weights. The Weighted Degree Contractibility problem is to test if a weighted graph G can be contracted to a weighted graph of minimum weighted degree at least d by using at most k weighted contractions. We show that this problem is NP-complete and that it is fixed-parameter tractable when parameterized by k. In addition, we pinpoint a relationship with the problem of finding a minimal edge-cut of maximum size in a graph and study the parameterized complexity of this problem and its variants.

1. Introduction

Throughout the paper we consider undirected finite graphs that have no loops. Unless we explicitly indicate this, they do not have multiple edges either. We denote the vertex set and the edge set of a graph G by V_G and E_G, respectively. If no confusion is possible, we may omit subscripts. We refer the reader to the text book of Diestel [10] for undefined graph terminology and to the monographs of Downey and Fellows [12] and Niedermeier [27] for more on parameterized complexity.

A graph modification problem has as input a graph G and an integer k. The question is whether G can be modified to belong to some specified graph class that satisfies further properties by using at most k operations of a certain specified type such as deleting a vertex or deleting an edge. In our paper the permitted operation is the *contraction* of an edge, which removes both end-vertices of the edge and replaces them by a new vertex adjacent to precisely those vertices to which the two end-vertices were adjacent.

We continue a very recent study [17–19] of the following graph modification problem called Π-Contractibility, where Π is some prespecified graph class.

Π-Contractibility
Instance: a graph G and an integer k.
Question: can G be modified to a graph in Π by at most k contractions?

1.1. Previous results

Research on the Π-Contractibility problem dates back to the early eighties, when Watanabe, Ae and Nakamura [29,30] showed that Π-Contractibility is NP-complete if Π is finitely characterizable by 3-connected graphs. Their result was generalized by Asano and Hirata [2] who showed that Π-Contractibility is NP-complete whenever Π is a graph class that fulfills the following conditions. First, Π must be closed under contractions. Second, Π must be described by a property that is satisfied by infinitely many connected graphs and violated by infinitely many other connected graphs. Third, a graph belongs to Π if and only if each of its biconnected components belong to Π. Examples [2] of such graph classes Π include planar graphs, outerplanar graphs, series–parallel graphs, and also forests, chordal graphs, or more generally, graphs with no cycles of length at least ℓ for some fixed integer $\ell \geq 3$.

The problem Π-Contractibility is closely related to the problem H-Contractibility, which is to test whether a given graph G can be contracted to a fixed graph H (i.e., which is not part of the input). Brouwer and Veldman [8] showed that the H-Contractibility problem is NP-complete whenever H is a triangle-free graph that contains no vertex adjacent to all the other vertices. Their work has been extended by a series of other papers [20,23,24] showing both polynomial-time solvable and NP-complete cases. Determining a full complexity classification for H-Contractibility is open, although such results restricting the input graph G to be in a special graph class have been obtained [3,4,22].

If Π is the class of paths or cycles, then Π-Contractibility is polynomially equivalent to the problems of determining the length of a longest path and a longest cycle, respectively, to which a given graph can be contracted. The first problem has been shown to be NP-complete by van 't Hof, Paulusma and Woeginger [21] even for graphs with no induced path on 6 vertices. The second problem has been shown to be NP-complete by Hammack [16].

Eppstein [13] showed that it is NP-complete to decide if a graph contains a complete graph K_p as a minor for some given integer p. This problem is equivalent to deciding if a graph is contractible to K_p. As a direct consequence, Π-Contractibility is NP-complete if Π is the class of complete graphs. Due to a close relationship with the problem that is to test whether a given graph contains a so-called disconnected cut set, Martin and Paulusma [25] have shown that Π-Contractibility is NP-complete if Π is the class of bicliques $K_{p,q}$ with $p, q \geq 2$.

Recently, more papers appeared that study the Π-Contractibility problem, and in particular, its *parameterized* complexity where the parameter is the number k of edges that may be contracted. Heggernes et al. [19] gave an $4^{k+O(\log^2 k)} + n^{O(1)}$ time algorithm for Π-Contractibility if Π is the class of paths. Moreover, they showed that in this case the problem has a linear kernel. When Π is the class of trees, they showed that the problem can be solved in $4.88^k n^{O(1)}$ time and that a polynomial kernel does not exist unless coNP $\subseteq$ NP $\setminus$ poly. When the input graph is a chordal graph with n vertices and m edges, Heggernes et al. [17] could show that Π-Contractibility can be solved in $O(n+m)$ time when Π is the class of trees and in $O(nm)$ time when Π is the class of paths. When Π is the class of bipartite graphs, Heggernes et al. [18] observed that Π-Contractibility is NP-complete and showed that Π-Contractibility is fixed-parameter tractable when parameterized by k. Later on, Marx, O'Sullivan, and Razgon [26] obtained this result for bipartite graphs as a corollary from their result on generalized bipartization.

Bodlaender, Koster and Wolle [6] introduced the related notion of *contraction degeneracy* as a useful tool to improve lower bound heuristics for treewidth. The contraction degeneracy of a graph G is the largest minimum degree of any minor of G. When G is connected, the contraction degeneracy of G is equal to the largest minimum degree of any graph to which G can be contracted [6]. The Contraction Degeneracy problem is to test whether the contraction degeneracy of a given graph is at least d for some given integer d (see also [28] for extensions of this problem). Bodlaender, Koster and Wolle [6] proved that this problem is NP-complete, even for bipartite graphs, and that it is fixed-parameter tractable when parameterized by d. They also evaluated a number of heuristics for computing the contraction degeneracy.

1.2. Our results

In order to define the class Π of graphs that we consider, we need the following terminology. A vertex u in a graph $G = (V, E)$ is a *neighbor* of a vertex u if $uv \in E_G$. We let $N(u) = \{uv \mid v \in V\}$ denote the *neighborhood* of u. The *degree* of a vertex u is denoted $d(u) = |N(u)|$. We let $\delta = \min\{d(v) \mid v \in V\}$ denote the *minimum degree* of G. We study the Π-Contractibility problem where Π is the class of graphs of minimum (vertex) degree at least d for some integer d. Note that this class of graphs does not satisfy the first and third property of Asano and Hirata [2]. Moreover, for this class of graphs, we allow the integer d to be part of the input as well. This leads to the following problem.

Degree Contractibility
Instance: a graph G and two integers d and k.
Question: can G be modified to a graph of minimum degree at least d by at most
$\quad\quad k$ contractions?

We observe that the problem becomes equivalent to the Contraction Degeneracy problem when the input is restricted to connected graphs $G = (V, E)$ and $k \geq |E|$.

In Section 2 we show that Degree Contractibility is fixed-parameter tractable when parameterized by k and d. However, when either k or d is part of the input, Degree Contractibility becomes hard in the following sense. First, if k is part of the

Table 1
An overview of our results for the problems DC and WDC.

Input	Parameter	DC	WDC
d, k		NP-complete	NP-complete
d	k	W[1]-hard	FPT
k	d	para-NP-complete	para-NP-complete
	d, k	FPT	FPT

input, then DEGREE CONTRACTIBILITY is NP-complete for any fixed $d \geq 14$. Second, if d is part of the input, then DEGREE CONTRACTIBILITY is W[1]-hard when parameterized by k. These results complement the result of Amini, Sau and Saurabh [1] who showed that detecting a subgraph that has at most k vertices and minimum degree at least d is W[1]-hard for any fixed $d \geq 4$ when parameterized by k.

In Section 3 we study a weighted version of DEGREE CONTRACTIBILITY. In order to define this variant, let $G = (V, E)$ be a *weighted* graph, i.e., with some edge weighting $w: E \rightarrow \mathbb{R}_{>0}$ where $\mathbb{R}_{>0}$ denotes the set of positive real numbers. The *weighted degree* $d^w(u)$ of a vertex u is the sum of the weights of the edges incident with u in G, i.e., $d^w(u) = \sum_{v \in N(u)} w(uv)$. We let $\delta_w = \min\{d^w(v) \mid v \in V\}$ denote the *minimum weighted degree* of G. The *weighted contraction* of an edge $e = uv$ is a contraction of e where the weights on the edges incident with the new vertex x_{uv} are defined as follows:

- $w(x_{uv}y) = w(uy)$ if y is adjacent to u and not adjacent to v;
- $w(x_{uv}y) = w(vy)$ if y is adjacent to v and not adjacent to u;
- $w(x_{uv}y) = w(uy) + w(vy)$ if y is adjacent to u and v.

We can now state the new variant.

WEIGHTED DEGREE CONTRACTIBILITY
Instance: a weighted graph G and two integers d and k.
Question: can G be modified to a weighted graph of minimum weighted degree
at least d by at most k weighted contractions?

Because the weight of an edge $x_{uv}y$ with y adjacent to both u and v is the accumulated weight of the two original edges uy and vy, DEGREE CONTRACTIBILITY is not a special (unweighted) case of WEIGHTED DEGREE CONTRACTIBILITY. However, we can make the following observation. A *simple contraction* is the operation on loopless multigraphs that identifies both end-vertices of the edges, keeps multiple edges, but removes the loop that was created. The WEIGHTED DEGREE CONTRACTIBILITY problem for *integer* edge weights on a graph G is equivalent to the variant of DEGREE CONTRACTIBILITY, where simple contractions are used on the loopless multigraph G' obtained from G by replacing each edge uv by $w(uv)$ parallel edges.

Contrary to the aforementioned W[1]-hardness result for DEGREE CONTRACTIBILITY when parameterized by k, accumulating the weights after contracting an edge results in the problem not being hard anymore, i.e., we prove that WEIGHTED DEGREE CONTRACTIBILITY is fixed-parameter tractable when parameterized by k even when d is part of the input. If both d and k are parts of the input, then WEIGHTED DEGREE CONTRACTIBILITY is NP-complete in the strong sense for integer edge weights, even in the case when $k \geq |E|$. The latter case is equivalent to the case, in which there is no upper bound imposed on the number of weighted contractions. We denote this special case as the problem

WEIGHTED CONTRACTION DEGENERACY
Instance: a weighted graph G and an integer d.
Question: can G be modified to a weighted graph of minimum weighted degree
at least d by weighted contractions?

We prove that this problem is fixed-parameter tractable when parameterized by d. Both this result and the aforementioned NP-completeness result are based on an equivalence, which we pinpoint, between WEIGHTED CONTRACTION DEGENERACY and the problem that is to test whether a connected graph has a minimal edge-cut of some given size.

Table 1 summarizes our results for the DEGREE CONTRACTIBILITY (DC) problem and the WEIGHTED DEGREE CONTRACTIBILITY (WDC) problem. In this table, the para-NP-completeness of the problem WEIGHTED DEGREE CONTRACTIBILITY parameterized by d immediately follows from our result that WEIGHTED DEGREE CONTRACTIBILITY is NP-complete in the strong sense for integer edge weights, which allows us to fix $d = 1$ after dividing all edge weights by d. However, when we restrict the edge weighting to be integer, the corresponding problem is still open.

In Section 4 we identify two related problem settings. In the first setting weighted plane graphs are considered, where the edge weightings define weighted degrees of the faces. In the second setting, parity constraints are considered instead of degree constraints. We show how these problem settings are related with our previous problems, and we study them using results of the previous sections.

1.3. Preliminaries

Let $G = (V, E)$ be a (weighted) graph. A subset $U \subseteq V$ is a *clique* if there is an edge in G between any two vertices of U, and U is an *independent set* if there is no edge in G between any two vertices of U. We write $G[U]$ to denote the subgraph of G

induced by $U \subseteq V$, i.e., the graph on vertex set U and an edge between any two vertices if and only if there is an edge between them in G. We let G/e denote the (weighted) graph obtained from G by the (weighted) contraction of e. If a (weighted) graph H is obtained from G by a sequence of (weighted) contractions, then H is a *(weighted) contraction* of G. For a weighted graph G with an edge weighting w and $X \subseteq E$, we write $w(X) = \sum_{e \in X} w(e)$.

Let G and H be two graphs. An *H-witness structure* $\mathcal{W}$ is a vertex partition of G into $|V_H|$ (nonempty) sets $W(x)$ called *H-witness bags*, such that

(i) each $W(x)$ induces a connected subgraph of G;
(ii) for all $x, y \in V_H$ with $x \neq y$, bags $W(x)$ and $W(y)$ are adjacent in G if and only if x and y are adjacent in H;

By contracting all bags to singletons we observe that H is a contraction of G if and only if G has an H-witness structure such that conditions (i)–(ii) hold. Note that a graph may have more than one H-witness structure.

2. Contractions

First, we observe that DEGREE CONTRACTIBILITY is FPT when parameterized by k and d.

Proposition 1. DEGREE CONTRACTIBILITY *can be solved in time* $O(d^k(n + m))$ *for graphs with n vertices and m edges.*

Proof. Let G be a graph with n vertices and m edges. We give the following branching algorithm. Let $d_G(u) < d$ for some vertex $u \in V_G$. We consider all edges e incident with u, and call our algorithm recursively for G/e and parameter $k' = k - 1$. The algorithm returns Yes, if for at least one of the new instances the answer is Yes, and it returns No otherwise. Since for each recursive call of our algorithm, we create at most $d - 1$ instances of the problem, and the depth of the recursion is at most k, the algorithms runs in time $O(d^k(n + m))$. $\square$

A graph G is *r-degenerate* for some integer r if $\delta(H) \leq r$ for every subgraph H of G. A graph class $\mathcal{G}$ is *contraction-closed* if $G/e \in \mathcal{G}$ for every graph $G \in \mathcal{G}$ and every $e \in E_G$. Proposition 1 has the following consequence.

Corollary 1. *Let* $\mathcal{G}$ *be a contraction-closed graph class so that all graphs in* $\mathcal{G}$ *are r-degenerate for some integer* $r \geq 0$. *Then* DEGREE CONTRACTIBILITY *can be solved in time* $O(r^k(n + m))$ *for every* $G \in \mathcal{G}$ *with n vertices and m edges.*

Proof. If $d > r$, then we cannot modify a graph $G \in \mathcal{G}$ to a graph of minimum degree at least d by edge contractions. Otherwise, we apply Proposition 1. $\square$

Corollary 1 holds, for example, for the class of planar graphs which are 5-degenerate, or more general, for bounded-genus graph classes and excluded-minor graph classes.

For general graphs, we observe that DEGREE CONTRACTIBILITY is in XP when parameterized by k, i.e., it can be solved in $n^{f(k)}$ time for n-vertex graphs by checking all sequences of at most k contractions (here f is some function depending only on k). However, we show that it is unlikely to be solvable in FPT-time.

Theorem 1. DEGREE CONTRACTIBILITY *parameterized by k is* W[1]*-hard.*

Proof. The problem MULTICOLORED CLIQUE is to test whether a graph with a proper k-coloring contains a clique of size k with exactly one vertex from each color class. Fellows et al. [11] proved that this problem is W[1]-hard when parameterized by k. Consequently, its "complementary" problem, the problem MULTICOLORED INDEPENDENT SET, which is to test whether a graph with a partition $X_1, \ldots, X_k$ of the vertex set, where each X_i is a clique, has an independent set of size k with exactly one vertex from each X_i, is W[1]-hard as well when parameterized by k. Our aim is to reduce DEGREE CONTRACTIBILITY to this problem.

Let (G, k) with a partition $X_1, \ldots, X_k$ of V_G be an instance of MULTICOLORED INDEPENDENT SET. Let clique $X_i = \{x_{i1}, \ldots, x_{in_i}\}$ for $i \in \{1, \ldots, k\}$ where we assume without loss of generality that $n_i \geq 2$. Let $d = n(4k + 3) + 1$. From G we construct a graph G' in the following way. Recall that connecting two vertices means adding an edge between them.

1. Construct a clique W with vertices $w_1, \ldots, w_{d+1}$.
2. Connect every x_{ij} with $w_1, \ldots, w_{t_{ij}}$ where $t_{ij} = d - d_G(x_{ij}) - n_i - 4k - 2$.
3. Add vertices $y_1, \ldots, y_k$.
4. For every x_{ij}, construct a clique Q_{ij} with vertices $r_{ij}, a_{ij}^{(1)}, \ldots, a_{ij}^{(2k+1)}, b_{ij}^{(1)}, \ldots, b_{ij}^{(2k+1)}$ and connect every vertex of Q_{ij} with x_{ij} and y_i. Moreover, connect every r_{ij} with $w_1, \ldots, w_{d-(4k+3)}$, every $a_{ij}^{(s)}$ with $w_1, \ldots, w_{d-(4k+3)}$, and every $b_{ij}^{(s)}$ with $w_{4k+5}, \ldots, w_{d+1}$.
5. Construct a clique C with vertices $c_1, \ldots, c_{k+2}, z_1, \ldots, z_k$.
6. For $h = 1, \ldots, k + 2$, connect c_h with $w_1, \ldots, w_{d-2k+1}$
7. For $i = 1, \ldots, k$, connect z_i with every vertex of X_i.

Stages 1–4 of the construction are shown in Fig. 1(a), and Stages 5–7 are shown in Fig. 1(b). We let $k' = k(2k + 3)$ and claim that G has an independent set with exactly one vertex from each X_i if and only if G' can be modified to a graph H with minimum degree at least d by at most k' contractions.
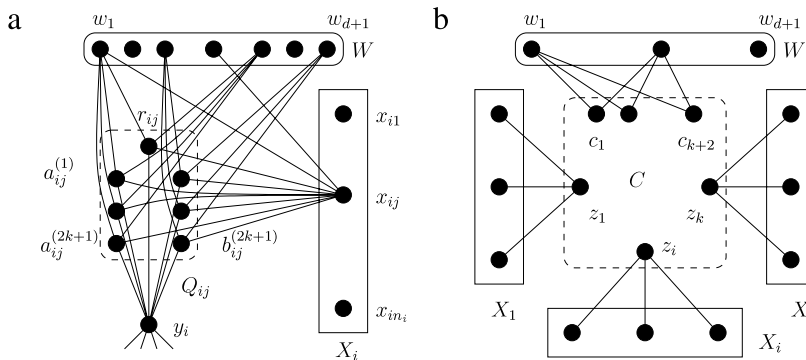
Fig. 1. The construction of G.

First suppose that $\{x_{1j_1}, \ldots, x_{kj_k}\}$ is an independent set in G. For $i = 1, \ldots, k$, let $A_i = \{a_{ij_i}^{(1)} b_{ij_i}^{(1)}, \ldots, a_{ij_i}^{(2k+1)} b_{ij_i}^{(2k+1)}, r_{ij_i} y_i, x_{ij_i} z_i\}$ be a set of $2k + 3$ edges in G'. We contract every edge in every A_i. Then the total number of contractions is $k(2k + 3) = k'$. Moreover, the resulting graph has minimum degree at least d.

Now suppose that G' can be modified to a graph H with minimum degree at least d by at most k' contractions. Let $\mathcal{W}$ be an H-witness structure of G'. For each bag W of $\mathcal{W}$, we choose an arbitrary spanning tree of $G'[W]$. Let $A \subseteq E_{G'}$ denote the union of the sets of edges of these trees. Because we obtain H by contracting the edges of A, we find that $|A| \leq k'$.

Claim 1. $A = A_1 \cup \cdots \cup A_k$, where each $A_i = \{x_{ij}z_i, y_if, g_1h_1, \ldots, g_{2k+1}h_{2k+1}\}$ with $\{f, g_1, \ldots, g_{2k+1}, h_1, \ldots, h_{2k+1}\} = Q_{ij}$.

We prove Claim 1 as follows. Let $1 \leq i \leq k$. Because $d_{G'}(y_i) < d$, at least one edge incident with y_i must be included in A_i. Assume that $y_if \in A_i$ for some $f \in Q_{ij}$. Note that after contracting y_if, all $4k + 2$ vertices of $Q_{ij} \setminus \{f\}$ have degrees less that d. Hence, at least $2k + 1$ edges incident with these vertices must be contracted. We also note that $d_{G'}(z_i) < d$. Therefore, at least one edge incident with z_i must be in A. Suppose that $z_it \in A$ for some $t \in C$. Then, after contracting z_it, all other $2k$ vertices of C have degrees less than d. Hence, we must contract at least k edges incident with these vertices. Because the total number of contractions is $k' = k(2k + 3)$ and we also need to contract at least $2k + 3$ edges for every $h \neq i$, this is not possible. We conclude that $z_ix_{ij'} \in A$ for some $1 \leq j' \leq n_i$ and that $A_i = \{x_{ij}z_i, y_if, g_1h_1, \ldots, g_{2k+1}h_{2k+1}\}$ with $\{f, g_1, \ldots, g_{2k+1}, h_1, \ldots, h_{2k+1}\} = Q_{ij}$. We now consider x_{ij} and observe that by contracting the edges $g_1h_1, \ldots, g_{2k+1}h_{2k+1}$ we decreased the degree of x_{ij} by $2k + 1$. Hence, $j' = j$ and Claim 1 follows.

Due to Claim 1, we can define the set $\{x_{1j_1}, \ldots, x_{kj_k}\}$ with $x_{ij_i}z_i \in A_i$ for $i = 1, \ldots, k$. We prove that this is an independent set in G. In order to obtain a contradiction, assume that there is an edge $x_{ij_i}x_{i'j_{i'}} \in E_G$. Recall that $d_{G'}(x_{ij}) = d_G(x_{ij}) + (n_i - 1) + (4k + 3) + 1 + t_{ij} = d + 1$. Contracting those edges of A_i that have both end-vertices in Q_{ij} decreases the degree of x_{ij} by $2k + 1$. Moreover, after contracting the edges in A_i and $A_{i'}$, the edges $z_iz_{i'}$ and $x_{ij_i}x_{i'j_{i'}}$ have been replaced by one edge. Because z_i is adjacent to all vertices in $C \setminus \{z_i\}$, this means that the degree of the vertex of H obtained by contracting $x_{ij}z_i$ is at most $d + 1 - (2k + 1) - 2 + (2k + 1) = d - 1$. This is not possible. Hence, $\{x_{1j_1}, \ldots, x_{kj_k}\}$ is an independent set in G with a vertex, namely x_{ij_i}, from each X_i, as desired. This completes the proof of Theorem 1. $\square$

If we parameterize the problem only by d, then Degree Contractibility becomes hard even if d is a fixed integer.

Theorem 2. *For any fixed $d \geq 14$, Degree Contractibility is NP-complete.*

Proof. The inclusion of the problem in NP is obvious. For simplicity, we prove NP-hardness for $d = 14$. We reduce from the NP-complete Set Cover problem [14]. This problem is defined as follows.

Given a set $U = \{u_1, \ldots, u_m\}$, a family of subsets $X_1, \ldots, X_n \subseteq U$ and an integer r, are there at most r subsets that *cover* U, i.e., their union is U?

It is known [14] that this problem remains NP-complete even if

(i) each X_i has cardinality 3, and
(ii) each u_j is included in either two or three subsets of $X_1, \ldots, X_n$.

We consider an instance $(U, X_1, \ldots, X_n)$ of Set Cover with restrictions (i) and (ii). We construct a graph G in the following way; also see Fig. 2. We say that we *connect* a vertex with some other vertex if we add an edge between them.

1. Construct a clique with 13 vertices $w_1, \ldots, w_{13}$.
2. Add two new vertices s, t and connect each of them with $w_1, \ldots, w_{13}$.
3. For $i = 1, \ldots, n$, add a vertex x_i and connect it with s, t.
4. For $i = 1, \ldots, n$, add two adjacent vertices $p_i^{(1)}, p_i^{(2)}$, connect $p_i^{(1)}$ with $s, w_1, \ldots, w_{11}, x_i$, and connect $p_i^{(2)}$ with $s, w_3, \ldots, w_{13}, x_i$.
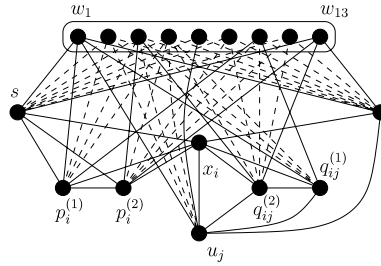5. For $j = 1, \ldots, m$, add a vertex u_j and connect it with t.

Fig. 2. The construction of G.

6. Connect x_i and u_j whenever $u_j \in X_i$. In that case also add two adjacent vertices $q_{ij}^{(1)}$, $q_{ij}^{(2)}$, connect $q_{ij}^{(1)}$ with x_i, u_j, w_1, $\ldots$, w_{11} and connect $q_{ij}^{(2)}$ with x_i, u_j, w_3, $\ldots$, w_{13}.
7. For $j = 1, \ldots, m$, connect u_j with $w_1, \ldots, w_8$ if u_j occurs in two subsets of $X_1, \ldots, X_n$, and connect u_j with $w_1, \ldots, w_6$ if u_j occurs in three subsets.

We set $k = n + r$ and claim that U can be covered by at most r subsets of $\{X_1, \ldots, X_n\}$ if and only if G can be modified to a graph with minimum degree at least $d = 14$ by at most k contractions.

First suppose that $X_{i_1}, \ldots, X_{i_r}$ is a set cover of U, i.e., $U = X_{i_1} \cup \cdots \cup X_{i_r}$. For $j = 1, \ldots, r$, we contract the edges $s x_{i_j}$ and $p_{i_j}^{(1)} p_{i_j}^{(2)}$. We also contract the edge $x_i t$ for every $i \notin \{i_1, \ldots, i_r\}$. The total number of contractions is $2r + (n - r) = n + r = k$. Moreover, the resulting graph is readily seen to have minimum degree at least 14, as desired.

Now suppose G can be modified to a graph H with minimum degree at least $d = 14$ by at most k contractions. Let $\mathcal{W}$ be an H-witness structure of G. For each bag W of $\mathcal{W}$, we choose an arbitrary spanning tree of $G[W]$. Let $A \subseteq E_G$ denote the union of the sets of edges of these trees. Because H is obtained by contracting the edges of A, we find that $|A| \leq k$.

For each X_i, we define a set of edges $E_i \subseteq E_G$ as follows. The set E_i includes all edges incident with x_i, $p_i^{(1)}$, $p_i^{(2)}$, and all edges incident with $q_{ij}^{(1)}$, $q_{ij}^{(2)}$ for every $u_j \in X_i$. Moreover, we choose one vertex $u_j \in X_i$ and also add all (other) edges incident with u_j to E_i. The sets $E_1, \ldots, E_n$ have the following properties.

1. $E_i \cap E_j = \emptyset$ for $1 \leq i < j \leq n$.
2. $E_i \cap A \neq \emptyset$ for $i = 1, \ldots, n$.
3. The number of sets E_i with $|E_i \cap A| \geq 2$ is at most r.

Property 1 is true by definition. Property 2 follows from the fact that $d_G(x_i) = 13 < 14 = d$; therefore, at least one edge incident with x_i must be contracted. Property 3 follows from properties 1 and 2 and the aforementioned observation that $|A| \leq k = n + r$.

Let $I = \{i \mid |E_i \cap A| \geq 2\}$. We claim that $\cup_{i \in I} X_i = U$. In order to obtain a contradiction, assume that there is a vertex $u_j \in U \setminus \cup_{i \in I} X_i$. Then, for each X_i with $u_j \in X_i$, we find that E_i contains a unique edge $e_i \in E_i \cap A$. Because $d_G(x_i) = 13 < d$, e_i is incident with x_i. If $e_i = s x_i$, then contracting e_i decreases the degree of $p_i^{(1)}$ and $p_i^{(2)}$. Because they both have degree 14, at least one edge incident with them must be contracted as well. Hence, $e_i \neq s x_i$. Similarly, if $e_i = x_i p_i^{(1)}$ then contracting e_i decreases the degree of $p_i^{(2)}$. Hence, $e_i \neq x_i p_i^{(1)}$. We apply the same arguments on the other edges in E_i and conclude that the only possibility is $e_i = x_i t$. Now we consider two cases.

Case 1. u_j is included in exactly two sets X_{i_1}, X_{i_2}. Then edges $x_{i_1} t$, $x_{i_2} t$ are contracted, whereas all other edges incident with x_{i_1}, x_{i_2} and also edges $p_{i_1}^{(1)} p_{i_1}^{(2)}$, $p_{i_2}^{(1)} p_{i_2}^{(2)}$, $q_{i_1 j}^{(1)} q_{i_1 j}^{(2)}$, $q_{i_2 j}^{(1)} q_{i_2 j}^{(2)}$ are not contracted. Moreover, no edges incident with u_j are contracted, because these belong to $E_{i_1} \cup E_{i_2}$. However, then u_j has degree at most $13 < d$ in H, a contradiction.

Case 2. u_j is included in three sets $X_{i_1}, X_{i_2}, X_{i_3}$. By the same arguments as in Case 1, we find that the degree of u_j in H is at most $13 < d$, a contradiction.

We conclude that $\{X_i \mid i \in I\}$ is a set cover, which contains at most r sets due to Property 3. This completes the proof of Theorem 2. $\square$

While it can be easily seen that for any fixed $d \leq 3$, DEGREE CONTRACTIBILITY can be solved in polynomial time, determining the complexity for $4 \leq d \leq 13$ is an open question.

3. Weighted contractions

3.1. Weighted Degree Contractibility parameterized by k

We first show that WEIGHTED DEGREE CONTRACTIBILITY is in FPT when parameterized by k. Recall that x_{uv} denotes the vertex obtained from u and v after contracting an edge uv in a graph.

Theorem 3. Weighted Degree Contractibility *can be solved in time* $O(2^k k^{2k}(n+m))$ *for weighted graphs with n vertices and m edges.*

Proof. Let d and k be two integers, and let G be a weighted graph with n vertices and m edges. Below we present our algorithm for deciding whether G can be modified to a weighted graph of minimum weighted degree at least d by at most k weighted contractions. Let $U = \{u \in V_G \mid d_G^w(u) < d\}$ and let $r = |U|$. Trivially, if $r = 0$, then the answer is Yes. If $r \geq 1$, then we branch according to the following four cases.

Case 1. $r > 2k$.
The algorithm returns No. The reason is that at least one edge incident with each vertex of U must be contracted to get a graph of minimum weighted degree at least d, and every edge is incident with at most two vertices of U.

Case 2. $r \leq 2k$ and there is a vertex $u \in U$ with $d_G(u) \leq k$.
At least one edge incident with u must be contracted to obtain a graph of minimum degree at least d. Hence, for each edge e incident with u, we call our algorithm recursively for G/e and parameter $k' = k - 1$. The algorithm returns Yes if for at least one of the new instances the answer is Yes, and No otherwise.

Case 3. $k < r \leq 2k$ and $d_G(u) \geq k + 1$ for all $u \in U$.
If G can be contracted to a graph of minimum weighted degree at least d, then at least one edge with both its end-vertices in U must be contracted. Note that there at most $k(2k-1)$ such edges. If there are no such edges, then the algorithm returns No. Otherwise, for each $e = xy$ with $x, y \in U$, we call our algorithm recursively for G/e and parameter $k' = k - 1$. The algorithm returns Yes if for at least one of the new instances the answer is Yes, and No otherwise.

Case 4. $r \leq k$ and $d_G(u) \geq k + 1$ for all $u \in U$.
Let $U = \{u_1, \ldots, u_r\}$. Each u_i is adjacent to at least two vertices in $V_G \setminus U$. For $i = 1, \ldots, r$, we do the following. Let y, z be two neighbors of u in $V_G \setminus U$, where we assume that $w(u_i y) \leq w(u_i z)$. Let $G' = G/u_i y$. Then we deduce that

$$
\begin{aligned}
d_{G'}^w(x_{u_i y}) &= d_G^w(u_i) + d_G^w(y) - 2w(u_i y) \\
&\geq w(u_i y) + w(u_i z) + d_G^w(y) - 2w(u_i y) \\
&= d_G^w(y) - w(u_i y) + w(u_i z) \\
&\geq d_G^w(y) \\
&\geq d.
\end{aligned}
$$

Hence, we contract $u_i y$ and recursively proceed with G' and $U' = U \setminus \{u_i\}$. Note that the weighted contraction of $u_i y$ does not change the weighted degrees of the other vertices. Consequently, each vertex in U' is adjacent to at least two vertices of weighted degree at least d in G', and U' is the set of vertices of weighted degree at most $d - 1$ in G'. Then after processing u_r, we obtain a graph of minimum degree at least d by using $r \leq k$ weighted contractions. Hence, our algorithm always returns Yes in this case.

To estimate the running time, observe that for each recursive call of our algorithm, we create at most $k(2k-1)$ instances of the problem, and the depth of the recursion is at most k. Hence, the algorithm runs in time $O(2^k k^{2k}(n+m))$. □

3.2. Weighted Contraction Degeneracy

Recall that we call the special case of Weighted Degree Contractibility. in which there is no upper bound on the number of weighted contractions, i.e., in which $k = |E_G|$, the Weighted Contraction Degeneracy problem. In this section we prove that Weighted Degree Contractibility is NP-complete in the strong sense but FPT when parameterized by d if all edge weights are integers. We first introduce some extra terminology.

Let $G = (V, E)$ be a connected graph. For a proper subset $U \subset V$, the set of edges that have one end-vertex in U and the other one in $\overline{U} = V \setminus U$ is called an *edge-cut* denoted $E(U, \overline{U})$. An edge-cut C is *minimal* if G has no edge cut $C' \subset C$. The following lemma is well known (see e.g [10]).

Lemma 1. *Let $G = (V, E)$ be a connected graph and $U \subset V$. Then $E(U, \overline{U})$ is a minimal edge-cut of G if and only if $G[U]$ and $G[\overline{U}]$ are both connected.*

For the proofs of our results we also need the following lemma.

Lemma 2. *Let G be a connected weighted graph with an edge weighting w, and let $d \in \mathbb{R}_{>0}$. Then G has a weighted contraction H with $\delta_w(H) \geq d$ if and only if G has a minimal edge-cut C with $w(C) \geq d$.*

Proof. First suppose that H is a weighted contraction of G with $\delta_w(H) \geq d$. Let $\mathcal{W}$ be a corresponding H-witness structure. Let x be a vertex of H that is not a cut-vertex. Then the subgraphs of G induced by $U = W(x)$ and by $\overline{U}$, respectively, are connected. Lemma 1 tells us that $C = E(U, \overline{U})$ is a minimal edge-cut of G. Because $d_H^w(x) \geq d$, we deduce that $w(C) \geq d$.

Now suppose that G has a minimal edge-cut $C = E(U, \overline{U})$ with $w(C) \geq d$. Then $G[U]$ and $G[\overline{U}]$ are connected graphs due to Lemma 1. Contracting the edges of $G[U]$ and $G[\overline{U}]$ yields a graph H that has two vertices and one edge with weight at least d. Hence, $\delta_w(H) \geq d$. $\quad\square$

Lemma 2 implies that the WEIGHTED CONTRACTION DEGENERACY problem is equivalent to the MAXIMUM MINIMAL CUT problem that is to test whether a connected graph G with an edge weighting w has a minimal edge-cut C with $w(C) \geq d$ for some given integer d.

A problem is said to be NP-complete *in the strong sense*, if it remains NP-complete even when all of its numerical parameters are bounded by a polynomial in the size of the input. We prove that WEIGHTED DEGREE CONTRACTIBILITY is NP-complete in the strong sense.

Theorem 4. WEIGHTED CONTRACTION DEGENERACY *with integer edge weights is* NP-*complete in the strong sense.*

Proof. It is clear that the problem is in NP. In order to prove NP-hardness, we reduce from the MAX-CUT problem. This NP-complete problem [7] is to test whether a connected graph G has an edge-cut C with at least s edges for some given integer s.

Given an instance (G, s) of MAX-CUT, we construct a weighted graph G' as follows. We add two new adjacent vertices u and v to G by making each of them adjacent to all vertices of G. We set $w(uv) = n + m$, and $w(e) = 1$ for all other edges in G'. We let $d = 2n + m + s$. In this way we obtained an instance (G', d) of WEIGHTED CONTRACTION DEGENERACY. Observe that all numerical parameters of this instance, i.e., all edge weights and d are polynomially bounded in n and m.

We claim that G has an edge-cut C with $|C| \geq s$ if and only if G' has a minimal edge-cut C' with $w(C') \geq d$.

First suppose that G has an edge-cut $C = E_G(U, \overline{U})$ with $|C| \geq s$. We define $U' = U \cup \{u\}$ and $\overline{U'} = (V_G \setminus U) \cup \{v\}$. Because both u and v are adjacent to all vertices of G, we find that both $G'[U']$ and $G'[\overline{U'}]$ are connected. Then Lemma 1 tells us that $C' = E_{G'}(U', \overline{U'})$ is a minimal edge-cut of G'. We also deduce that $w(C') \geq w(uv) + n + s = n + m + n + s = d$.

Now suppose that G' has a minimal edge-cut $C' = E_{G'}(U^*, \overline{U^*})$ with $w(C') \geq d$. If $u, v \in U^*$ or (symmetrically) $u, v \in \overline{U^*}$, then $|C'| \leq |E_G| + 2n = 2n + m < d$. Because C' only contains edges of weight 1, we obtain $w(C') < d$, which is not possible. Hence, either $u \in U^*, v \in \overline{U^*}$ or $v \in U^*, u \in \overline{U^*}$. We may assume without loss of generality that $u \in U^*, v \in \overline{U^*}$. Then $C = E_G(U^* \setminus \{u\}, \overline{U^*} \setminus \{v\})$ is an edge-cut in G with

$$|C| = w(C') - w(uv) - n \geq d - (n + m) - n = 2n + m + s - n - m - n = s.$$

To conclude the proof, it remains to observe that, by Lemma 2, G' has a weighted contraction H with $\delta_w(H) \geq d$ if and only if G' has a minimal edge-cut C' with $w(C') \geq d$. $\quad\square$

Due to Theorem 4, we immediately obtain the following result by taking $k = |E|$.

Corollary 2. WEIGHTED DEGREE CONTRACTIBILITY *with integer edge weights is* NP-*complete in the strong sense.*

Recall that CONTRACTION DEGENERACY is in FPT when parameterized by d [6]. We now show that WEIGHTED CONTRACTION DEGENERACY is also FPT when parameterized by d. For doing this we first give some extra terminology.

A *tree decomposition* of a graph G is a pair $(\mathcal{X}, T)$ where T is a tree, the vertices of which are called *nodes*, and $\mathcal{X} = \{X_i \mid i \in V_T\}$ is a collection of subsets (called *bags*) of V_G such that the following three conditions are satisfied:

1. $\bigcup_{i \in V_T} X_i = V_G$;
2. for each edge $xy \in E_G$, the vertices x, y are in a bag X_i for some $i \in V_T$;
3. for each $x \in V_G$, the set $\{i \mid x \in X_i\}$ induces a connected subtree of T.

The *width* of tree decomposition $(\mathcal{X}, T)$ is $\max_{i \in V_T} \{|X_i| - 1\}$. The *treewidth* of a graph G is the minimum width over all tree decompositions of G.

Theorem 5. WEIGHTED CONTRACTION DEGENERACY *with integer edge weights is in* FPT *when parameterized by* d.

Proof. Let $G = (V, E)$ be a weighted graph and d be a nonnegative integer. We may assume without loss of generality that G is connected, as otherwise we can consider each connected component of G separately. We use Bodlaender's algorithm [5] to check in linear time whether the treewidth of G is at most $2d - 2$.

First suppose that the treewidth of G is at most $2d - 2$. Because all edge weights are integers, for a given d, we can express WEIGHTED CONTRACTION DEGENERACY in monadic second order logic using Lemmas 1 and 2. Then we apply the well-known result of Courcelle [9] to solve the problem in linear time.

Now suppose that the treewidth of G is at least $2d - 1$. We claim that G contains a minimal edge-cut C with $w(C) \geq d$. In order to obtain a contradiction, we assume that all minimal edge-cuts of G have weight less than d.

We recursively construct a tree decomposition of G as follows. During its construction we maintain the following property. Suppose that at some moment we have constructed a tree decomposition $(\mathcal{X}, T)$ of a subgraph of G induced by a subset $U \subset V$. Then for each minimal edge-cut $C \subseteq E_G(U, \overline{U})$ of G the set

$$Z = \{z \in U \mid z \text{ is incident with an edge of } C\}$$

must be included in some bag of $(\mathcal{X}, T)$. To ensure this property, initially, we set $U = \{u\}$ for an arbitrary vertex $u \in V$. Then we recursively extend U until we get V as follows. Let $C \subseteq E_G(U, \overline{U})$ be a minimal edge-cut of G. Define the sets

$Z = \{z \in U \mid z \text{ is incident with an edge of } C\}$ and

$Z' = \{z' \in \overline{U} \mid z' \text{ is incident with an edge of } C\}.$

Let X_i be a bag of the tree decomposition of $G[U]$ that contains Z. We set $U' = U \cup Z'$. To obtain a tree decomposition of $G[U']$, we add a pendant node j adjacent to node i and define the bag X_j as $Z \cup Z'$. Then $|Z \cup Z'| \leq 2|C| \leq 2d - 2$, where the latter inequality follows from the fact that $|C| < d$, as all edge weights are positive integers and we assume that $w(C) < d$. For every minimal edge-cut $C' \subseteq E_G(U', \overline{U'})$, the set of vertices in U' incident with an edge of C' is either contained in Z or in Z'. In the first case we take the bag X_i, and in the second case we take the bag X_j. Hence, we may proceed with U' and the obtained tree decomposition of $G[U']$. This eventually leads to a tree decomposition of G that has width at most $2d - 2$; a contradiction.

By this claim and Lemma 2, we conclude that the answer is always a Yes if the treewidth of G is at least $2d - 1$.   □

4. Concluding remarks

We leave the problem of determining the complexity of Weighted Degree Contractibility with integer edge weights, when parameterized by d, as an open problem. We conclude our paper with two additional results.

4.1. Weighted Face Degree Subgraph

We show that our algorithm from Theorem 3 can be applied for weighted faces in plane graphs; for the definitions of a plane graph and the geometric dual of a plane graph and other related notions we refer the reader to the text-book of Diestel [10].

The *weighted face degree* of a face f of a plane weighted graph G is the sum of all the weights of the edges of G incident with f. The Weighted Face Degree Subgraph problem is to test whether a plane weighted graph G can be modified to a plane weighted graph of minimum weighted face degree at least d by using at most k edge removals.

Theorem 6. Weighted Face Degree Subgraph *is in* FPT *when parameterized by k.*

Proof. Given an instance of Weighted Face Degree Subgraph with a plane weighted graph G and an integer d we do as follows. Let G^* denote the geometric dual of G. There is a one-to-one correspondence between the edges of G and the edges of G^*. Let e^* be the edge of G that corresponds to the edge e in G. We assign weights to the edges of G^* in the following way: $w_{G^*}(e^*) = w_G(e)$.

An *embedded contraction* of an edge e of a plane graph is a contraction of e that respects the embedding and keeps multiple edges if they appear (that is, if the endpoints of e have common neighbors). We observe that the dual of the graph obtained from a plane graph G by removing an edge e is the graph obtained from G^* by an embedded contraction of e^*.

We apply our algorithm from Theorem 3 for Weighted Degree Contractibility for G^* and degree d. Note that there is a one-to-one correspondence between the faces of G and the vertices of G^*. Therefore, weighted contractions can simulate the face degree transformations of a graph with embedded contractions and multiple edges. Due to the equivalence between edge removals in a plane graph and embedded edge contractions in its dual, the sequence of k edges of G^* that is a solution to the Weighted Degree Contractibility problem can be transformed into a sequence of k edge removals in G. Hence, Weighted Face Degree Subgraph can be solved in FPT time.   □

4.2. Eulerian Simple-Contractibility

As future work, one may consider other variants of constrained contractibility problems, for example by imposing parity constrains. To show that this may be interesting, we spot an algorithm that solves such a problem.

Recall that a simple contraction is the operation on loopless multigraphs that identifies both end-vertices of the edge and keeps multiple edges but removes the loop that was created. A connected multigraph is *Eulerian* if all its vertices have even degree. The problem Eulerian Simple-Contractibility is to find the minimum number of simple contractions that transforms a connected multigraph into an Eulerian graph. While it has never been stated explicitly in the literature, Eulerian Simple-Contractibility admits a polynomial-time algorithm. We show this below.

Let G be a multigraph. A vertex of G is *odd* (or *even*) when its degree is odd (or even). An *odd-vertex pairing* is a subset of edges, whose removal from G yields a multigraph that contains no odd vertices. Hadlock [15] considered *minimum* odd-vertex pairings and proved the following result.

Lemma 3 ([15]). *Let P be a subset of edges of a multigraph G. Then P is a minimum odd-vertex pairing if and only if P forms a collection of edge-disjoint paths with the odd vertices of G as endpoints, using each as an endpoint once, and with minimum sum of path lengths.*

We use Lemma 3 to show the following lemma.

Lemma 4. *Let P be a subset of edges of a multigraph G. Then P is a minimum set of edges whose simple contractions transform G into an Eulerian graph if and only if P is a minimum odd-vertex pairing of G.*

Proof. First suppose that P is a minimum set of edges whose simple contractions transform G into an Eulerian graph. We will prove that P is a minimum odd-vertex pairing by induction on $|P|$.

Let $|P| = 1$. Then the endpoints of the only edge e in P must be odd in G, and they must be the only two odd vertices in G, because P is a minimum set of edges whose simple contractions transform G into an Eulerian graph. Hence, P is an odd-vertex pairing. Because $|P| = 1$, we deduce that P is minimum.

Let $|P| > 1$. Let C be a connected component of P. Because P is minimum, C is a tree. Consider an edge vw in P such that v is a leaf of C (and w is its neighbor in C). Let x_{vw} be the new vertex obtained as the result of a simple contraction of vw. Also, let P' be the resulting set of edges obtained from P, and let G' be the resulting multigraph obtained from G, after performing the simple contraction of vw. Then P' is a minimum set of edges whose simple contractions transform G' into an Eulerian graph. By our induction hypothesis, P' is a minimum odd-vertex pairing of G'.

If v is even in G, then the simple contractions of all the edges in P except the edge incident with this leaf would transform G into an Eulerian graph; a contradiction with the choice of P. Hence, v is an odd vertex in G. If w is odd in G, then x_{vw} is even in G', because v is odd in G. This is not possible, because P' is a minimum odd-vertex pairing of G', and by Lemma 3 this means that P' forms a collection of edge-disjoint paths with the odd vertices of G' as endpoints, using each as an endpoint once, Hence, w is even in G. Consequently, P forms a collection of edge-disjoint paths with the odd vertices of G as endpoints, using each as an endpoint once. Moreover, P is minimum because we need at least one edge to cover v. By Lemma 3, we then find that P is a minimum odd-vertex pairing of G.

Now suppose that P is a minimum odd-vertex pairing. Then Lemma 3 tells us that P forms a collection of edge-disjoint paths with the odd vertices of G as endpoints, using each as an endpoint once, and with minimum sum of path lengths. We observe that the vertex obtained by the simple contraction of an edge $xy \in E$ is odd if and only if one of $\{x, y\}$ is even and the other one is odd. Hence, the simple contractions of all edges in P transform G into an Eulerian graph. Suppose that there exists a smaller set P^* of edges whose simple contractions transform G into an Eulerian graph. Then we may assume without loss of generality that P^* is minimum. However, as shown in the forward implication, P^* is a minimum odd-vertex pairing as well. This is not possible. Hence, P is minimum. This completes the proof of Lemma 4. □

Hadlock [15] observed that a minimum odd-vertex pairing can be computed in polynomial time by a reduction to a maximum matching problem. This observation and Lemma 4 imply the following theorem.

Theorem 7. *The* Eulerian Simple-Contractibility *problem can be solved in polynomial time.*

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