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On the stability of the FIFO protocol for non-adaptive routing *

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Happy birthday Giorgio, ALCOM-* forever!

Abstra t

In this paper, we analyze the stability properties of the FIFO protocol in the Adversarial Queueing model for packet routing. We show a graph for which FIFO is stable for any adversary with injection rate $r \leq 0.1428$. We generalize this results to show upper bounds for stability of any network under the FIFO protocol. We also design a network and an adversary for which FIFO is non-stable for any $r \geq 0.771$, improving the previous known bounds in [1] and in the preliminary version of the present work [3].

1 Introduction

An important issue in parallel and distributed computing is the efficient routing of packages in networks. A possible setting for study packet routing represents the network as a digraph, where nodes represent processors and arcs represent communication channels. When a packet is injected into the network, it contains its destination and at each step it must decide the best edge to traverse (adaptive routing). In an alternative model, the packet includes also the path to follow to its destination (non-adaptive routing).

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In this paper we only deal with the non-adaptive model of routing. In a single step, only one packet can traverse any given arc. To regulate the traffic of packets through each edge, at each arc there is a queue to assign the priorities to the packets through the arc. First-in-First-out (FIFO) is one of the simplest queueing policies and has been used to provide best-effort services in packet-switched networks. One crucial aspect of FIFO's performance is that of stability: A network is said to be stable if there is a bound on the total size of packets in the network at all times. Recall that FIFO belongs to the class of protocols usually denoted as *greedy protocols*. A protocol is said to be greedy if whenever the queue of an arc is non empty, the protocol must send a packet through the arc.

The adversarial queueing model of Borodin et al. [2], was developed as a robust model of queueing theory in network traffic, and replaces stochastic by worst case inputs. See [5] for a nice survey on the adversarial model. Adversarial Queueing Theory considers the time evolution of a packet-routing network as a game between an adversary and a protocol. The adversary, at each time step, may inject a set of packets at some nodes. For each packet, the adversary specifies a network path that the packet must traverse, when the packet arrives to its final destination, it is absorbed by the system. If more than one packet wish to cross an edge e in the current time step, then FIFO resolves the conflict, maintaining a queue per edge. We shall say that a packet targets edge e if the edge belongs to its destination path, regardless whether that the packet path finishes at e.

A crucial parameter of the adversary is its *injection rate*. The rate of an adversary in this model, is specified by a pair (r, b) where $b \geq 1$ is a natural number and $0 < r < 1$. The adversary must obey the following rule:

> Of the packets that the adversary injects in any time interval I, at most $\lceil r|I| \rceil + b$ can have paths that contain any particular edge.

Such a model allows for adversarial injection of packets that are "bursty".

The previous definition of injection rate, is the one described in [1] except that in our model the adversary can select the initial configuration of the system, which was not assumed in the model described in [1]. Without lost of generality, we will also assume that $b = 1$.

The motivation to study the behavior of packet communication networks, is to determine the conditions of *stability*, the fact that the number of packets in the system remains bounded, as the system dynamically evolves in time. Andrews et al. [1] solved several open questions posed in [2]. They also showed the existence of graphs and protocols that are not universally stable.

In particular, they showed that for the network in Figure 2, the protocol FIFO is non-stable for values of $r \geq 0.85$. Later, Goel [4] proved that FIFO is not stable for the network in Figure 1. This result is a corollary of his main theorem, showing that this network is not universally stable. However, the paper does not provide any upper bounds to r. It must be observed, that although we give as references for [1] and [2] the journal versions, which appeared at the same time, the work of Borodin et al. was presented at STOC-96, while the work of Andrews et all, was presented a few months latter at FOCS-96.

In [1], Andrews et al. proved the existence of a finite set of basic undirected graphs such that a graph G is stable for every r if and only if none of these graphs is a minor of G. Goel [4] presented three simple graphs H_1, H_2, H_3, which form this set for any directed graph. This result ensure the decidability of the question: *is G universally stable?* It remains an open problem the decidability of the following question: *is G stable for the FIFO protocol?* However, until now, it was not known if there is an $r_0 > 0$ such that, even these simple graphs, are stable for FIFO for any $r \leq r_0$.

A question raised at the end of [1], refers to the existence of a *threshold* rate $r_0 > 0$ such that any FIFO network is stable against every adversary of rate $r \leq r_0$. In the first section of this paper, we give evidence that their question may be answered in the positive, we show that for every network, FIFO is stable against any adversary with a small injection rate, where r depends only on the specific network. Before presenting the general result, we begin by showing that FIFO is stable for the particular network in Figure 1, for any adversary with rate $r \leq 0.1428$. We feel this proof will help to understand the general case.

Our second result is the design of a simple network and an adversary making the network unstable for $r \geq 0.771$, thus improving the previous known bounds from [1] and [3]. Our improvement is based on some new ideas of a) suitably exploiting "initial" paths and b) controlling the injection rounds in the construction.

For simplicity, we will omit floors and ceilings, and sometimes we will count steps and packets roughly. By carrying these through the computation we are loosing some additive constants but gaining in clarity. Given a network $\mathcal{N}$, and an edge $e \in \mathcal{N}$, $Q(e)$ we will denote the queue at e, and $e(t)$ will denote the size of $Q(e)$ at time t.

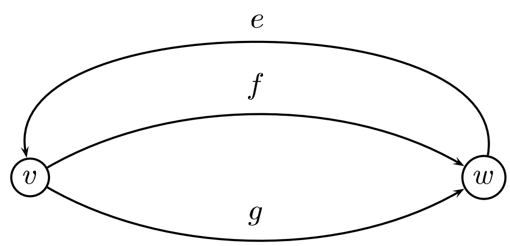

Figure 1: Goel's network $\mathcal{N}_1$.

2 An upper bound for the stability for FIFO

Let us consider the network $\mathcal{N}_1$ in Figure 1. As already said, this network was considered in [4], to prove non-stability under any greedy protocol, and as a corollary the non-stability of the FIFO protocol. We prove here, that given any initial packet configuration for $\mathcal{N}_1$, and for every adversary $\mathcal{A}$ with injection rate $r \leq r_1$, for a given r_1 to be computed, the number of packets in the system remains bounded. So we can conclude that FIFO is stable for $\mathcal{N}_1$, under any adversary with rate $r \leq r_1$.

Theorem 1. *For the network $\mathcal{N}_1$, given any initial configuration, FIFO is stable against any adversary with injection rate* $0 < r \leq 0.1428$.

Proof. We will split the time into periods. In the first part of each period we analyze the evolution of the system from the initial time to the first step in which all the initial packets have arrived to their last queue. The second part allows enough time to guarantee that all initial packets have been served and that the system configuration reproduces the initial ones.

Notice that the in degree of edge e is two, therefore this edge can get an input flow bigger than any of the other two edges. So the largest queue builded in $\mathcal{N}_1$ must occur at e. We analyze the flow of initial packets with final destination e. The analysis of edges g and f can be done similarly.

Assume that at time 0, the system has a configuration of $f(0)$ packets in $Q(f)$, $g(0)$ packets in $Q(g)$ and $e(0)$ packets in $Q(e)$. Make $M = \max\{f(0), g(0), e(0)\}$. Let us run the system for M steps, which correspond to the first part of the period.

Let $P(t)$ denote the number of packets in the system at time t, let $P_e(t)$, $f_e(t)$ and $g_e(t)$ denote respectively the number of packets in $\mathcal{N}_1$, $Q(f)$ and $Q(g)$, that will pass through e. Then $P(0) = f(0) + g(0) + e(0)$ and $P_e(0) =$

$f_e(0) + g_e(0) + e(0)$. As the protocol is FIFO, at time M, the number of packets from $P_e(0)$ which still have not been served by $Q(e)$ is at most $f_e(0)+g_e(0)$, and they are queued in $Q(e)$. At the same time, the maximum number of new injections of packets targeted with e is at most rM. Thus, $e(M) \leq rM + f_e(0) + g_e(0)$. Let

$$s = rM + f_e(0) + g_e(0), \tag{1}$$

then at time $M+s$ all the $P_e(0)$ old packets will have been served by $Q(e)$.

Consider the time intervals with duration: $s, rs, r^2s, \dots r^ks$, where k is such that $r^ks \geq 1$ and $r^{k+1}s < 1$. During these $k+1$ intervals of time, the total number of packets injected, targeted with e is

$$r\sum_{i=0}^{k} r^i s < sr\sum_{i=0}^{\infty} r^i = \frac{rs}{1-r}.$$

Let t_1 denote the time at the end of these intervals (which includes the initial time M). Then,

$$P_e(t_1) \leq \frac{r}{1-r}s + rM \tag{2}$$

By the previous definitions, $f_e(0)+g_e(0) \leq P_e(0) \leq P(0)$ and $M \leq P(0)$. Substituting in (1) we get $s \leq rP(0) + P(0) = (1+r)P(0)$. Plugging this bound for s and M into (2), we get

$$P_e(t_1) \leq \frac{r}{1-r}(1+r)P(0) + rP(0) \leq \left(\frac{2r}{1-r}\right)P(0).$$

To guarantee the stability of the system, it must be that the expression inside the parenthesis must be less than 1, which holds for a value $r_1 \leq 0.3334$.

On the other hand, at time t_1 the total number of packets in the system can be estimated in the following way: As all queues behave in the worst case as $Q(e)$, each queue can get at most one new packet per injection, and $\mathcal{N}_1$ has three queues, we get

$$P(t_1) \leq 3rM + \frac{3r}{1-r}s.$$

Substituting the value of s, we get

$$P(t_1) \leq 3rP(0) + \frac{3r}{1-r}(1+r)P(0) = \left(\frac{6r}{1-r}\right)P(0).$$

To guarantee the stability of the system, the quantity inside the parenthesis must be less than or equal to 1, which need a value $r_2 \leq 0.1428$.

Taking $r_0 = \min\{r_1, r_2\}$, we have shown that $P(t_1) \leq P(0)$ for any value of $r \leq r_0$.

Repeat the argument, getting an infinite time sequence $t_1, t_2, t_3, \cdots$. In the period between time 0 and t_1, the queues of the edges in $\mathcal{N}_1$ are bounded. As at each new period starting at t_{i+1}, the number of packets with a specific target, is a non-increasing function, we have proved the Theorem. $\square$

The previous argument, can be extended to work in full generality, for any network $\mathcal{N}$ with k edges, maximum in-degree α, and maximum directed path length β,

Theorem 2. *For any network $\mathcal{N}$, given any initial configuration, there exist an $0 < r_{\mathcal{N}} < 1$, such that FIFO is stable with respect any adversary with injection rate $r \leq r_{\mathcal{N}}$, where $r_{\mathcal{N}}$ is independent of the initial configuration and it is a function of the maximum in-degree, the maximum directed path length and the number of edges.*

Proof. Assume that $\mathcal{N}$ has k edges, maximum in-degree α, and maximum directed path length β and that the injection rate of $\mathcal{A}$ is r. Notice that if $\alpha = 1$ then we have a tree or a ring, for which it is known that such a network is universally stable [1], so we assume $\alpha > 1$.

Let us denote the queues as $Q_1, Q_2, \ldots, Q_k$ and their loads at time $t \geq 0$ as $q_1(t), q_2(t), \ldots, q_k(t)$. Let $P(0) = \sum q_i(0)$ be the initial load.

We will construct an infinite sequence of time periods, t_i, at which $P(t_i) \leq P(0)$ thus keeping the network stable. Again, we will refer to the packets at time 0, as the *old packets*.

The fact that we are using a FIFO protocol implies that after a certain time all the old packets will leave the system. We will compute a bound to this time.

Consider the worst case of an old packet being last in a queue Q_j at time 0 and targeted with the largest simple path in the network. Rename the queues in this simple path as $Q_j \equiv Q_{j_0}, \ldots = Q_{j_{\beta-1}}$

Note that at time $M_1 = q_{j_0}(0)$ all packets of this queue will have been served. Thus these packets have passed to the next queues in the path. Moreover, they can be delayed by at most rM_1 new injections. Furthermore the size of any Q_{j_i} is bounded above by $(\alpha + r)M_1$.

We repeat the same procedure, each time considering the last queue in the path that still contains old packets. After $\beta - 2$ additional steps $(M_2, M_3, \ldots M_{\beta-1})$ all the old packets would disappear or being in Q_{j_β}.

Define $q(t) = \max_{i=0}^{k}\{q_i(t)\}$. Working in the previous way, an absolute bound for the delay of the last old packet in Q_j is $M = M_1 + \cdots + M_{\beta-1}$, where for every $0 < i < \beta$, we have $M_i \leq q(\sum_{j<i} M_j)$, with $M_0 = 0$. Moreover,during a period of $q(t)$ steps starting at time t, we have $q(t+q(t)) \leq (\alpha + r)q(t)$. Solving the recurrence, we have that the total time,

$$M \leq \sum_{i=0}^{\beta-1} (\alpha + r)^i q(0). \tag{3}$$

Consider consecutive time periods M, rM, r^2M, $\ldots$, r^lM, where l is such that $r^lM \geq 1$ and $r^{l+1}M < 1$. Let t_1 be the time at which r^lM finishes. The packets in $\mathcal{N}$ at time t_1 are all new, therefore the number of packets per queue is at most

$$rM + r^2M + \ldots + r^lM \ \leq \ rM \ + \ \frac{r}{1-r}M,$$

Therefore, $P(t_1) \leq \frac{2-r}{1-r} rkM$. Substituting the value of M from (3), we have

$$P(t_1) \leq \frac{2-r}{1-r} rk \left(\sum_{i=0}^{\beta-1} (\alpha + r)^i \right) P(0).$$

For the stability condition, we need $P(t_1)$ to be less than $P(0)$, which implies that we must choose an r such that,

$$\frac{2-r}{1-r} rk \left(\sum_{i=0}^{\beta-1} (\alpha + r)^i \right) \leq 1,$$

which is equivalent to finding in the real interval $(0, 1)$, the root $r_{\mathcal{N}}$ of

$$-2Zk(\alpha + Z)^{(\beta+1)} + 2Zk + Z^2k(\alpha + Z)^{(\beta+1)} - Z^2k + \alpha + 2Z - 1 - \alpha Z - Z^2.$$

By the Bolzano's theorem, this polynomial has a root between 0 and 1, which is $r_{\mathcal{N}}$. □

Using the polynomial obtained in the proof of the previous theorem, we can prove the stability under FIFO for any network. However, doing a particular analysis for specific networks, we can get better lower bounds for the values of the r's that make the network stable under FIFO protocol. For example, in [1] to prove the non-stability of the FIFO protocol, they use the network in Figure 2. Applying the previous techniques we get that the

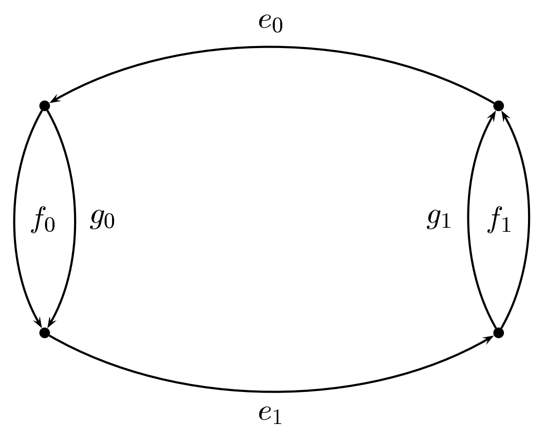

Figure 2: AAFKLL network $\mathcal{N}_2$.

network is stable for $r \leq 0.0026732592$, while a direct proof gives a value for stability of $r \leq 0.03889$. In Section 3, we will use the network in Figure 3, to lower the injection rate for non-stability of FIFO. Applying again the techniques developed in Theorem 2, we get a value $r \leq 0.0003932323554$ for stability, while a more careful analysis will yield $r \leq 0.01062737157$.

3 A lower bound for the non-stability of FIFO

Using the network $\mathcal{N}_2$ described in Figure 2, Andrews et al. [1] gave a lower bound of 0.85, to prove the non-stability of FIFO. In [3], we slightly lowered their bound to 0.8357, on the network given in Figure 3. In this section, we use the same network, to lower the injection rate bound to 0.771, so that FIFO is non-stable for values of $r \geq 0.771$.

Theorem 3. *There is a network $\mathcal{N}_3$ and an adversary $\mathcal{A}$ of rate r, such that the ($\mathcal{N}$, $\mathcal{A}$) system is non-stable, starting from a non-empty initial configuration, for any $r \geq 0.771$.*

Proof. We consider the network $\mathcal{N}_3$, and the following hypothesis.
Inductive Hypothesis: At the beginning of phase j, there are s packets queued in the queues $e_0, f_2^{'}, f_3^{'}$ requiring to traverse edges e_0, g, f_2, all the packets in queue $f_3^{'}$ are of this type, and the number of packets queued in $f_3^{'}$ queue is bigger than the number of packets queued in $f_2^{'}$ queue

We will construct an adversary $\mathcal{A}$ such that at the beginning of phase $j+1$ the inductive hypothesis will hold, for the symmetric edges, with an

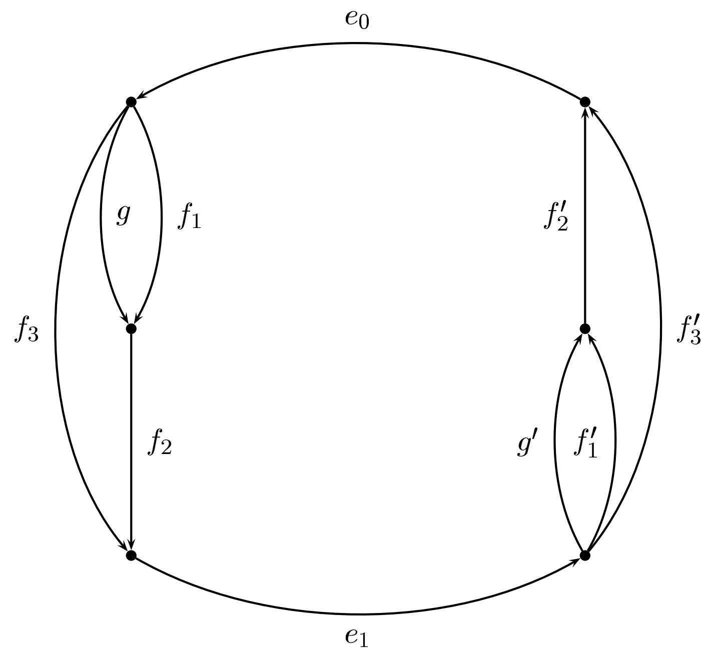

Figure 3: The network $\mathcal{N}_3$ for the non-stability of FIFO

increased value of s. That is, there will be $s' > s$ packets queued in the queues f_2, f_3, e_1 requiring to traverse edges $e_1, g^{'}, f_2^{'}$, all the packets in queue f_3 are of this type, and the number of packets queued in f_3 queue is bigger than the number of packets queued in f_2 queue.

From the inductive hypothesis, initially, there are s packets (called $S-flow$) in the queues $e_0, f_2^{'}, f_3^{'}$ requiring to traverse edges e_0, g, f_2.

During phase j the adversary plays three rounds of injections:

Round 1: For s steps, the adversary injects in $f_3^{'}$ queue a set X of rs packets wanting to traverse $f_3^{'}, e_0, f_1, f_2, e_1, g^{'}, f_2^{'}$. These packets are blocked by the $S-flow$. Notice that it has been assumed that all the packets queued in $f_3^{'}$ queue form part of the S-flow, and that the number of packets queued in $f_3^{'}$ queue at the beginning of this round is bigger than the number of $S-flow$ packets queued in $f_2^{'}$ queue.

At the same time, the $S-flow$ is delayed by the adversary's rs single injections S_1 in queue g. The S_1 packets get mixed with the packets in the $S-flow$, and some of them will remain queued at g queue.

Notice that, because of the FIFO policy, the packets of S, S_1 mix in consecutive blocks according to their initial proportion of their sizes (fair mixing property). Since $|S| = s$ and $|S_1| = rs$, these proportions are $\frac{1}{r+1}$ and $\frac{r}{r+1}$, respectively. Thus, during the s steps of round 1, the packets of S, S_1, which arrive to g are, respectively,

$$s\frac{1}{r+1} = \frac{s}{r+1}, s\frac{r}{r+1} = \frac{rs}{r+1}.$$

From this flow only s packets traverse edge g. Therefore, the remaining packets of each type are:

- for S_{rem}: $s - \frac{s}{r+1} = \frac{rs}{r+1}$
- for $S_{1,rem}$: $rs - \frac{rs}{r+1} = \frac{r^2 s}{r+1}$

Round 2: For the next rs steps, the adversary injects a set Y of $r^2 s$ packets requiring edges $f_3^{'}, e_0, f_3, e_1, g^{'}, f_2^{'}$. These packets are blocked by the set X. At the same time, the adversary pushes a set S_2 of single injections in the queue f_1, where $|S_2| = r^2 s$ and a set S_3 of single injections in the queue f_2, where $|S_3| = r^2 s$.

Because of the FIFO policy, the packets of X, S_2 mix in consecutive blocks according to their initial proportion of their sizes. Since $|X| = rs$ and $|S_2| = r^2 s$, these proportions are $\frac{1}{r+1}$ and $\frac{r}{r+1}$, respectively. Thus, during the rs steps of this round, the packets of X, S_2, that pass f_1 are, respectively,

$$X_{pass} = \frac{rs}{r+1}, S_{2,pass} = \frac{r^2 s}{r+1}.$$

Therefore, the remaining packets of each type are:

- for X_{rem}: $rs - \frac{rs}{r+1} = \frac{r^2 s}{r+1}$
- for $S_{2,rem}$: $r^2 s - \frac{r^2 s}{r+1} = \frac{r^3 s}{r+1}$

Note that in queue g, there are the remaining $S - flow$ and the remaining $S_1 - flow$ packets. Since their total number is rs (which is equal to the duration of the round), the $S_{1,rem} - flow$ does not delay the $S_{rem} - flow$. Note also that, because the $S_{1,rem}$ packets are absorbed after they pass only g, only the S_{rem} packets require edge f_2. As a result the stream arriving from g to f_2 contains empty spaces at the positions of the $S_{1,rem}$ packets. Therefore, during round 2, three different flows of packets arrive to the f_2 queue:

- the $X_{pass} - flow$, where $|X_{pass}| = \frac{rs}{r+1}$. This flow is mixed with $S_{2,pass} - flow$. However, since their total number is rs (that is equal to the duration of the round), $S_{2,pass} - flow$ does not delay the $X_{pass} - flow$. Note also that, because the $S_{2,pass} - flow$ is absorbed after they pass f_1, only the $X_{pass} - flow$ requires edge f_2. As a result the stream arriving from f_1 to f_2 contains empty spaces at the positions of the $S_{2,pass}$ packets.

- the $S_{rem} - flow$, where $|S_{rem}| = \frac{rs}{r+1}$. Notice that the stream arriving from this edge also contains empty spaces, from the S_1 flows.

- the S_3 single-injected packets, where $|S_3| = r^2 s$.

Since the total number of packets that have to traverse f_2 in the three flows is:

$$T = \frac{r^3 s + r^2 s + 2rs}{r+1}$$

and the empty spaces and new injections arrives regularly, the corresponding proportions are:

- for X_{pass}: $\frac{X_{pass}}{T} = \frac{1}{r^2+r+2}$
- for S_{rem}: $\frac{S_{rem}}{T} = \frac{1}{r^2+r+2}$
- for S_3: $\frac{S_3}{T} = \frac{r^2+r}{r^2+r+2}$

Thus, the remaining packets in f_2 queue from each flow at the end of round 2 are:

- for X_{pass}: $\frac{rs}{r+1} - \frac{rs}{r^2+r+2} = rs\frac{r^2+1}{(r+1)(r^2+r+2)}$
- for S_{rem}: $\frac{rs}{r+1} - \frac{rs}{r^2+r+2} = rs\frac{r^2+1}{(r+1)(r^2+r+2)}$
- for S_3: $r^2 s - rs\frac{r^2+r}{r^2+r+2} = rs\frac{r^3+r}{r^2+r+2}$

Round 3: For the next $r^2 s$ steps, the adversary injects a set Z of $r^3 s$ packets requiring edges f_3, e_1, g', f_2'. The set Z is mixed with the set Y in consecutive blocks according to their initial proportion of their sizes. These proportions are $Y : \frac{1}{r+1}$ and $Z : \frac{r}{r+1}$. Thus, during the $r^2 s$ steps of this round, the packets Y, Z that pass f_3 are respectively,

$$\begin{array}{rcccl} Y_{pass} & = & r^2 s \frac{1}{r+1} & = & \frac{r^2 s}{r+1} \\ Z_{pass} & = & r^2 s \frac{r}{r+1} & = & \frac{r^3 s}{r+1} \end{array}$$

Therefore, the remaining packets in f_3 queue are:

- for Y_{rem}: $r^2 s - \frac{r^2 s}{r+1} = \frac{r^3 s}{r+1}$
- for Z_{rem}: $r^3 s - \frac{r^3 s}{r+1} = \frac{r^4 s}{r+1}$.

Furthermore, notice that all the packets share the same destination path.

During this period the number of $X_{pass} - flow$ packets that traverses f_2 is

$$r^2 s \frac{1}{r^2 + r + 2}$$

Thus, the remaining $X_{pass} - flow$ and S_{rem} packets that are still in f_2 queue at the end of this round are:

$$S_{rem} = X_{pass,rem} = \frac{r^3 s + rs}{(r+1)(r^2+r+2)} - \frac{r^2 s}{r^2+r+2} = \frac{rs - r^2 s}{(r+1)(r^2+r+2)}$$

Also, the remaining $S_3 - flow$ packets that remain in f_2 queue at the end of this round are:

$$S_{3,rem} = \frac{r^4 s + r^2 s}{r^2+r+2} - \frac{r^4 s + r^3 s}{r^2+r+2} = \frac{r^2 s - r^3 s}{r^2+r+2}$$

Furthermore, all the X_{rem} packets that are queued in f_1 at the beginning of this round traverse f_1 and are queued in f_2 because the total size of packets in f_1 is

$$|X_{rem}| + |S_{2,rem}| = \frac{r^2 s}{r+1} + \frac{r^3 s}{r+1} = r^2 s$$

which is equal to the duration of this round.

From the inductive hypothesis, the assumption that the number of packets requiring to traverse edges f_3, e_1, g', f_2' is bigger than the number of packets requiring to traverse edges f_2, e_1, g', f_2' should be hold.

However in queue f_2, there is a number of S_{rem} and $S_{3,rem}$ packets at the end of this round, that are mixed with $X_{pass,rem}$ packets, while the X_{rem} packets are queued after S_{rem}, $S_{3,rem}$ and $X_{pass,rem}$ packets in queue f_2. Because of this mixture $X_{pass,rem}$ packets are delayed in the next phase. So, we should take them into account for the following comparison:

$$\begin{array}{rcl} Q(f_3) & > & Q(f_2) \\ \Longrightarrow \frac{r^3 s}{r+1} + \frac{r^4 s}{r+1} & > & 2\frac{rs-r^2 s}{(r+1)(r^2+r+2)} + \frac{r^2 s}{r+1} + \frac{r^2 s - r^3 s}{r^2+r+2} \\ \Longrightarrow r & \geq & 0.755 \end{array}$$

where $Q(f_3)$ and $Q(f_2)$ are the number of packets in f_3 and f_2 queues respectively.

Thus, for $r \geq 0.755$, we have proved that under the *inductive hypothesis*, at the end of step 3, the number of packets queued in f_3 queue is bigger than the number of packets queued in f_2 queue, furthermore all the packets queued in f_3 get destination $f_3, e_1, g^{'}, f_2^{'}$.

Now, we will find the appropriate lower bound for injection rate in order to prove that the the number $s^{'}$ of packets queued in the queues f_2, f_3, e_1 requiring to traverse edges $e_1, g^{'}, f_2^{'}$ at the end of step 3 is bigger than s.

At the end of the round, the number of packets that are in queues f_2, f_3, e_1 requiring to traverse edges $e_1, g^{'}, f_2^{'}$ is:

$$s^{'} = r^3 s + r^2 s + \frac{r^2 s}{r+1} + \frac{r^3 s + rs}{(r+1)(r^2+r+2)} - r^2 s$$

In order to have instability $s^{'} > s$ should be hold. Therefore,

$$\begin{array}{rcl} r^3 s + \frac{r^2 s}{r+1} + \frac{r^3 s + rs}{(r+1)(r^2+r+2)} & > & s \\ \Longrightarrow r^6 + 2r^5 + 4r^4 + 3r^3 & > & 2r+2 \end{array}$$

The above inequality has as a result $r \geq 0.771$. Thus, in order to fulfill the *inductive hypothesis*, we take the maximum of 0.771 and 0.755. Therefore, for $r \geq 0.771$ the network in Figure 3 is unstable. This concludes our proof. □

4 Open problems

As we already said, Andrews et al. posed the open question on the existence of an threshold constant r_0 such that any network with FIFO protocol is stable against any adversary with injection ratio $r \leq r_0$ [1]. In this work we have showed that for any network $\mathcal{N}$ with FIFO protocol, there exists a $r_{\mathcal{N}}$ such that $\mathcal{N}$ is stable against any adversary with $r \leq r_{\mathcal{N}}$. The original open problem still remains open, and the values given in this paper indicates that even for a particular network, the gap is still is too large.

Another interesting open area relates to decidability issues of FIFO protocol. In particular, the decidability of the following problems is open:

Given a network $\mathcal{N}$ and a r, $0 < r < 1$, is $\mathcal{N}$ stable against any adversary with injection ratio r?

Given a network $\mathcal{N}$, does it have a non trivial threshold for FIFO stability? that is, is there an r, $0 < r_{\mathcal{N}} < 1$ such that $\mathcal{N}$ is stable against any adversary with injection ratio $r < r_{\mathcal{N}}$ and non-stable against any adversary with injection ratio $r > r_{\mathcal{N}}$

We have mentioned that several networks are Universally stable for any greedy protocol. Another interesting question would be to show the existence of an specific network, which is not universally stable but for which the network is always stable for FIFO.

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