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Fast sub-exponential Algorithms and Compactness in Planar Graphs

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Abstract We provide a new theory, alternative to bidimensionality, of sub-exponential parameterized algorithms on planar graphs, which is based on the notion of *compactness*. Roughly speaking, a parameterized problem is (r,q)-compact when all the faces and vertices of its YES-instances are "r-radially dominated" by some vertex set whose size is at most q times the parameter. We prove that if a parameterized problem can be solved in $c^{\text{branchwidth}(G)}n^{O(1)}$ steps and is (r,q)-compact, then it can be solved by a $c^{r\cdot 2.122\cdot\sqrt{q\cdot k}}n^{O(1)}$ step algorithm (where k is the parameter). Our framework is general enough to unify the analysis of almost all known sub-exponential parameterized algorithms on planar graphs and improves or matches their running times. Our results are based on an improved combinatorial bound on the branchwidth of planar graphs that bypasses the grid-minor exclusion theorem. That way, our approach encompasses new problems where bidimensionality theory fails to provide sub-exponential parameterized algorithms.

Keywords: Parameterized Algorithms, Branchwidth, Planar Graphs.

1 Introduction

A parameterized problem can be defined as a language $\Pi \subseteq \Sigma^* \times \mathbb{N}$. Its inputs are pairs $(I,k) \in \Sigma^* \times \mathbb{N}$, where I can be seen as the main part of the problem and k is some parameter of it. A problem $\Pi \subseteq \Sigma^* \times \mathbb{N}$ is *fixed parameter tractable* when it admits an $f(k)\cdot n^{O(1)}$ -time algorithm. In that case, Π is classified in the parameterized complexity class FPT and, when we insist to indicate the parameter dependance (i.e., the function f), we also say that that $\Pi \in f(k)$-FPT.

Sub-exponential parameterized algorithms. A central problem in parameterized algorithm design is to investigate in which cases and under which input restrictions a parameterized problem belongs to FPT and, if so, to find algorithms with the simplest possible parameter dependance. When $f(k) = 2^{o(k)}$, a parameterized problem is said to admit a *sub-exponential parameterized algorithm* (for a survey on this topic, see [14]).

In [4], Cai and Juedes proved that several parameterized problems most probably do not belong in $2^{o(k)}$-FPT. Among them, one can distinguish core problems such as the standard parameterizations of Vertex Cover, Dominating Set, and Feedback Vertex Set. However, it appears that many problems admit sub-exponential parameterized algorithms when their inputs are restricted to planar graphs or other sparse graph classes. Moreover, the results of [4] indicated that this is indeed the best we may expect when the planarity restriction is imposed. The first sub-exponential parameterized algorithm on planar graphs appeared in [1] for Dominating Set, Independent Dominating Set, and Face Cover. After that, many other problems were classified in $2^{c\sqrt{k}}$-FPT, while there was a considerable effort towards improving the constant c for each one of them [1, 6, 9, 12, 15, 16, 20, 25–27].

Bidimensionality theory. A major advance towards a theory of sub-exponential parameterized algorithms was made with the introduction of Bidimensionality, in [10]. Bidimensionality theory offered a generic condition for classifying a parameterized problem in $2^{c\sqrt{k}}$-FPT. It also provided a machinery for estimating a (reasonably small) value c for each particular problem. Moreover, it also provided meta-algorithmic results in approximation algorithms [11, 18] and kernelization [19] (for a survey on bidimensionality, see [7]).

According to [10], a problem $\Pi \subseteq \mathcal{P} \times \mathbb{N}$ on planar graphs is *minor-bidimensional* with density α if the following two conditions are satisfied.

(a) the graph G' is a minor of the graph G, then $(G, k) \in \Pi \Rightarrow (G', k) \in \Pi$.
(b) there exists a $\delta > 0$ such that for every $k \in \mathbb{N}$ it holds that $(\Gamma_{\sqrt{k/\delta}}, k) \in \Pi$.

In the above definition, we denote by $\mathcal{P}$ the class of all planar graphs and we use the term Γ_w for the $(\lceil w \rceil \times \lceil w \rceil)$-grid. Also, we say that G' is a *minor* of G, denoted as $G' \leqslant_{\mathrm{m}} G$, if G' can be obtained by some subgraph of G after a series of edge contractions. We stress that there is an analogous definition for the case where, in **(a)**, we replace minors by contractions only, but, for simplicity, we avoid giving more definitions on this point.

Branchwidth (along with its twin parameter of treewidth) has been a powerful tool in parameterized algorithm design. Roughly speaking, branchwidth is a measure of the topological resemblance of a graph to the structure of a tree. We use the term $\mathbf{bw}(G)$ for the branchwidth of a graph G and we postpone its formal definition until Section 2.

We say that a problem $\Pi \subseteq \mathcal{P} \times \mathbb{N}$ is λ*-single exponentially solvable with respect to branchwidth* if there exists an algorithm that solves it in $2^{\lambda \cdot \mathbf{bw}(G)} n^{O(1)}$ steps. The main idea of [10] was to make use of the grid-minor exclusion theorem in [31] asserting that, for every planar graph G, $\mathbf{bw}(G) \leqslant 4 \cdot \mathbf{gm}(G)$, where $\mathbf{gm}(G) = \max\{w \mid \Gamma_w \leqslant_{\mathrm{m}} G\}$. This result was recently improved by Gu and Tamaki in [23] who proved that $\mathbf{bw}(G) \leqslant 3 \cdot \mathbf{gm}(G)$. This implies that for a bidimensional problem with density δ on planar graphs, all YES-instances have branchwidth at most $\frac{3}{\delta}\sqrt{k}$ and this reduces the problem to its variant where the inputs are now bounded by $\frac{3}{\delta}\sqrt{k}$. An optimal branch decomposition of a planar graph can be constructed in $O(n^3)$ steps, (see [22, 33]) and it is possible to transform it to a tree decomposition of width at most 1.5 times more [32]. Therefore, the main algorithmic consequence of bidimensionality, as restricted to planar graphs[1], is the following.

Proposition 1. *If $\Pi \subseteq \mathcal{P} \times \mathbb{N}$ is minor-bidimensional with density δ and λ-single exponentially solvable with respect to branchwidth, then $\Pi \in 2^{(3\lambda/\delta)\cdot\sqrt{k}}$-*$\mathsf{FPT}$*.*

The above result, along with its contraction-bidimensionality counterpart, defined in [10] (see also [17]), reduce the solution of bidimensional problems to the easier task of designing dynamic programming algorithms on graphs with small branchwidth (or treewidth). Dynamic programming is one of the most studied and well developed topics in parameterized algorithms and there is an extensive bibliography on what is the best value of λ that can be achieved for each problem (see e.g., [34]). Especially for planar graphs, there are tools that can make dynamic programming run in single exponential time, even if this is not, so far, possible for general graphs [13]. Lower bounds on the value of λ for problems such as Dominating Set appeared recently in [28]. Finally, meta-algorithmic conditions for single exponential solvability with respect to branchwidth have very recently appeared in [30].

A consequence of Proposition 1 and its contraction-counterpart was a massive classification of many parameterized problems in $2^{c\cdot\sqrt{k}}$-FPT and, in many cases, with estimations of c that improved all previously existing bounds. The remaining question was whether it is possible to do even better and when. Indeed, a more refined problem-specific combinatorial analysis improved some of the bounds provided by the bidimensionality framework (see also [12]). For instance, Planar Dominating Set [20], Face Cover, Planar Vertex Feedback Set, and Cycle Packing [27] where classified in $2^{c\cdot\sqrt{k}}$-FPT where $c = 15.3, 10.1, 15.11$, and 26.3 respectively, improving all previous results on those problems.

Our results. In this paper, we provide an alternative theory for the design of fast sub-exponential parameterized algorithms.

Let us give first some definitions from [3]. Given a plane graph $G = (V, E)$ and a set $S \subseteq V$, we define $\mathbf{R}^r_G(S)$ as the set of all vertices or faces of G whose radial distance from some vertex of S is at most r. The *radial distance* between two vertices x, y is one less than the minimum length of an alternating sequence of vertices and faces starting from x and ending in y, such that every two consecutive elements in this sequence are incident to each other.

[1] The results in [10] apply for more general graph classes and have been further extended in [8, 17].

The notion of compactness has been introduced in [3] in the context of automated kernel design, while some preliminary concepts had already appeared in [24]. It encompasses a wide number of parameterized problems on graphs; some of them are listed in Table 1.

A parameterized problem $\Pi \subseteq \mathcal{P} \times \mathbb{N}$ is (q,r)-*compact* if for all $(G = (V,E),k) \in \Pi$ and for some planar embedding of a graph H, that is either G or its dual, there is a set $S \subseteq V$ such that $|S| \leqslant q \cdot k$ and $\mathbf{R}^r_H(S)$ contains all faces and vertices of H. Intuitively, a parameterized problem is compact if the input graph of every YES-instance can be "covered" by a collection of $O(k)$ balls each of constant radius.

	q	r		q	r
Dominating Set	1	3	Cycle Domination	1	4
l-Dominating Set	1	$2l+1$	Edge Dominating Set	2	2
l-Threshold Dominating Set	1	3	Clique Transversal	1	3
Perfect Code	1	3	Independent Dominating Set	1	3
Red Blue Dominating Set	1	3	Odd Set	1	3
Independent Directed Domination	1	3	Face Cover	1	2
Vertex Cover	1	2	Vertex Triangle Covering$^{(c)}$	1	3
Almost Outerplanar$^{(a)}$	1	3	Edge Triangle Covering$^{(c)}$	2	2
Almost Series-Parallel$^{(a)}$	1	3	l-Cycle Transversal$^{(b)}$	1	l
Connected l-Dominating Set	1	$2l+1$	l-Scattered Set	1	$2l+1$
Connected Vertex Cover	1	2	Cycle Packing	3	3
Feedback Vertex Set$^{(b)}$	1	2	Induced Matching	2	3
Feedback Edge Set$^{(b)}$	1	2	Max Internal Spanning Tree	1	3
Connected Feedback Vertex Set	1	3	Triangle Packing$^{(c)}$	1	3
Minimum-Vertex Feedback Edge Set	1	3	Minimum Leaf Out-branching	1	3
Connected Dominating Set	1	3	Max Full Degree Spanning Tree	1	3

Table 1. Examples of (q,r)-compact parameterized problems. $^{(a)}$triconnected instances. $^{(b)}$biconnected instances. $^{(c)}$for instances where each vertex of the input graph belongs in some triangle.

We use the term $\overline{\Pi}$ for the set of NO-instances of the parameterized problem Π. We present below the main algorithmic contribution of our paper.

Theorem 1. *Let $\Pi \subseteq \mathcal{P} \times \mathbb{N}$ be a parameterized problem on planar graphs. If Π is λ-single exponentially solvable with respect to branchwidth and either Π or $\overline{\Pi}$ is* (q,r)-compact*, then $\Pi \in 2^{\lambda \cdot r \cdot 2.122 \cdot \sqrt{q \cdot k}}$*-FPT.

The advantages of our approach, compared with those of bidimensionality theory, are the following:

– It applies to *many* problems where bidimensionality fails. This typically happens for problems whose YES-instances are not closed under taking of minors (or contractions) such as Independent Vertex Cover, Independent Dominating Set, Perfect Code, and threshold Dominating Set.

– When applied, it *always* gives better bounds than those provided by bidimensionality theory. A direct comparison of the combinatorial bounds implies that the constants in the exponent provided by compactness are $\sqrt{4.5}/3 \approx 70\%$ of those emerging by the grid-minor exclusion theorem of Gu and Tamaki in [23].
– Matches or improves *all* problem-specific time upper bounds known so far for sub-exponential algorithms in planar graphs (including the results in [9, 12, 20, 26, 27]) and unifies their combinatorial analysis to a single theorem.

Theorem 1 follows from the following theorem that we believe is of independent combinatorial interest.

Theorem 2. *If* (G, k) *is a* YES*-instance of an* (r, q)*-compact parameterized problem, then* $\mathbf{bw}(G) \leqslant r \cdot \sqrt{4.5 \cdot q \cdot k}$*.*

Theorem 2 is our main combinatorial result and the rest of the sections of this paper are devoted to its proof. In Table 1, we give a list of examples of compact parameterized problems together with the corresponding values of q and r. Deriving these estimations of q and r is an easy exercise whose details are omitted. (In the Appendix we give hints for some of them.)

The paper is organized as follows. In Section 2, we give some necessary definitions and preliminary results. Section 3 is dedicated to the proof of Theorem 2. Finally, some open problems are given in Section 4. The proofs of the Lemmata marked with $(\star)$ have been moved in the Appendix.

2 Basic definitions and preliminaries

All graphs in this paper are simple (i.e., have no multiple edges or loops). For any set $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by the vertices in S. Given two graphs G_1 and G_2, we define $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. We also denote by $G \setminus S$ the graph $G[V(G) \setminus S]$. We also denote the distance between two vertices x, y in G by $\mathbf{dist}_G(x, y)$. A *subdivision* of a graph H is any graph that can be obtained from H if we apply a sequence of subdivisions to some (possibly none) of its edges (a subdivision of an edge is the operation of replacing an edge $e = \{x, y\}$ by a (x, y)-path of length two). We say that a graph H is a topological minor of a graph G (we denote it by $H \leqslant_{\mathrm{t}} G$) if some subdivision of H is a subgraph of G.

Plane graphs. In this paper, we mainly deal with plane graphs (i.e. graphs embedded in the plane $\mathbb{R}^2$ without crossings). For simplicity, we do not distinguish between a vertex of a plane graph and the point of the plane used in the drawing to represent the vertex or between an edge and the open line segment representing it. Given a plane graph G, we denote its dual by G^*. A *parameter* on plane graphs is any function $\mathbf{p}$ mapping plane graphs to $\mathbb{N}$. Given such a parameter $\mathbf{p}$, we define its *dual* parameter $\mathbf{p}^*$ so that $\mathbf{p}^*(G) = \mathbf{p}(G^*)$.

Given a plane graph G, we denote by $F(G)$ the set of the faces of G (i.e., the connected components of $\mathbb{R}^2 \setminus G$, that are open subsets of the plane). We use the notation $A(G)$ for the set $V(G) \cup F(G)$ and we say that $A(G)$ contains the *elements* of G. If $a_i, i = 1, 2$ is an edge or an element of G, we say that a_1 is *incident* to a_2 if $a_1 \subseteq \overline{a}_2$ or $a_2 \subseteq \overline{a}_1$, where $\overline{x}$ is the closure of the set x. For every face $f \in F(G)$, we denote by $\mathbf{bd}(f)$ the *boundary* of f, i.e., the set $\overline{f} \setminus f$ where $\overline{f}$ is the closure of f.

A *triangulation* H of a plane graph G is a plane graph H where $V(H) = V(G)$, $E(G) \subseteq E(H)$, and where H is triangulated, i.e., every face of H (including the exterior face) has exactly three edges incident upon it.

We use the term *arc* for any subset of the plane homeomorphic to the closed interval $[0, 1]$. Given a plane graph G, an arc I that does not intersect its edges (i.e., $I \cap G \subseteq V(G)$) is called *normal*. The *length* $|I|$ of a normal arc I is equal to the number of elements of $A(G)$ that it intersects minus one. If x and y are the elements of $A(G)$ intersected by the extreme points a normal arc I, then we also call I *normal* (x, y)*-arc*. A *noose* of the plane, where G is embedded, is a Jordan curve that does not intersect the edges of G. We also denote by $V(N)$ the set of vertices of G met by N, i.e., $V(N) = V(G) \cap N$. The *length* $|N|$ of a noose N is $|V(N)|$, i.e., is the number of the vertices it meets.

Let G be a plane graph and let r be a non-negative integer. Given two elements $x, y \in A(G)$, we say that they are *within radial distance at most* r if there is a normal (x, y)-arc of the plane of length at most r and we denote this fact by $\mathbf{rdist}_G(x, y) \leqslant r$.

Observation 1 *Let G be a triangulated plane graph and let $x, y \in V(G)$. Then $2 \cdot \mathbf{dist}_G(x, y) \leqslant \mathbf{rdist}_G(x, y)$.*

Given a vertex set $S \subseteq V(G)$ and a non-negative integer r, we denote by $\mathbf{R}^r_G(S)$ the set of all elements of G that are within radial distance at most r from some vertex in S. We say that a set $S \subseteq V(G)$ is an *r-radial dominating set* of G (or, alternatively we say that S *r-radially dominates* G) if $\mathbf{R}^r_G(S) = A(G)$. We define

$$\mathbf{rds}(G, r) = \min\{k \mid G \text{ contains an } r\text{-radial dominating set of size at most } k\}.$$

The following observation follows easily from the definitions.

Observation 2 *The parameter* $\mathbf{rds}$ *is closed under topological minors. In other words, if $H, G \in \mathcal{P}$, $r \in \mathbb{N}$, and $H \leqslant_{\mathrm{t}} G$, then $\mathbf{rds}(H, r) \leqslant \mathbf{rds}(G, r)$.*

Branchwidth. Let G be a graph on n vertices. A *branch decomposition* (T, μ) of a graph G consists of an unrooted ternary tree T (i.e., all internal vertices are of degree three) and a bijection $\mu : L \to E(G)$ from the set L of leaves of T to the edge set of G. We define for every edge e of T the *middle set* $\omega(e) \subseteq V(G)$, as follows: Let T^e_1 and T^e_2 be the two connected components of $T \setminus e$. Then, let G^e_i be the graph induced by the edge set $\{\mu(f) : f \in L \cap V(T^e_i)\}$ for $i \in \{1, 2\}$.

The *middle set* is the intersection of the vertex sets of G_1^e and G_2^e, i.e., $\omega(e) = V(G_1^e) \cap V(G_2^e)$. The *width* of (T,μ) is the maximum order of the middle sets over all edges of T (in case T has no edges, then the width of (T,μ) is equal to 0). The *branchwidth*, denoted by $\mathbf{bw}(G)$, of G is the minimum width over all branch decompositions of G.

We now state a series of results on branchwidth that are useful for our proofs.

Proposition 2 (See e.g., [32, (4.1)]). *The parameter* $\mathbf{bw}$ *is closed under topological minors, i.e., if* $H \leqslant_{\mathrm{t}} G$, *then* $\mathbf{bw}(H) \leqslant \mathbf{bw}(G)$.

Proposition 3 ([29, 33]). *If* G *is a plane graph with a cycle, then* $\mathbf{bw}(G) = \mathbf{bw}^*(G)$.

Proposition 4 ([21]). *If* G *is a* n*-vertex planar graph, then* $\mathbf{bw}(G) \leqslant \sqrt{4.5 \cdot n}$.

Triconnected components. Let G be a graph, let $S \subseteq V(G)$, and let $V_1,\ldots,V_q$ be the vertex sets of the connected components of $G \setminus S$. We define $\mathcal{C}(G,S) = \{G_1,\ldots,G_q\}$ where G_i is the graph obtained from $G[V_i \cup S]$ if we add all edges between vertices in S.

Given a graph G, the set $\mathcal{Q}(G)$ of its *triconnected components* is recursively defined as follows:

- If G is 3-connected or a clique of size $\leqslant 3$, then $\mathcal{Q}(G) = \{G\}$.
- If G contains a separator S where $|S| \leqslant 2$, then $\mathcal{Q}(G) = \bigcup_{H \in \mathcal{C}(G,S)} \mathcal{Q}(H)$.

Observation 3 *Let* G *be a graph. All graphs in* $\mathcal{Q}(G)$ *are topological minors of* G.

The following lemma follows easily from Observation 3 and [20, Lemma 3.1].

Lemma 1. *If* G *is a graph that contains a cycle, then* $\mathbf{bw}(G) = \max\{\mathbf{bw}(H) \mid H \in \mathcal{Q}(G)\}$.

Sphere-cut decompositions. Let G be a plane graph. A branch decomposition (T,μ) of G is called a *sphere-cut decomposition* if for every edge e of T there exists a noose N_e, such that (a) $\omega(e) = V(N_e)$, (b) $G_i^e \subseteq \Delta_i \cup N_e$ for $i = 1, 2$, where Δ_i is the open disc bounded by N_e, and (c) for every face f of G, $N_e \cap f$ is either empty or connected (i.e., if the noose traverses a face then it traverses it once).

The following theorem is a useful tool when dealing with branch decompositions of planar graphs.

Proposition 5 ([33, Theorem (5.1)]). *Let* G *be a planar graph without vertices of degree one and with branchwidth at most* k *embedded on a sphere. Then there exists a sphere-cut decomposition of* G *of width at most* k.

3 Radially extremal sets and branchwidth

Before we proceed with the proof of Theorem 2, we prove first some auxiliary results.

Let G be a plane graph, $y \in \mathbb{N}$, and $S \subseteq V(G)$. We say that S is *y-radially scattered* if for any $a_1, a_2 \in S$, $\mathbf{rdist}_G(a_1, a_2) \geqslant y$. We say that S is *r-radially extremal in* G if S is an r-radial dominating set of G and S is $2r$-radially scattered in G.

The proof of Theorem 2 is based on a suitable "padding" of a graph that is r-radially dominated so that it becomes r-radially extremal. This is done by the following lemma.

Lemma 2. *Let G be a 3-connected plane graph and S be an r-radial dominating set of G. Then G is the topological minor of a triangulated 3-connected plane graph H where S is r-radially extremal in H.*

We postpone the proof of Lemma 2 until after presentation of the following two lemmata.

Lemma 3 ($\star$). *Let G be a 3-connected plane graph and S an r-radial dominating set of G. Then G has a triangulation H where S is an r-radial dominating set of H.*

Given a number $k \in \mathbb{N}$, we denote by $\mathcal{A}^{\langle k \rangle}$ the set of all sequences of numbers in $\mathbb{N}$ with length k. We denote $\mathbf{0}_k$, the sequence containing k zero's. Given $\alpha^i = (a^i_1, \ldots, a^i_k),\ i = 1, 2$, we say that $\alpha^1 \prec \alpha^2$ if, there is some integer $j \in 1, \ldots, k$, such that $a^1_h = a^2_h$ for all $h \leqslant j$ and $a^1_j < a^2_j$. For example $(1, 1, 2, 4, 15, 3, 82, 2) \prec (1, 1, 3, 1, 6, 29, 1, 3)$. A sequence $A = (\alpha^i \mid i \in \mathbb{N})$ of sequences in $\mathcal{A}^{\langle k \rangle}$ is *properly decreasing* if for any two consecutive elements α^j, α^{j+1} of A it holds that $\alpha^j \prec \alpha^{j+1}$. We will use the following known lemma.

Observation 4 *For every $k \in \mathbb{N}$, every properly decreasing sequence of sequences in $\mathcal{A}^{\langle k \rangle}$ is finite.*

Given a graph G and a subset $S \subseteq V(G)$, we call a path an *S-path* if its extremes are vertices of S. The proof of the following lemma is based on Observations 1 and 4.

Lemma 4 ($\star$). *Let G be a triangulated plane graph and let S be an r-radial dominating set of G. Then G is the topological minor of a graph H that is $2r$-radially extremal.*

Proof (of Lemma 2). Applying first Lemma 3, we obtain a planar triangulation H of G where the set S is an r-radial dominating. Then, applying Lemma 4, we obtain a triangulated graph H' that is a topological minor of H and such that S is $2r$-radially scattered in H'. The lemma follows as G is a topological minor of H' and H' is triangulated and thus 3-connected. $\square$

We are now ready to prove our main combinatorial result.

Lemma 5. *Let r be a positive integer and let G be a plane graph. Then* $\mathbf{bw}(G) \leqslant r \cdot \sqrt{4.5 \cdot \mathbf{rds}(G,r)}$.

Proof. We use induction on r. If $r = 1$ then $|V(G)| = \mathbf{rds}(G,1)$ and the result follows from Proposition 4. Assume now that the lemma holds for values smaller that r and we will prove that it also holds for r, where $r \geqslant 2$. Let G be a plane graph where $\mathbf{rds}(G,r) \leqslant k$. Using Lemma 1, we choose $H \in \mathcal{Q}(G)$ such that $\mathbf{bw}(H) = \mathbf{bw}(G)$ (we may assume that G contains a cycle, otherwise the result follows trivially). By Observations 2 and 3, $\mathbf{rds}(H,r) \leqslant \mathbf{rds}(G,r) \leqslant k$. Let S be an r-radial dominating set of H where $|S| \leqslant k$. From Lemma 2, H is the topological minor of a 3-connected plane graph H_1 where S is r-radially extremal.

Let H_2 be the graph obtained if we remove from H_1 all vertices of S. By the 3-connectivity of H_1, it follows that, for any $v \in S$, the graph $H_1[N_{H_1}(v)]$ is a cycle and each such cycle is the boundary of some face of H_2. We denote by F the set of these faces and observe that F^* is a $(r-1)$-radial dominating set of H_2^* (we denote by F^* the vertices of H_2^* that are duals of the faces of F in H_2). Moreover, the fact that S is a $2r$-scattered dominating set in H_1, implies that F^* is a $2(r-1)$-scattered dominating set in H_2^*.

From the induction hypothesis and the fact that $|F^*| = |S|$, we obtain that $\mathbf{bw}(H_2^*) \leqslant (r-1) \cdot \sqrt{4.5 \cdot k}$. This fact, along with Proposition 3, implies that $\mathbf{bw}(H_2) \leqslant (r-1) \cdot \sqrt{4.5 \cdot k}$.

In graph H_2, for any face $f_i \in F$, let $(x_0^i, \ldots, x_{m_i-1}^i)$ be the cyclic order of the vertices in its boundary cycle (as H_1 is 3-connected we have that $m_i \geqslant 3$). We also denote by x^i the vertex in H_1 that was removed in order f_i to appear in H_2. Let (T,τ) be a sphere cut decomposition of H_2 of width $\leqslant (r-1) \cdot \sqrt{4.5 \cdot k}$. By Proposition 5, we may assume that (T,τ) is a sphere-cut decomposition. We use (T,τ) in order to construct a branch decomposition of H_1, by adding new leaves in T and mapping them to the edges of $E(H_1) \setminus E(H_2) = \bigcup_{i=1,\ldots,|F|}\{\{x^i, x_h^i\} \mid h = 0, \ldots, m_i - 1\}$ in the following way: For every $i = 1, \ldots, |F|$ and every $h = 0, \ldots, m_i - 1$, we set $t_h^i = \tau^{-1}(\{x_h^i, x_{h+1 \bmod m_i}^i\})$ and let $e_h^i = \{y_h^i, t_h^i\}$ be the unique edge of T that is incident to t_h^i. We subdivide e_h^i and we call the subdivision vertex s_h^i. We also add a new vertex z_h^i and make it adjacent to s_h^i. Finally, we extend the mapping of τ by mapping the vertex z_h^i to the edge $\{x^i, x_h^i\}$ and we use the notation (T',τ') for the resulting branch decomposition of H_1.

Claim. The width of (T',τ') is at most $r \cdot \sqrt{4.5 \cdot k}$.

Proof. We use the functions ω and ω' to denote the middle sets of (T,τ) and (T',τ') respectively. Let e be an edge of T'. If e is not an edge of T (i.e., is an edge of the form $\{z_h^i, s_h^i\}$ or $\{t_h^i, s_h^i\}$ or $\{y_h^i, s_h^i\}$), then $|\omega'(e)| \leqslant 3$, therefore we may fix our attention to the case where e is also an edge of T. Let N_e be the noose of H_2 meeting the vertices of $\omega(e)$. We distinguish the following cases.

Case 1. N_e does not meet any face of F, then clearly $\omega'(e) = \omega(e)$. Thus $|\omega'(e)| \leqslant (r-1) \cdot \sqrt{4.5 \cdot k}$.
Case 2. If N_e meets only one, say f_i, of the faces of F, then the vertices in $\omega'(e)$ are the vertices of a noose N'_e of H_1 meeting all vertices of $\omega(e)$ plus x^i. Therefore, $\omega'(e) = \omega(e) \cup \{x^i\}$ and thus $|\omega'(e)| \leqslant (r-1) \cdot \sqrt{4.5 \cdot k} + 1$.
Case 3. N_e meets $p \geqslant 2$ faces of F. We denote by $\{f'_0, \ldots, f'_{p-1}\}$ the set of these faces and let $J_0, \ldots, J_{p-1}$ be the normal arcs corresponding to the connected components of $N_i - \bigcup_{i=0,\ldots,p-1} f'_i$. Let also $I_0, \ldots, I_{p-1}$ be the normal arcs corresponding to the closures of the connected components of $N_i \cap (\bigcup_{i=0,\ldots,p-1} f'_i)$, assuming that $I_i \subseteq \overline{f'_i}$, for $i = 0, \ldots, p-1$. Recall that F^* is a $(r-1)$-scattered dominating set of H_2^*. This implies that each J_i meets at least $r-1$ vertices of H_2 and therefore $p \cdot (r-1) \leqslant \omega(e) \leqslant (r-1) \cdot \sqrt{4.5 \cdot k}$. We conclude that $p \leqslant \sqrt{4.5 \cdot k}$. Observe now that the vertices of $\omega'(e)$ are the vertices of a noose N'_e of H_1 where $N'_e = (\bigcup_{i=0,\ldots,p-1} J_i) \cup (\bigcup_{i=0,\ldots,p-1} I'_i)$ and such that, for each $i = 0, \ldots, p-1$, I'_i is a replacement of I_i so that it is still a subset of $\overline{f}'_i$, has the same extremes as I_i, and also meets the unique vertex in $S \cap f'_i$. As N'_e meets in H_1 all vertices of $N_e \cap V(H_2)$ plus p more, we obtain that $|\omega'(e)| \leqslant (r-1) \cdot \sqrt{4.5 \cdot k} + p \leqslant r \cdot \sqrt{4.5 \cdot k}$.

According to the above case analysis, $|\omega'(e)| \leqslant \max\{3, \sqrt{4.5 \cdot k} + 1, r \cdot \sqrt{4.5 \cdot k}\} = r \cdot \sqrt{4.5 \cdot k}$ and the claim follows.

We just proved that $\mathbf{bw}(H_1) \leqslant r \cdot \sqrt{4.5 \cdot k}$. As H is a topological minor of H_1, from Proposition 2, we also have that $\mathbf{bw}(H) \leqslant r \cdot \sqrt{4.5 \cdot k}$. The lemma follows as $\mathbf{bw}(G) = \mathbf{bw}(H)$. □

Proof (of Theorem 2). By the definition of compactness, G has an embedding such that either G or G^* contains an r-radial dominating set of size at most $q \cdot k$. Without loss of generality, assume that this is the case for G (here we use Proposition 3). Then $\mathbf{rds}(G, r) \leqslant q \cdot k$ and, from Lemma 5, $\mathbf{bw}(G) \leqslant r \cdot \sqrt{4.5 \cdot q \cdot \mathbf{p}(G)}$. □

4 Conclusions and open problems

The concept of compactness for parameterized problems appeared for the first time in [3] in the context of kernelization. In this paper, we show that it can also be used to improve the running time analysis of a wide family of sub-exponential parameterized algorithms. Essentially, we show that such an analysis can be done without the grid-minor exclusion theorem. Instead, our better combinatorial bounds emerge from the result in [21] that, in turn, is based on the "planar separators theorem" of Alon, Seymour, and Thomas in [2]. This implies that any improvement of the constant $\sqrt{4.5}$ in [2] would improve all the running times emerged from the framework of our paper.

It follows that there are bidimensional parameterized problems that are not compact and vice versa. For instance, Independent Dominating Set is compact but not bidimensional while Longest Path is bidimensional but not compact. Is it possible to extend both frameworks to a more powerful theory, at least in the context of sub-exponential parameterized algorithms?

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Appendix

Estimation of q and r for some compact parameterized problems

The DOMINATING SET problem is $(1,3)$-compact because in graph $G=(V,E)$, dominated by a set S, each face $f \in F(G)$ is either incident to some vertex $v \in S$ (thus $\mathbf{rdist}_G(v,f)=1$) or is incident to a vertex $x \in V \setminus S$ that is dominated by some vertex $v \in S$. As $\mathbf{rdist}_G(v,x)=2$ and $\mathbf{rdist}_G(f,x)=1$, we conclude that $r=3$. For the FACE COVER problem, we observe that the dual $G^*=(V^*,E^*)$ of G in a YES-instance (G,k) contains a set of vertices F^* where each vertex in $V^* \setminus F^*$ (resp. face of G^*) is adjacent (resp. incident) to some vertex in F^*. For the FEEDBACK VERTEX SET problem we restrict ourselves to 2-connected planar graphs. Such a restriction does not harm the generality of our analysis as, by Lemma 1 in Section 3, one may concentrate to the branchwidth of the biconnected –or even triconnected– inputs of the problem. Then, it is enough to check that if a set S intersects all cycles of a graph G, then each face or G should be incident to a vertex in S. Then, $r=3$ follows from the fact that each other vertex is incident to some face. For the case of CYCLE PACKING, $q=3$ because according to a recent result of [5], every planar graph G with at most k disjoint cycles contains a set of size $3k$ meeting all the cycles of G. For TRIANGLE COVERING we assume first that every vertex in G is incident to a triangle. If S is a vertex set meeting each triangle of such a graph G, then each face should be incident to a vertex that, in turn, is adjacent to some triangle of G. Removing from G of all vertices that are non incident to a triangle is an easy preprocessing step that can be done in $O(n)$ steps (see [3] for a similar application).

Omitted proofs

Proof (of Lemma 3). We apply, for any face $f \in F(G)$ that is not a triangle, an edge addition that does not harm r-radial domination by S. Let t be the minimum radial distance of f from some vertex, say v, of S. Clearly, $t \leqslant r$ and there exist a vertex u in $V_f = V(G) \cap \mathbf{bd}(f)$ where $\mathbf{rdist}_G(v,u)=t-1$. Let also w be a vertex incident to f that is not a neighbor of u. We claim that if G' is the graph occurring after adding the edge $\{u,w\}$ in G, then S is also an r-radial dominating set of G' (notice that $\{u,w\} \notin E(G)$ because of the 3-connectivity of G, therefore G' remains a simple 3-connected graph). Suppose to the contrary that z_q is an element of G' such that for any $x \in S$, $\mathbf{rdist}_{G'}(z_q,x) > r$. Clearly, z_q cannot be one of the two new f_1, f_2 faces of G' that replaced the face f of G as they both contain u in their boundary and thus $\mathbf{rdist}_{G'}(v,f_i) \leqslant \mathbf{rdist}_{G'}(v,u) + 1 \leqslant \mathbf{rdist}_G(v,u)+1 = t \leqslant r,\ i=1,2$, a contradiction. Suppose now that z_q is an element of $A(G')$ that is also an element of $A(G)$ and let I be a normal (z_0,z_q)-arc in G of length $q \leqslant r$ for some $z_0 \in S$. Let also $\{z_0,\ldots,z_q\}$ be the set of elements of $A(G)$ met by I ordered as they appear in I from z_0 to z_q. Clearly, I cannot exist in G'. This means that $f \cap I \neq \emptyset$ and therefore $f = z_i$ for some $i, 1 < i < q$. Moreover, $y_j \in \mathbf{bd}(f_i) - \{v, y\}$, for $j = i-1, i+1$. Observe now that $\mathbf{rdist}_{G'}(v, z_q) \leqslant \mathbf{rdist}_{G'}(v, u) + \mathbf{rdist}_{G'}(u, z_{i+1}) + \mathbf{rdist}_{G'}(z_{i+1}, z_q) \leqslant \mathbf{rdist}_{G}(v, u) + \mathbf{rdist}_{G'}(u, z_{i+1}) + \mathbf{rdist}_{G}(z_{i+1}, z_q) = t-1+2+q-(i+1) \leqslant t+q-i$. By the minimality of the radial distance between v and f, we have that $t \leqslant i$, therefore $\mathbf{rdist}_{G'}(v, z_q) \leqslant q \leqslant r$, a contradiction. □

Proof (of Lemma 4). Given a triangulated plane graph H and $S \subseteq V(G)$, we consider the sequence

$$\mathbf{q}(H) = (a_1, \dots, a_{r-1})$$

where, for $j = 1, \dots, r-1$, a_j is the number of S-paths of length j in H. Notice that if $\mathbf{q}(G) = \mathbf{0}_{r-1}$, then all S-paths have distance at least r. As G is triangulated, Observation 1 implies that S is $2r$-radially scattered in G.

If $\mathbf{0}_{r-1} \prec \mathbf{q}(G)$, we show how to transform G to a new graph G' satisfying the following properties: **(i)** G' is triangulated, **(ii)** $G' \leqslant_{\mathrm{t}} G$, **(iii)** G' is r-radially dominated by S, and **(iv)** $\mathbf{q}(G') \prec \mathbf{q}(G)$.

From Observation 4, this transformation cannot be applied forever. Therefore, it will end up with a graph G_{final} where $\mathbf{q}(G_{\mathrm{final}}) = \mathbf{0}_{r-1}$. Then the lemma follows as S is an r-radially scattered set of G_{final} and $G_{\mathrm{final}} \leqslant_{\mathrm{t}} G$. Below, we describe this transformation.

Let $P = (v_0, \dots, v_t)$ be a minimum length S-path in G. Clearly, $\mathbf{0}_{r-1} \prec \mathbf{q}(G)$ implies that $t \leqslant r-1$. Also, we set $l = \lfloor t/2 \rfloor$ and denote by f_1, f_2 the two triangular faces of H that are incident to the edge $e = \{v_l, v_{l+1}\}$. We denote by y_1 and y_2 the two vertices that are incident to f_1 or f_2 but not to e and assume that y_1 (resp. y_2) is incident to f_1 (resp. f_2). We transform G as follows: first subdivide in G the edge $\{v_l, v_{l+1}\}$, call v_{new} the subdivision vertex, and then add the edges $\{y_1, v_{\mathrm{new}}\}$ and $\{y_2, v_{\mathrm{new}}\}$. We denote by G' the resulting graph and we prove that it satisfies Properties **(i)**–**(iv)**.

Properties **(i)** and **(ii)** hold directly by the construction of G'. For property **(iii)**, we assume, towards a contradiction, that some element $a \in A(G')$ is not r-radially dominated by any vertex in S. Notice that $a \neq v_{\mathrm{new}}$ as $\mathbf{dist}_{G'}(v_{\mathrm{new}}, v_t) \leqslant 1 + \mathbf{dist}_{G'}(v_{l+1}, v_t) \leqslant 1 + \mathbf{dist}_{G}(v_{l+1}, v_t) = 1 + t - \lfloor t/2 \rfloor - 1 \leqslant \lceil t/2 \rceil$, thus, from Observation 1, $\mathbf{rdist}_{G'}(v_{\mathrm{new}}, v_t) \leqslant 2 \cdot \lceil t/2 \rceil \leqslant r$. Notice also that $\mathbf{rdist}_{G'}(v_0, v_l) \leqslant \mathbf{rdist}_{G}(v_0, v_l) \leqslant 2 \cdot \lfloor t/2 \rfloor \leqslant r-1$ and $\mathbf{rdist}_{G'}(v_q, v_{l+1}) \leqslant \mathbf{rdist}_{G}(v_q, v_{l+1}) \leqslant 2 \cdot (t - \lfloor t/2 \rfloor - 1) \leqslant 2 \cdot \lfloor t/2 \rfloor \leqslant r-1$, therefore each of the new faces of G' that are either incident to v_l or incident to v_{l+1} is r-radially dominated by either v_0 or v_t respectively. This means that a is also an element of $A(G)$. Let $P' = (a_0, \dots, a_q)$ be a path in G such that $a_0 \in S$ and either $a_q = a$ (in case a is a vertex) or a_q is incident to a (in case a is a face); in the first case $q \leqslant \lfloor r/2 \rfloor$ and in the second $q \leqslant \lfloor (r-1)/2 \rfloor$. As P' is not a path of G', some, say $\{a_j, a_{j+1}\}$, of its edges should be the subdivided edge $\{v_l, v_{l+1}\}$. We will end up with a contradiction by proving the existence in G' of a (x, a_q)-path of length $\leqslant q$ for some $x \in \{v_0, v_t\} \subseteq S$. We examine two cases:

Case I. $a_j = v_l$ and $a_{j+1} = v_{l+1}$. By the minimality of the choice of P, we deduce that $l \leqslant j$ (otherwise $\mathbf{dist}_G(a_0, v_t) \leqslant \mathbf{dist}_G(a_0, v_l)+1+\mathbf{dist}_G(v_{l+1}, v_t) < l+1+(t-l-1) \leqslant t$) and this means that $\mathbf{dist}_{G'}(v_{l+1}, a_q) \leqslant \mathbf{dist}_G(v_{l+1}, a_q) \leqslant q-j-1 \leqslant q-l-1 = q-\lfloor t/2 \rfloor -1$. Observe that $\mathbf{dist}_{G'}(v_{l+1}, v_t) \leqslant \mathbf{dist}_G(v_{l+1}, v_t) \leqslant t-l-1 = \lceil t/2 \rceil - 1$. Therefore, $\mathbf{dist}_{G'}(v_t, a_q) \leqslant q - \lfloor t/2 \rfloor - 1 + \lceil t/2 \rceil - 1 \leqslant q$, a contradiction.
Case II. $a_j = v_{l+1}$ and $a_{j+1} = v_l$. Now, by the minimality of P we have that $t-l-1 \leqslant j$ (otherwise $\mathbf{dist}_G(a_0, v_0) \leqslant \mathbf{dist}_G(a_0, a_j) + 1 + \mathbf{dist}_G(v_l, v_0) < (t-l-1)+1+l \leqslant t$) and thus $\mathbf{dist}_{G'}(v_l, a_q) \leqslant \mathbf{dist}_G(v_l, a_q) \leqslant q-j-1 \leqslant q-t+l = q - \lceil t/2 \rceil$. As $\mathbf{dist}_{G'}(v_0, v_l) \leqslant \mathbf{dist}_G(v_0, v_l) = l = \lfloor t/2 \rfloor$, we conclude that $\mathbf{dist}_{G'}(v_0, a_q) \leqslant \lfloor t/2 \rfloor + q - \lceil t/2 \rceil \leqslant q$, a contradiction and property **(iii)** holds for G'.

For Property **(iv)**, we need to prove that $\mathbf{q}(G') \prec \mathbf{q}(G)$. As all S-paths in G' that avoid v_{new} also exist in G, we have to prove that there is at least one S-path of length t in G that is not in G' and that no new paths of length t appear in G'. Indeed P is a path of length t that does not exist in G'. What remains is to prove that no new paths of length t appear in G'. Suppose to the contrary that $P' = (x_0, \ldots, x_t)$ is such a path. Clearly, P' should meet the vertex v_{new} and assume that $v_{\text{new}} = x_i$. The cases where the set $\{x_{i-1}, x_{i+1}\}$ is one of $\{v_l, y_1\}, \{v_l, y_2\}, \{v_{l+1}, y_1\}, \{v_{l+1}, y_2\}$ are excluded as, in such a case, the existence of the path $(x_0, x_{i-1}, x_{i+1}, \ldots, x_t)$ in G contradicts the minimality of the choice of P. Therefore, $\{x_{i-1}, x_{i+1}\} = \{y_1, y_2\}$ and, w.l.o.g., we assume that $x_{i-1} = y_1$ and $x_{i+1} = y_2$. Then either $\mathbf{dist}_G(x_0, x_{i-1}) \leqslant \lfloor (t-2)/2 \rfloor$ or $\mathbf{dist}_G(x_{i+1}, x_t) \leqslant \lfloor (t-2)/2 \rfloor$. W.l.o.g., we assume that $\mathbf{dist}(x_0, x_{i-1}) \leqslant \lfloor (t-2)/2 \rfloor$. Then $\mathbf{dist}_G(x_0, v_t) \leqslant \mathbf{dist}_G(x_0, y_1) + 1 + \mathbf{dist}_G(v_{l+1}, v_t) \leqslant \lfloor (t-2)/2 \rfloor + 1 + (t-l-1) = \lfloor (t-2)/2 \rfloor + 1 + t - \lfloor t/2 \rfloor - 1 = \lfloor t/2 \rfloor - 1 + 1 + \lceil t/2 \rceil - 1 = t-1$, a contradiction to the minimality of P. $\square$