# Coverability and Fast Sub-exponential Parameterized Algorithms in Planar Graphs*

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**Abstract**

We provide a new framework, alternative to bidimensionality, of sub-exponential parameterized algorithms on planar graphs, which is based on the notion of *coverability*. Roughly speaking, a parameterized problem is $(r, q)$-coverable when all the faces and vertices of its YES-instances are "$r$-radially dominated" by some vertex set whose size is at most $q$ times the parameter. Our results are based on suitably bounding the branchwidth, $\mathbf{bw}(G)$, of the input graph $G$. In particular, we prove that if a parameterized problem can be solved in $c^{\mathbf{bw}(G)}n^{O(1)}$ steps and is $(r, q)$-coverable, then it can be solved by a $c^{r \cdot 2.122 \cdot \sqrt{q \cdot k}}n^{O(1)}$ step algorithm (where $k$ is the parameter). Our framework is general enough to unify the analysis of almost all known sub-exponential parameterized algorithms on planar graphs and improves or matches their running times. Our combinatorial bound on the branchwidth of planar graphs bypasses the grid-minor exclusion theorem. That way, our approach encompasses new problems where bidimensionality theory do not directly provide sub-exponential parameterized algorithms.

**Keywords:** Parameterized Algorithms, Branchwidth, Planar Graphs, Coverable Problems, Radial Domination, Scattered Set.

## 1 Introduction

A parameterized problem can be defined as a language $\Pi \subseteq \Sigma^* \times \mathbb{N}$. Its inputs are pairs $(I, k) \in \Sigma^* \times \mathbb{N}$, where $I$ can be seen as the main part of the problem and $k$ is some parameter of it. A problem $\Pi \subseteq \Sigma^* \times \mathbb{N}$ is *fixed parameter tractable* when it admits an $f(k) \cdot n^{O(1)}$ -time algorithm. In that case, $\Pi$ is classified in the parameterized complexity class $\mathsf{FPT}$ and, when we insist to indicate the parameter dependance (i.e., the function $f$), we also say that that $\Pi \in f(k)$-$\mathsf{FPT}$.

In this paper we deal with *parameterized problem on planar graphs* where the main part of the problem $I$ encodes a planar graph $G$. For this reason we see such a problem $\Pi$ as a subset of $\mathcal{P} \times \mathbb{N}$, where $\mathcal{P}$ is the set of all planar graphs. A pair $(G, k) \in \mathcal{P} \times \mathbb{N}$ we say that $(G, k)$ is a YES-*instance* of $\Pi$ if $(G, k) \in \Pi$, otherwise we say that it is a NO-*instance* of $\Pi$.

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**Sub-exponential parameterized algorithms.** A central problem in parameterized algorithm design is to investigate in which cases and under which input restrictions a parameterized problem belongs to $\mathsf{FPT}$ and, if so, to find algorithms with the best possible parameter dependance. When $f(k) = 2^{o(k)}$, a parameterized problem is said to admit a *sub-exponential parameterized algorithm* (for a survey on this topic, see [24]).

In [8], Cai and Juedes proved that several parameterized problems not belong to $2^{o(k)}$-$\mathsf{FPT}$, unless Exponential Time Hypothesis[1] fails. Among them, one can distinguish core problems such as the standard parameterizations of Vertex Cover, Dominating Set, and Feedback Vertex Set. However, it appears that many problems admit sub-exponential parameterized algorithms when their inputs are restricted to planar graphs or other sparse graph classes. moreover, the results of [8] indicated that this is indeed the best we may expect when the planarity restriction is imposed.

The first sub-exponential parameterized algorithm on planar graphs appeared in [1] for Dominating Set, Independent Dominating Set, and Face Cover. After that, many other problems were classified in $2^{O(\sqrt{k})}$-$\mathsf{FPT}$, while there was a considerable effort towards improving the constant hidden in the "$O$"-notation for each one of them [1, 44, 10, 15, 26, 27, 43, 35, 45, 19].

**Bidimensionality theory.** A major advance towards a theory of sub-exponential parameterized algorithms was made with the introduction of Bidimensionality, in [16]. Bidimensionality theory offered a generic condition for classifying a parameterized problem in $2^{O(\sqrt{k})}$-$\mathsf{FPT}$. It also provided a machinery for estimating a (reasonably small) value $c$ for each particular problem. moreover, it also provided meta-algorithmic results in approximation algorithms [17, 32] and kernelization [28] (for a survey on bidimensionality, see [12]). We stress that alternative approaches for the design of sub-exponential parameterized algorithms have been appeared in [2, 22, 37, 58] (see also [31] for a recent innovative approach).

According to [16], a parameterized problem on planar graphs $\Pi \subseteq \mathcal{P} \times \mathbb{N}$ is *minor-bidimensional* with density $\delta$ if the following two conditions are satisfied.

**(a)** If the graph $G'$ is a minor of the graph $G$, then $(G, k) \in \Pi \Rightarrow (G', k) \in \Pi$.

**(b)** There exists a $\delta > 0$ such that for every $k \in \mathbb{N}$ it holds that $(\boxplus_{\sqrt{k}/\delta}, k) \notin \Pi$.

In the above definition, we use the term $\boxplus_w$ for the $(\lceil w \rceil \times \lceil w \rceil)$-grid. Also, we say that $G'$ is a *minor* of $G$, denoted as $G' \leqslant_{\mathrm{m}} G$, if $G'$ can be obtained by some subgraph of $G$ after a series of edge contractions[2]. We stress that there is an variant of the above definition, called *contraction bidimenisionality* in the case where we take contractions instead of minors (we avoid the precise definition in this paper as it is not necessary for our results). For more on Bidimensionality Theory, see [18, 13, 59].

[1]The Exponential Time Hypothesis states that 3-SAT cannot be solved by a subexponential time algorithm.

[2]The result of the *contraction* of en edge $e = \{x, y\}$ in $G$ is the graph obtained if we remove $x$ and $y$ from $G$, add a new vertex $v_{x,y}$, and make it adjacent with all vertices of $V(G) \setminus \{x, y\}$ that are adjacent with $x$ or $y$ in $G$.

Branchwidth (along with its twin parameter of treewidth) has been a powerful tool in parameterized algorithm design. Roughly speaking, branchwidth is a measure of the topological resemblance of a graph to the structure of a tree. We use the term $\mathbf{bw}(G)$ for the branchwidth of a graph $G$ and we postpone its formal definition until Section 2.

We say that a problem $\Pi \subseteq \mathcal{P} \times \mathbb{N}$ is $\lambda$-*single exponentially solvable with respect to branchwidth* if there exists an algorithm that solves it in $2^{\lambda \cdot \mathbf{bw}(G)} n^{O(1)}$ steps. The main idea of [16], towards designing sub-exponential parameterized algorithms, was to make use of the grid-minor exclusion theorem in [52] asserting that, for every planar graph $G$, $\mathbf{bw}(G) \leqslant 4 \cdot \mathbf{gm}(G)$, where $\mathbf{gm}(G) = \max\{w \mid \boxplus_w \leqslant_{\mathrm{m}} G\}$. This result was recently improved by Gu and Tamaki in [40] who proved that $\mathbf{bw}(G) \leqslant 3 \cdot \mathbf{gm}(G)$. This implies that for a bidimensional problem with density $\delta$ on planar graphs, all YES-instances have branchwidth at most $\frac{3}{\delta}\sqrt{k}$ and this reduces the problem to its variant where the branchwidth of the inputs are now bounded by $\frac{3}{\delta}\sqrt{k}$. As a preprocessing step, an optimal branch decomposition of a planar graph can be constructed[3] in $O(n^3)$ steps, (see [39, 57]). Therefore, the main algorithmic consequence of bidimensionality, as restricted to planar graphs[4], is the following.

**Proposition 1.** *If* $\Pi \subseteq \mathcal{P} \times \mathbb{N}$ *is minor-bidimensional with density* $\delta$ *and is* $\lambda$*-single exponentially solvable with respect to branchwidth, then* $\Pi \in 2^{(3\lambda/\delta)\cdot\sqrt{k}}$-$\mathsf{FPT}$.

The above result, along with its contraction-bidimensionality counterpart, defined in [16] (see also [29, 30, 33, 38]), reduce the solution of bidimensional problems to the easier task of designing dynamic programming algorithms on graphs with small branchwidth (or treewidth). Dynamic programming is one of the most studied and well developed topics in parameterized algorithms and there is an extensive bibliography on what is the best value of $\lambda$ that can be achieved for each problem (see e.g., [61, 11, 5, 34]). Especially for planar graphs, there are tools that can make dynamic programming run in single exponential time, even if this is not, so far, possible for general graphs [21, 4]. Lower bounds on the value of $\lambda$ for problems such as Dominating Set appeared recently in [46, 47]. Finally, meta-algorithmic conditions for single exponential solvability with respect to branchwidth have appeared in [50].

A consequence of Proposition 1 and its contraction-counterpart was a massive classification of many parameterized problems in $2^{c\cdot\sqrt{k}}$-$\mathsf{FPT}$ and, in many cases, with estimations of $c$ that improved all previously known bounds. The remaining question was whether it is possible to do even better and when. Indeed, a more refined problem-specific combinatorial analysis improved some of the bounds provided by the bidimensionality framework (see also [19]). For instance, such refinements appeared in Planar Dominating Set [35], Face Cover, Planar Vertex Feedback Set, and Cycle Packing [45], that where classified in $2^{c\cdot\sqrt{k}}$-$\mathsf{FPT}$ where $c = 15.3, 10.1, 15.11$, and $26.3$

[3] A possible alternative is to use the $\mathsf{FPT}$-approximation algorithm in [6]. This may reduce the $O(n^3)$ contribution in this preprocessing step to a *linear* one with the cost of slightly worst constants in the parametrized dependence of the algorithm of Proposition 1.

[4] The results in [16] apply for more general graph classes and have been further extended in [29, 30, 14].

respectively, improving all previous results on those problems.

**Our results.** In this paper, we provide an alternative theory for the design of fast sub-exponential parameterized algorithms.

Let us give first some definitions from [7]. As we mainly deal with plane graphs, we use the notation $F(G)$ for its faces and we refer to the members of $F(G) \cup V(F)$ as the *elements* of $G$. The *radial distance* between two elements $x, y$ of $G$ is one less than the minimum length of an alternating sequence of vertices and faces starting from $x$ and ending in $y$, such that every two consecutive elements in this sequence are incident to each other. Given a plane graph $G = (V, E)$ and a set $S \subseteq V$, we define $\mathbf{R}^r_G(S)$ as the set of all vertices or faces of $G$ whose radial distance from some vertex of $S$ is at most $r$ (for more definitions and notation, see Section 2).

A parameterized problem on planar graphs $\Pi \subseteq \mathcal{P} \times \mathbb{N}$ is $(q, r)$*-coverable* if for all $(G = (V, E), k) \in \Pi$ and for some planar embedding of a graph $H$, that is either $G$ or its dual, there is a set $S \subseteq V$ such that $|S| \leqslant q \cdot k$ and $\mathbf{R}^r_H(S)$ contains all faces and vertices of $H$. Intuitively, a parameterized problem is coverable if the input graph of every YES-instance can be "covered" by a collection of $O(k)$ balls each of constant radius.

| | $q$ | $r$ |
|---|---|---|
| Dominating Set | 1 | 3 |
| $l$-Dominating Set | 1 | $2l+1$ |
| $l$-Threshold Dominating Set | 1 | 3 |
| Perfect Code | 1 | 3 |
| Red Blue Dominating Set | 1 | 3 |
| Independent Directed Domination | 1 | 3 |
| Vertex Cover | 1 | 2 |
| Almost Outerplanar$^{(a)}$ | 1 | 3 |
| Almost Series-Parallel$^{(a)}$ | 1 | 3 |
| Connected $l$-Dominating Set | 1 | $2l+1$ |
| Connected Vertex Cover | 1 | 2 |
| Feedback Edge Set$^{(b)}$ | 1 | 2 |
| Feedback Vertex Set$^{(b)}$ | 1 | 3 |
| Connected Feedback Vertex Set | 1 | 3 |
| Minimum-Vertex Feedback Edge Set | 1 | 3 |
| Connected Dominating Set | 1 | 3 |

| | $q$ | $r$ |
|---|---|---|
| Cycle Domination | 1 | 4 |
| Edge Dominating Set | 2 | 2 |
| Clique Transversal | 1 | 3 |
| Independent Dominating Set | 1 | 3 |
| Odd Set | 1 | 3 |
| Face Cover | 1 | 2 |
| Vertex Triangle Covering$^{(c)}$ | 1 | 3 |
| Edge Triangle Covering$^{(c)}$ | 2 | 2 |
| $l$-Cycle Transversal$^{(b)}$ | 1 | $l$ |
| $l$-Scattered Set | 1 | $2l+1$ |
| Cycle Packing | 3 | 3 |
| Induced Matching | 2 | 3 |
| Max Internal Spanning Tree | 1 | 3 |
| Triangle Packing$^{(c)}$ | 1 | 3 |
| Minimum Leaf Out-branching | 1 | 3 |
| Max Full Degree Spanning Tree | 1 | 3 |

Table 1: Examples of $(q, r)$-coverable parameterized problems. $^{(a)}$Triconnected instances. $^{(b)}$Biconnected instances. $^{(c)}$For instances where each vertex of the input graph belongs in some triangle.

The notion of coverability has been introduced in [7] in the context of automated kernel design, while some preliminary concepts had already appeared in [41]. It encompasses a wide number of parameterized problems on graphs; some of them are listed in Table 1. Notice also that every $(q, r)$-coverable problem whose YES/NO instances are closed under taking of minors is bidimensional. In the next paragraph, we justify some of the constants depicted in this table (partially extracted from [7]).

**Examples of $(q, r)$-coverable problems.** The DOMINATING SET problem is $(1, 3)$-coverable because in graph $G = (V, E)$, dominated by a set $S$, each face $f \in F(G)$ is either incident to some vertex $v \in S$ (thus $\mathbf{rdist}_G(v, f) = 1$) or is incident to a vertex $x \in V \setminus S$ that is dominated by some vertex $v \in S$. As $\mathbf{rdist}_G(v, x) = 2$ and $\mathbf{rdist}_G(f, x) = 1$, we conclude that $r = 3$. For the FACE COVER problem, we keep in mind that input graphs are plane and we observe that the dual $G^* = (V^*, E^*)$ of $G$ in a YES-instance $(G, k)$ contains a set of vertices $F^*$ where each vertex in $V^* \setminus F^*$ (resp. face of $G^*$) is adjacent (resp. incident) to some vertex in $F^*$. For the FEEDBACK VERTEX SET problem we restrict ourselves to 2-connected planar graphs. Such a restriction does not harm the generality of our analysis as, by Lemma 1 in Section 3, one may concentrate to the branchwidth of the biconnected –or even triconnected– inputs of the problem. Then, it is enough to check that if a set $S$ intersects all cycles of a graph $G$, then each face of $G$ should be incident to a vertex in $S$. Then, $r = 3$ follows from the fact that each other vertex is incident to some face. For the case of CYCLE PACKING, $q = 3$ because, according to a recent result of [9], every planar graph $G$ with at most $k$ disjoint cycles contains a set of size $3k$ meeting all the cycles of $G$. For TRIANGLE COVERING we assume first that every vertex in $G$ is incident to a triangle. If $S$ is a vertex set meeting each triangle of such a graph $G$, then each face should be incident to a vertex that, in turn, is adjacent to some triangle of $G$. Removing from $G$ of all vertices that are not incident to a triangle is an easy preprocessing step that can be done in $O(n)$ steps (see [7] for a similar application).

**Our results.** We define $\overline{\Pi} = (G, k) \in (\mathcal{P} \times \mathbb{N}) \setminus \Pi$ , i.e., $\overline{\Pi}$ is the set of NO-instances of $\Pi$. We present below the main algorithmic contribution of our paper.

**Theorem 1.** *Let* $\Pi \subseteq \mathcal{P} \times \mathbb{N}$ *be a parameterized problem on planar graphs. If* $\Pi$ *is* $\lambda$*-single exponentially solvable with respect to branchwidth and either* $\Pi$ *or* $\overline{\Pi}$ *is* $(q, r)$-coverable*, then* $\Pi \in 2^{\lambda \cdot r \cdot 2.122 \cdot \sqrt{q \cdot k}}$-FPT.

The advantages of our approach, compared with those of bidimensionality theory, are the following:

- It applies to *many* problems where bidimensionality does not apply directly. This typically happens for problems whose YES-instances are not closed under taking of minors (or contractions) such as INDEPENDENT VERTEX COVER, INDEPENDENT DOMINATING SET, PERFECT CODE, and THRESHOLD DOMINATING SET.
- When applied, it *always* gives better bounds than those provided by bidimensionality theory. A direct comparison of the combinatorial bounds implies that the constants in the exponent provided by coverability are $\sqrt{4.5}/3 \approx 70\%$ of those emerging by the grid-minor exclusion theorem of Gu and Tamaki in [40].
- Matches or improves *all* problem-specific time upper bounds known so far for sub-exponential algorithms in planar graphs (including the results in [15, 35, 45, 44, 19]) and unifies their combinatorial analysis to a single theorem.

Theorem 1 follows from the following theorem that we believe is of independent combinatorial interest.

**Theorem 2.** *If $(G, k)$ is a* YES*-instance of a $(q, r)$-coverable parameterized problem on planar graphs, then $\mathbf{bw}(G) \leqslant r \cdot \sqrt{4.5 \cdot q \cdot k}$.*

Theorem 2 is our main combinatorial result and the rest of the sections of this paper are devoted to its proof.

The paper is organized as follows. In Section 2, we give some necessary definitions and preliminary results. Sections 3 and 4 are dedicated to the proof of Theorem 2. Finally, some conclusion and directions for further research are given in Section 5.

## 2 Basic definitions and preliminaries

All graphs in this paper are simple (i.e., have no multiple edges or loops). Given a graph $G$ we denote by $V(G)$ (resp. $E(G)$) the set of its vertices (resp. edges). For any set $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of $G$ induced by the vertices in $S$. Given two graphs $G_1$ and $G_2$, we define $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. We also use the notation $G \setminus S$ for the graph $G[V(G) \setminus S]$. Given $x, y \in V(G)$ an *$(x, y)$-path* is any subgraph of $G$ where $x$ and $y$ have degree 1 and all the other vertices (if any) have degree 2. The *length* of such a path is the number of its edges. The distance between two vertices $x, y$ in $G$ is denoted by $\mathbf{dist}_G(x, y)$ and is the minimum length of a $(x, y)$-path in $G$ or is infinite if no such as path exists. The *subdivision* of an edge $e$ in a graph is the operation of replacing an edge $e = \{x, y\}$ by a $(x, y)$-path of length two. A *subdivision* of a graph $H$ is any graph that can be obtained from $H$ if we apply a sequence of subdivisions to some (possibly none) of its edges. We say that a graph $H$ is a *topological minor* of a graph $G$ (we denote it by $H \leqslant_{\mathrm{t}} G$) if some subdivision of $H$ is a subgraph of $G$.

**Plane graphs.** In this paper, we mainly deal with plane graphs (i.e. graphs embedded in the plane $\mathbb{R}^2$ without crossings). For simplicity, we do not distinguish between a vertex of a plane graph and the point of the plane used in the drawing to represent the vertex or between an edge and the open line segment representing it. Given a plane graph $G$, we denote its dual by $G^*$. A *parameter* on plane graphs is any function $\mathbf{p}$ mapping plane graphs to $\mathbb{N}$. Given such a parameter $\mathbf{p}$, we define its *dual* parameter $\mathbf{p}^*$ so that $\mathbf{p}^*(G) = \mathbf{p}(G^*)$.

Given a plane graph $G$, we denote by $F(G)$ the set of the faces of $G$ (i.e., the connected components of $\mathbb{R}^2 \setminus G$, that are open subsets of the plane). We use the notation $A(G)$ for the set $V(G) \cup F(G)$ and we say that $A(G)$ contains the *elements* of $G$. If $a_i, i = 1, 2$ is an edge or an element of $G$, we say that $a_1$ is *incident* to $a_2$ if $a_1 \subseteq \overline{a}_2$ or $a_2 \subseteq \overline{a}_1$, where $\overline{x}$ is the closure of the set $x$. For every face $f \in F(G)$, we denote by $\mathbf{bd}(f)$ the *boundary* of $f$, i.e., the set $\overline{f} \setminus f$, where $\overline{f}$ is the closure of $f$.

A *triangulation* $H$ of a plane graph $G$ is a plane graph $H$ where $V(H) = V(G)$, $E(G) \subseteq E(H)$, and where $H$ is triangulated, i.e., every face of $H$ (including the exterior face) has exactly three edges incident upon it. Notice that each plane graph with at least 3 vertices has some triangulation. Notice that, in general, a triangulation of a simple graphs is not necessarily simple (i.e. there is a triangulation of $K_{2,r}$ with an edge with multiplicity $r$). However, there is always a simple one and, in this paper, we consider only such triangulations.

We use the term *arc* for any subset of the plane homeomorphic to the closed interval $[0, 1]$. Given a plane graph $G$, an arc $I$ that does not intersect its edges (i.e., $I \cap G \subseteq V(G)$) is called *normal.* The *length* $|I|$ of a normal arc $I$ is equal to the number of elements of $A(G)$ that it intersects minus one. If $x$ and $y$ are the elements of $A(G)$ intersected by the extreme points a normal arc $I$, then we also call $I$ *normal* $(x, y)$*-arc.* A *noose* of the plane, where $G$ is embedded, is a Jordan curve that does not intersect the edges of $G$. We also denote by $V(N)$ the set of vertices of $G$ met by $N$, i.e., $V(N) = V(G) \cap N$. The *length* $|N|$ of a noose $N$ is $|V(N)|$, i.e., is the number of the vertices it meets.

Let $G$ be a plane graph and let $r$ be a non-negative integer. Given two elements $x, y \in A(G)$, we say that they are *within radial distance at most* $r$ if there is a normal $(x, y)$-arc of the plane of length at most $r$ and we denote this fact by $\mathbf{rdist}_G(x, y) \leqslant r$.

**Observation 1.** *Let $G$ be a triangulated plane graph and let $x, y \in V(G)$. Then $2 \cdot \mathbf{dist}_G(x, y) \leqslant \mathbf{rdist}_G(x, y)$.*

Given an edge set $F \subseteq E(G)$, we define *the subgraph of $G$ induced by $F$* as the graph whose vertex set consists of the endpoints of the edges in $F$ and whose edge set is $F$. Given a vertex set $S \subseteq V(G)$ and a non-negative integer $r$, we denote by $\mathbf{R}^r_G(S)$ the set of all elements of $G$ that are within radial distance at most $r$ from some vertex in $S$. We say that a set $S \subseteq V(G)$ is an *$r$-radial dominating set* of $G$ (or, alternatively we say that $S$ *$r$-radially dominates* $G$) if $\mathbf{R}^r_G(S) = A(G)$. We define

$$\mathbf{rds}(G, r) = \min\{k \mid \ G \text{ contains an } r\text{-radial dominating set of size at most } k\}.$$

The following observation follows easily from the definitions.

**Observation 2.** *The parameter* $\mathbf{rds}$ *is closed under topological minors. In other words, if $H, G \in \mathcal{P}$, $r \in \mathbb{N}$, and $H \leqslant_{\mathrm{t}} G$, then $\mathbf{rds}(H, r) \leqslant \mathbf{rds}(G, r)$.*

**Branchwidth.** Let $G$ be a graph on $n$ vertices. A *branch decomposition* $(T, \mu)$ of a graph $G$ consists of an unrooted ternary tree $T$ (i.e., all internal vertices are of degree three) and a bijection $\mu : L \to E(G)$ from the set $L$ of leaves of $T$ to the edge set of $G$. We define for every edge $e$ of $T$ the *middle set* $\omega(e) \subseteq V(G)$, as follows: Let $T^e_1$ and $T^e_2$ be the two connected components of $T \setminus e$. Then, let $G^e_i$ be the graph induced by the edge set $\{\mu(f) : f \in L \cap V(T^e_i)\}$ for $i \in \{1, 2\}$. The *middle set* is the intersection of the vertex sets of $G^e_1$ and $G^e_2$, i.e., $\omega(e) = V(G^e_1) \cap V(G^e_2)$. The *width* of $(T, \mu)$ is the maximum order of the middle sets over all edges of $T$ (in case $T$ has no edges, then the width of $(T, \mu)$ is equal to 0). The *branchwidth*, denoted by $\mathbf{bw}(G)$, of $G$ is the minimum width over all branch decompositions of $G$.

We now state a series of results on branchwidth that are useful for our proofs.

**Proposition 2** (See e.g., [51, (4.1)]). *The parameter* $\mathbf{bw}$ *is closed under topological minors, i.e., if* $H \leqslant_{\mathrm{t}} G$*, then* $\mathbf{bw}(H) \leqslant \mathbf{bw}(G)$.

**Proposition 3** ([49, 57]). *If $G$ is a planar graph with a cycle, then* $\mathbf{bw}(G) = \mathbf{bw}^*(G)$.

**Proposition 4** ([36]). *If $G$ is a $n$-vertex planar graph, then* $\mathbf{bw}(G) \leqslant \sqrt{4.5 \cdot n}$.

**Triconnected components.** Let $G$ be a graph and let $S \subseteq V(G)$. We say that $S$ is a *separator* of $G$ if $G$ has less connected components that $G \setminus S$. Given that $V_1, \ldots, V_q$ are the vertex sets of the connected components of $G \setminus S$, we define $\mathcal{C}(G, S) = \{G_1, \ldots, G_q\}$ where, for $i \in \{1, \ldots, q\}$. $G_i$ is the graph obtained from $G[V_i \cup S]$ if we add all edges between vertices in $S$.

Given a graph $G$, the set $\mathcal{Q}(G)$ of its *triconnected components* is recursively defined as follows:

- If $G$ is 3-connected or a clique of size $\leqslant 3$, then $\mathcal{Q}(G) = \{G\}$.
- If $G$ contains a separator $S$ where $|S| \leqslant 2$, then $\mathcal{Q}(G) = \bigcup_{H \in \mathcal{C}(G,S)} \mathcal{Q}(H)$.

Notice that all graphs in $\mathcal{Q}(G)$ are either cliques on at most 3 vertices or 3-connected graphs (graphs without any separator of less than 3 vertices). We wish to remark that the study of triconnected components of plane graphs dates back to the work of Saunders Mac Lane in [48] (see also [60]). Also, given $G$, $\mathcal{Q}(G)$ can be constructed in linear time using the celebrated algorithm of Hopcroft and Tarjan in [42].

**Observation 3.** *Let $G$ be a graph. All graphs in $\mathcal{Q}(G)$ are topological minors of $G$.*

The following lemma follows easily from Observation 3 and [35, Lemma 3.1].

**Lemma 1.** *If $G$ is a graph that contains a cycle, then* $\mathbf{bw}(G) = \max\{\mathbf{bw}(H) \mid H \in \mathcal{Q}(G)\}$.

**Sphere-cut decompositions.** Let $G$ be a plane graph. A branch decomposition $(T, \mu)$ of $G$ is called a *sphere-cut decomposition* if for every edge $e$ of $T$ there exists a noose $N_e$, such that

(a) $\omega(e) = V(N_e)$,

(b) $G_i^e \subseteq \Delta_i \cup N_e$ for $i = 1, 2$, where $\Delta_i$ is the open disc bounded by $N_e$, and

(c) for every face $f$ of $G$, $N_e \cap f$ is either empty or connected (i.e., if the noose traverses a face then it traverses it once).

Sphere-cut decompositions have been introduced in [25] for improving the running time of dynamic programming algorithms on planar graphs (for extensions to more general graph classes, see [23, 56, 53, 54]). The following theorem is a useful tool when dealing with branch decompositions of planar graphs.

**Proposition 5** ([57, Theorem (5.1)]). *Let $G$ be a plane graph without vertices of degree one and with branchwidth at most $k$. Then there exists a sphere-cut decomposition of $G$ of width at most $k$.*

## 3 Radially extremal sets

Let $G$ be a plane graph, $y \in \mathbb{N}$, and $S \subseteq V(G)$. We say that $S$ is *$y$-radially scattered* if for any $a_1, a_2 \in S$, $\mathbf{rdist}_G(a_1, a_2) \geqslant y$. We say that $S$ is *$r$-radially extremal in $G$* if $S$ is an $r$-radial dominating set of $G$ and $S$ is $2r$-radially scattered in $G$.

This section is devoted to the proof of the following lemma.

**Lemma 2.** *Let $G$ be a 3-connected plane graph and $S$ be an $r$-radial dominating set of $G$. Then $G$ is the topological minor of a triangulated 3-connected plane graph $H$ where $S$ is $r$-radially extremal in $H$.*

Before we present the proof of Lemma 2, we need first some preliminary results and definitions.

**Lemma 3.** *Let $G$ be a 3-connected plane graph and $S$ be an $r$-radial dominating set of $G$. Then $G$ has a planar triangulation $H$ where $S$ is an $r$-radial dominating set of $H$.*

*Proof.* We apply, for any face $f \in F(G)$ that is not a triangle, an edge addition that does not harm $r$-radial domination by $S$. Let $t$ be the minimum radial distance of $f$ from some vertex, say $v$, of $S$. Clearly, $t \leqslant r$ and there exist a vertex $u$ in $V_f = V(G) \cap \mathbf{bd}(f)$ where $\mathbf{rdist}_G(v, u) = t - 1$. Let also $w$ be a vertex incident to $f$ that is not a neighbor of $u$. We claim that if $G'$ is the graph occurring after adding the edge $\{u, w\}$ in $G$, then $S$ is also an $r$-radial dominating set of $G'$ (notice that $\{u, w\} \notin E(G)$ because of the 3-connectivity of $G$, therefore $G'$ remains a simple 3-connected graph). Suppose to the contrary that $z_q$ is an element of $G'$ such that for any $x \in S$, $\mathbf{rdist}_{G'}(z_q, x) > r$. Clearly, $z_q$ cannot be one of the two new $f_1, f_2$ faces of $G'$ that replaced the face $f$ of $G$ as they both contain $u$ in their boundary and thus

$$\mathbf{rdist}_{G'}(v, f_i) \leqslant \mathbf{rdist}_{G'}(v, u) + 1 \leqslant \mathbf{rdist}_G(v, u) + 1 = t \leqslant r,\ \ i = 1, 2,$$

a contradiction. Suppose now that $z_q$ is an element of $A(G')$ that is also an element of $A(G)$ and let $I$ be a normal $(z_0, z_q)$-arc in $G$ of length $q \leqslant r$ for some $z_0 \in S$. Let also $\{z_0, \ldots, z_q\}$ be the set of elements of $A(G)$ met by $I$ ordered as they appear in $I$ from $z_0$ to $z_q$. Clearly, $I$ cannot exist in $G'$. This means that $f \cap I \neq \emptyset$ and therefore $f = z_i$

for some $i, 1 < i < q$. Moreover, $y_j \in \mathbf{bd}(f_i) - \{v, y\}$, for $j = i-1, i+1$. Observe now that

$$\begin{aligned}\mathbf{rdist}_{G'}(v, z_q) &\leqslant \mathbf{rdist}_{G'}(v, u) + \mathbf{rdist}_{G'}(u, z_{i+1}) + \mathbf{rdist}_{G'}(z_{i+1}, z_q) \leqslant \\ & \mathbf{rdist}_{G}(v, u) + \mathbf{rdist}_{G'}(u, z_{i+1}) + \mathbf{rdist}_{G}(z_{i+1}, z_q) = \\ & t - 1 + 2 + q - (i+1) \leqslant t + q - i.\end{aligned}$$

By the minimality of the radial distance between $v$ and $f$, we have that $t \leqslant i$, therefore $\mathbf{rdist}_{G'}(v, z_q) \leqslant q \leqslant r$, a contradiction. □

Given a number $k \in \mathbb{N}$, we denote by $\mathcal{A}^{\langle k \rangle}$ the set of all sequences of numbers in $\mathbb{N}$ with length $k$. We denote $\mathbf{0}_k$, the sequence containing $k$ zero's. Given $\alpha^i = (a^i_1, \ldots, a^i_k)$, $i = 1, 2$, we say that $\alpha^1 \prec \alpha^2$ if, there is some integer $j \in 1, \ldots, k$, such that $a^1_h = a^2_h$ for all $h \leqslant j$ and $a^1_j < a^2_j$. For example $(1, 1, 2, 4, 15, 3, 82, 2) \prec (1, 1, 3, 1, 6, 29, 1, 3)$. A sequence $A = (\alpha^i \mid i \in \mathbb{N})$ of sequences in $\mathcal{A}^{\langle k \rangle}$ is *properly decreasing* if for any two consecutive elements $\alpha^j, \alpha^{j+1}$ of $A$ it holds that $\alpha^j \prec \alpha^{j+1}$. We will use the following known observation.

**Observation 4.** *For every $k \in \mathbb{N}$, every properly decreasing sequence of sequences in $\mathcal{A}^{\langle k \rangle}$ is finite.*

Given a graph $G$ and a subset $S \subseteq V(G)$, an *$S$-path* is any $(x, y)$-path of $G$ where $x, y \in S$. The proof of the following lemma is based on Observations 1 and 4.

**Lemma 4.** *Let $G$ be a triangulated plane graph and let $S$ be an $r$-radial dominating set of $G$. Then $G$ is the topological minor of a graph $H$ that is $2r$-radially extremal.*

*Proof.* Given a triangulated plane graph $H$ and $S \subseteq V(G)$, we consider the sequence

$$\mathbf{q}(H) = (a_1, \ldots, a_{r-1})$$

where, for $j = 1, \ldots, r-1$, $a_j$ is the number of $S$-paths of length $j$ in $H$. Notice that if $\mathbf{q}(G) = \mathbf{0}_{r-1}$, then all $S$-paths have distance at least $r$. As $G$ is triangulated, Observation 1 implies that $S$ is $2r$-radially scattered in $G$.

If $\mathbf{0}_{r-1} \prec \mathbf{q}(G)$, we show how to transform $G$ to a new graph $G'$ satisfying the following properties:

**(i)** $G'$ is triangulated,

**(ii)** $G \leqslant_{\mathrm{t}} G'$,

**(iii)** $G'$ is $r$-radially dominated by $S$, and

**(iv)** $\mathbf{q}(G') \prec \mathbf{q}(G)$.

From Observation 4, this transformation cannot be applied forever. Therefore, it will end up with a graph $G_{\text{final}}$ where $\mathbf{q}(G_{\text{final}}) = \mathbf{0}_{r-1}$. Then the lemma follows as $S$ is an $r$-radially scattered set of $G_{\text{final}}$ and $G_{\text{final}} \leqslant_{\text{t}} G$. Below, we describe this transformation.

Let $P = (v_0, \dots, v_t)$ be a minimum length $S$-path in $G$. Clearly, $\mathbf{0}_{r-1} \prec \mathbf{q}(G)$ implies that $t \leqslant r-1$. Also, we set $l = \lfloor t/2 \rfloor$ and denote by $f_1, f_2$ the two triangular faces of $H$ that are incident to the edge $e = \{v_l, v_{l+1}\}$. We denote by $y_1$ and $y_2$ the two vertices that are incident to $f_1$ or $f_2$ but not to $e$ and assume that $y_1$ (resp. $y_2$) is incident to $f_1$ (resp. $f_2$). We transform $G$ as follows: first subdivide in $G$ the edge $\{v_l, v_{l+1}\}$, call $v_{\text{new}}$ the subdivision vertex, and then add the edges $\{y_1, v_{\text{new}}\}$ and $\{y_2, v_{\text{new}}\}$. We denote by $G'$ the resulting graph and we prove that it satisfies Properties **(i)**–**(iv)**.

Properties **(i)** and **(ii)** hold directly by the construction of $G'$. For property **(iii)**, we assume, towards a contradiction, that some element $a \in A(G')$ is not $r$-radially dominated by any vertex in $S$. Notice that $a \neq v_{\text{new}}$, as $\mathbf{dist}_{G'}(v_{\text{new}}, v_t) \leqslant 1 + \mathbf{dist}_{G'}(v_{l+1}, v_t) \leqslant 1 + \mathbf{dist}_G(v_{l+1}, v_t) = 1 + t - \lfloor t/2 \rfloor - 1 \leqslant \lceil t/2 \rceil$, thus, from Observation 1, $\mathbf{rdist}_{G'}(v_{\text{new}}, v_t) \leqslant 2 \cdot \lceil t/2 \rceil \leqslant r$. Notice also that

$$\mathbf{rdist}_{G'}(v_0, v_l) \leqslant \mathbf{rdist}_G(v_0, v_l) \leqslant 2 \cdot \lfloor t/2 \rfloor \leqslant r-1$$

and

$$\mathbf{rdist}_{G'}(v_t, v_{l+1}) \leqslant \mathbf{rdist}_G(v_t, v_{l+1}) \leqslant 2 \cdot (t - \lfloor t/2 \rfloor - 1) \leqslant 2 \cdot \lfloor t/2 \rfloor \leqslant r-1,$$

therefore each of the new faces of $G'$ that are either incident to $v_l$ or incident to $v_{l+1}$ is $r$-radially dominated by either $v_0$ or $v_t$ respectively. This means that $a$ is also an element of $A(G)$. Let $P' = (a_0, \dots, a_q)$ be a path in $G$ such that $a_0 \in S$ and either $a_q = a$ (in case $a$ is a vertex) or $a_q$ is incident to $a$ (in case $a$ is a face); in the first case $q \leqslant \lfloor r/2 \rfloor$ and in the second $q \leqslant \lfloor (r-1)/2 \rfloor$. As $P'$ is not a path of $G'$, some, say $\{a_j, a_{j+1}\}$, of its edges should be the subdivided edge $\{v_l, v_{l+1}\}$. We will end up with a contradiction by proving the existence in $G'$ of a $(x, a_q)$-path of length $\leqslant q$ for some $x \in \{v_0, v_t\} \subseteq S$. We examine two cases:

*Case I.* $a_j = v_l$ and $a_{j+1} = v_{l+1}$. By the minimality of the choice of $P$, we deduce that $l \leqslant j$ (otherwise $\mathbf{dist}_G(a_0, v_t) \leqslant \mathbf{dist}_G(a_0, v_l) + 1 + \mathbf{dist}_G(v_{l+1}, v_t) < l + 1 + (t - l - 1) \leqslant t$) and this means that

$$\mathbf{dist}_{G'}(v_{l+1}, a_q) \leqslant \mathbf{dist}_G(v_{l+1}, a_q) \leqslant q - j - 1 \leqslant q - l - 1 = q - \lfloor t/2 \rfloor - 1$$

Observe that $\mathbf{dist}_{G'}(v_{l+1}, v_t) \leqslant \mathbf{dist}_G(v_{l+1}, v_t) \leqslant t - l - 1 = \lceil t/2 \rceil - 1$. Therefore, $\mathbf{dist}_{G'}(v_t, a_q) \leqslant q - \lfloor t/2 \rfloor - 1 + \lceil t/2 \rceil - 1 \leqslant q$, a contradiction.

*Case II.* $a_j = v_{l+1}$ and $a_{j+1} = v_l$. Now, by the minimality of $P$ we have that $t - l - 1 \leqslant j$ (otherwise $\mathbf{dist}_G(a_0, v_0) \leqslant \mathbf{dist}_G(a_0, a_j) + 1 + \mathbf{dist}_G(v_l, v_0) < (t - l - 1) + 1 + l \leqslant t$) and thus

$$\mathbf{dist}_{G'}(v_l, a_q) \leqslant \mathbf{dist}_G(v_l, a_q) \leqslant q - j - 1 \leqslant q - t + l = q - \lceil t/2 \rceil.$$

As $\mathbf{dist}_{G'}(v_0, v_l) \leqslant \mathbf{dist}_G(v_0, v_l) = l = \lfloor t/2 \rfloor$, we conclude that $\mathbf{dist}_{G'}(v_0, a_q) \leqslant \lfloor t/2 \rfloor + q - \lceil t/2 \rceil \leqslant q$, a contradiction and property **(iii)** holds for $G'$.

For Property **(iv)**, we need to prove that $\mathbf{q}(G') \prec \mathbf{q}(G)$. As all $S$-paths in $G'$ that avoid $v_{\text{new}}$ also exist in $G$, we have to prove that there is at least one $S$-path of length $t$ in $G$ that is not in $G'$ and that no new paths of length $t$ appear in $G'$. Indeed $P$ is a path of length $t$ that does not exist in $G'$. What remains is to prove that no new paths of length $t$ appear in $G'$. Suppose to the contrary that $P' = (x_0, \ldots, x_t)$ is such a path. Clearly, $P'$ should meet the vertex $v_{\text{new}}$ and assume that $v_{\text{new}} = x_i$. The cases where the set $\{x_{i-1}, x_{i+1}\}$ is one of $\{v_l, y_1\}, \{v_l, y_2\}, \{v_{l+1}, y_1\}, \{v_{l+1}, y_2\}$ are excluded as, in such a case, the existence of the path $(x_0, x_{i-1}, x_{i+1}, \ldots, x_t)$ in $G$ contradicts the minimality of the choice of $P$. Therefore, $\{x_{i-1}, x_{i+1}\} = \{y_1, y_2\}$ and, w.l.o.g., we assume that $x_{i-1} = y_1$ and $x_{i+1} = y_2$. Then either $\mathbf{dist}_G(x_0, x_{i-1}) \leqslant \lfloor (t-2)/2 \rfloor$ or $\mathbf{dist}_G(x_{i+1}, x_t) \leqslant \lfloor (t-2)/2 \rfloor$. W.l.o.g., we assume that $\mathbf{dist}(x_0, x_{i-1}) \leqslant \lfloor (t-2)/2 \rfloor$. Then

$$\begin{aligned}\mathbf{dist}_G(x_0, v_t) \quad \leqslant \quad & \mathbf{dist}_G(x_0, y_1) + 1 + \mathbf{dist}_G(v_{l+1}, v_t) \leqslant \\ & \lfloor (t-2)/2 \rfloor + 1 + (t - l - 1) = \\ & \lfloor (t-2)/2 \rfloor + 1 + t - \lfloor t/2 \rfloor - 1 = \\ & \lfloor t/2 \rfloor - 1 + 1 + \lceil t/2 \rceil - 1 = t - 1,\end{aligned}$$

a contradiction to the minimality of $P$. □

We are now in position to proof Lemma 2.

*Proof of Lemma 2.* Applying first Lemma 3, we obtain a planar triangulation $H$ of $G$ where the set $S$ is a $r$-radial dominating. Then, applying Lemma 4, we obtain a triangulated graph $H$ that is a topological minor of $H'$ and such that $S$ is $2r$-radially scattered in $H'$. The lemma follows as $G$ is a topological minor of $H'$ and $H'$ is triangulated and thus 3-connected. □

# 4 A bound for branchwidth

We are now ready to prove our main combinatorial result.

**Theorem 3.** *Let $r$ be a positive integer and let $G$ be a plane graph. Then* $\mathbf{bw}(G) \leqslant r \cdot \sqrt{4.5 \cdot \mathbf{rds}(G, r)}$.

*Proof.* We use induction on $r$. If $r = 1$ then $|V(G)| = \mathbf{rds}(G, 1)$ and the result follows from Proposition 4. Assume now that the lemma holds for values smaller than $r$ and we will prove that it also holds for $r$, where $r \geqslant 2$. Let $G$ be a plane graph where $\mathbf{rds}(G, r) \leqslant k$. Using Lemma 1, we choose $H \in \mathcal{Q}(G)$ such that $\mathbf{bw}(H) = \mathbf{bw}(G)$ (we may assume that $G$ contains a cycle, otherwise the result follows trivially). By Observations 2 and 3, $\mathbf{rds}(H, r) \leqslant \mathbf{rds}(G, r) \leqslant k$. Let $S$ be an $r$-radial dominating set of $H$ where $|S| \leqslant k$. From Lemma 2, $H$ is the topological minor of a 3-connected plane graph $H_1$ where $S$ is $r$-radially extremal.

Let $H_2$ be the graph obtained if we remove from $H_1$ all vertices of $S$. By the 3-connectivity of $H_1$, it follows that, for any $v \in S$, the graph $H_1[N_{H_1}(v)]$ is a cycle and each such cycle is the boundary of some face of $H_2$. We denote by $F$ the set of these faces and observe that $F^*$ is a $(r-1)$-radial dominating set of $H_2^*$ (we denote by $F^*$ the vertices of $H_2^*$ that are duals of the faces of $F$ in $H_2$). moreover, the fact that $S$ is a $2r$-scattered dominating set in $H_1$, implies that $F^*$ is a $2(r-1)$-scattered dominating set in $H_2^*$.

From the induction hypothesis and the fact that $|F^*| = |S|$, we obtain that $\mathbf{bw}(H_2^*) \leqslant (r-1) \cdot \sqrt{4.5 \cdot k}$. This fact, along with Proposition 3, implies that $\mathbf{bw}(H_2) \leqslant (r-1) \cdot \sqrt{4.5 \cdot k}$.

In graph $H_2$, for any face $f_i \in F$, let $(x_0^i, \ldots, x_{m_i-1}^i)$ be the cyclic order of the vertices in its boundary cycle (as $H_1$ is 3-connected we have that $m_i \geqslant 3$). We also denote by $x^i$ the vertex in $H_1$ that was removed in order $f_i$ to appear in $H_2$. Let $(T, \tau)$ be a branch decomposition of $H_2$ of width $\leqslant (r-1) \cdot \sqrt{4.5 \cdot k}$. By Proposition 5, we may assume that $(T, \tau)$ is a sphere-cut decomposition.

We use $(T, \tau)$ in order to construct a branch decomposition of $H_1$, by adding new leaves in $T$ and mapping them to the edges of $E(H_1) \setminus E(H_2) = \bigcup_{i=1,\ldots,|F|} \{\{x^i, x_h^i\} \mid h = 0, \ldots, m_i - 1\}$ in the following way: For every $i = 1, \ldots, |F|$ and every $h = 0, \ldots, m_i - 1$, we set $t_h^i = \tau^{-1}(\{x_h^i, x_{h+1 \bmod m_i}^i\})$ and let $e_h^i = \{y_h^i, t_h^i\}$ be the unique edge of $T$ that is incident to $t_h^i$. We subdivide $e_h^i$ and we call the subdivision vertex $s_h^i$. We also add a new vertex $z_h^i$ and make it adjacent to $s_h^i$. Finally, we extend the mapping of $\tau$ by mapping the vertex $z_h^i$ to the edge $\{x^i, x_h^i\}$ and we use the notation $(T', \tau')$ for the resulting branch decomposition of $H_1$.

*Claim.* The width of $(T', \tau')$ is at most $r \cdot \sqrt{4.5 \cdot k}$.

*Proof.* We use the functions $\omega$ and $\omega'$ to denote the middle sets of $(T, \tau)$ and $(T', \tau')$ respectively. Let $e$ be an edge of $T'$. If $e$ is not an edge of $T$ (i.e., is an edge of the form $\{z_h^i, s_h^i\}$ or $\{t_h^i, s_h^i\}$ or $\{y_h^i, s_h^i\}$), then $|\omega'(e)| \leqslant 3$, therefore we may fix our attention to the case where $e$ is also an edge of $T$. Let $N_e$ be the noose of $H_2$ meeting the vertices of $\omega(e)$. We distinguish the following cases.

*Case 1.* $N_e$ does not meet any face of $F$, then clearly $\omega'(e) = \omega(e)$. Thus $|\omega'(e)| \leqslant (r-1) \cdot \sqrt{4.5 \cdot k}$.

*Case 2.* If $N_e$ meets only one, say $f_i$, of the faces of $F$, then the vertices in $\omega'(e)$ are the vertices of a noose $N_e'$ of $H_1$ meeting all vertices of $\omega(e)$ plus $x^i$. Therefore, $\omega'(e) = \omega(e) \cup \{x^i\}$ and thus $|\omega'(e)| \leqslant (r-1) \cdot \sqrt{4.5 \cdot k} + 1$.

*Case 3.* $N_e$ meets $p \geqslant 2$ faces of $F$. We denote by $\{f_0', \ldots, f_{p-1}'\}$ the set of these faces and let $J_0, \ldots, J_{p-1}$ be the normal arcs corresponding to the connected components of $N_i - \bigcup_{i=0,\ldots,p-1} f_i'$. Let also $I_0, \ldots, I_{p-1}$ be the normal arcs corresponding to the closures of the connected components of $N_i \cap (\bigcup_{i=0,\ldots,p-1} f_i')$, assuming that $I_i \subseteq \overline{f_i'}$, for $i = 0, \ldots, p-1$. Recall that $F^*$ is a $(r-1)$-scattered dominating set of $H_2^*$. This implies that each $J_i$ meets at least $r-1$ vertices of $H_2$ and therefore $p \cdot (r-1) \leqslant \omega(e) \leqslant (r-1) \cdot \sqrt{4.5 \cdot k}$. We conclude that $p \leqslant \sqrt{4.5 \cdot k}$. Observe now that the vertices of $\omega'(e)$ are the vertices of a noose $N'_e$ of $H_1$ where $N'_e = (\bigcup_{i=0,\dots,p-1} J_i) \cup (\bigcup_{i=0,\dots,p-1} I'_i)$ and such that, for each $i = 0, \dots, p-1$, $I'_i$ is a replacement of $I_i$ so that it is still a subset of $\overline{f}'_i$, has the same endpoints as $I_i$, and also meets the unique vertex in $S \cap f'_i$. As $N'_e$ meets in $H_1$ all vertices of $N_e \cap V(H_2)$ plus $p$ more, we obtain that

$$|\omega'(e)| \leqslant (r-1) \cdot \sqrt{4.5 \cdot k} + p \leqslant r \cdot \sqrt{4.5 \cdot k}.$$

According to the above case analysis, $|\omega'(e)| \leqslant \max\{3, \sqrt{4.5 \cdot k} + 1, r \cdot \sqrt{4.5 \cdot k}\} = r \cdot \sqrt{4.5 \cdot k}$ and the claim follows.

We just proved that $\mathbf{bw}(H_1) \leqslant r \cdot \sqrt{4.5 \cdot k}$. As $H$ is a topological minor of $H_1$, from Proposition 2, we also have that $\mathbf{bw}(H) \leqslant r \cdot \sqrt{4.5 \cdot k}$. The lemma follows as $\mathbf{bw}(G) = \mathbf{bw}(H)$. $\square$

We are now ready to prove Theorem 2.

*Proof of Theorem 2.* By the definition of coverability, $G$ has an embedding such that either $G$ or $G^*$ contains an $r$-radial dominating set of size at most $q \cdot k$. Without loss of generality, assume that this is the case for $G$ (here we use Proposition 3). Then $\mathbf{rds}(G, r) \leqslant q \cdot k$ and, from Theorem 3, $\mathbf{bw}(G) \leqslant r \cdot \sqrt{4.5 \cdot q \cdot \mathbf{p}(G)}$. $\square$

# 5 Conclusions and open problems

The concept of coverability for parameterized problems was introduced in [7]. In this paper, we show that it can also be used to improve the running time analysis of a wide family of sub-exponential parameterized algorithms. Essentially, we show that such an analysis can be done without the grid-minor exclusion theorem. Instead, our better combinatorial bounds emerge from the result in [36] that, in turn, is based on the "planar separators theorem" of Alon, Seymour, and Thomas in [3]. This implies that any improvement of the constant $\sqrt{4.5}$ in [3] would improve all the running times emerged from the framework of this paper.

It follows that there are bidimensional parameterized problems that are not coverable and vice versa. For instance, Independent Dominating Set is coverable but not bidimensional while Longest Path is bidimensional but not coverable. Is it possible to extend both frameworks to a more powerful theory, at least in the context of sub-exponential parameterized algorithms?

Recall that coverability appeared for the first time in the meta-algorithmic context of kernelization [7]. We believe that the combinatorial part of [7], based on the concept of *protrusion decomposition*, can be optimized using some ideas of the proofs of this paper. An important step in this direction has already been made by Michalis Samaris in his Master Thesis for $\mu\Pi\lambda\forall$, in [55]. The main result of [55], is a refinement of Theorem 3 where the resulting branch decompositions have additional structural properties, resembling those of protrusion decompositions in [7]. We believe that this approach may imply a significant improvement of the constants in the kernels derived from [7] for the case of planar graphs.

Another open issue is whether the framework we define in this paper can be extended to graphs embeddable to 2-dimensional surfaces (orientable or non-orientable) of genus $g$. For such graphs, it is straightforward to extend the definition of $(r, q)$-coverability. As most (but not all) of the concepts that we defined in this paper can also be extended, we conjecture that this may lead to an generalization of Theorem 2, classifying $(r, q)$-coverable problems on surface embeddable graphs in $c^{O(r\sqrt{q \cdot g \cdot k})}$-$\mathsf{FPT}$. However this might be a hard task. Also such research should additionally optimize the constant hidden in the $O$-notation, in order to compete with the parameterized dependencies guaranteed by Bidimensionality Theory in [20].

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