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Linear Kernels for (Connected) Dominating Set on H-minor-free graphs

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Abstract

In the DOMINATING SET problem we are given an n-vertex graph G with a positive integer k and we ask whether there exists a vertex subset D of size at most k such that every vertex of G is either in D or is adjacent to some vertex of D. In the connected variant, CONNECTED DOMINATING SET, we also demand the subgraph induced by D to be connected. Both variants are basic graph problems, known to be NP-complete, and many algorithmic approaches have been tried on them.

In this paper we study both problems on graphs excluding a fixed graph H as a minor from the kernelization point of view. Our main results are polynomial time algorithms that, for a given H-minor free graph G and positive integer k, output an H-minor free graph G' on $\mathcal{O}(k)$ vertices such that G has a (connected) dominating set of size k if and only if G' has. The polynomial time algorithm that obtains such equivalent instance is known as kernelization algorithm and its output is called a problem kernel. If the size of the output can be bounded by a polynomial (linear) function of k, then it is called polynomial (linear) kernel. Prior to our work, the only polynomial kernel for DOMINATING SET on graphs excluding a fixed graph H as a minor was due to Alon and Gutner [ECCC 2008, IWPEC 2009] and to Philip, Raman, and Sikdar [ESA 2009] but the size of their kernel is $k^{c(H)}$, where $c(H)$ is a constant depending on the size of H. Alon and Gutner asked explicitly, whether one can obtain a linear kernel for DOMINATING SET on H-minor free graphs. We answer this question in affirmative. For CONNECTED DOMINATING SET no polynomial kernel on H-minor free graphs was known prior to our work.

As a byproduct of our results we also obtain the first subexponentail time algorithm for CONNECTED DOMINATING SET running in time $2^{\mathcal{O}(\sqrt{k}\log k)}n^{\mathcal{O}(1)}$, as well as a simplification of a $2^{\mathcal{O}(\sqrt{k})}n^{\mathcal{O}(1)}$ algorithm for DOMINATING SET on H minor free graphs due to Demaine et al. [SODA 2003, J. ACM 2005]. All our results are based on a novel combination of the irrelevant vertex technique and divide-and-conquer.

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1 Introduction

In the DOMINATING SET (DS) problem, we are given a graph G and a non-negative integer k, and the question is whether G contains a set of k vertices whose closed neighborhood contains all the vertices of G. In the connected variant, CONNECTED DOMINATING SET (CDS), we also demand the subgraph induced by the dominating set to be connected. DS, together with its numerous variants, is one of the most classic and well-studied problems in algorithms and combinatorics [31]. A considerable part of the algorithmic study on this NP-complete problem has been focused on the design of parameterized algorithms. Formally, a *parameterization* of a problem is assigning an integer k to each input instance and a parameterized problem is *fixed-parameter tractable* (FPT) if there is an algorithm that solves the problem in time $f(k) \cdot |I|^{O(1)}$, where $|I|$ is the size of the input and f is an arbitrary computable function depending on the parameter k only. In general, DS is W[2]-complete and therefore it cannot be solved by a parameterized algorithm, unless an unexpected collapse occurs in the Parameterized Complexity (see [20, 23, 36]). However, there are interesting graph classes where FPT-algorithms exist for the DOMINATING SET problem. The project of widening the horizon where such algorithms exist spanned a multitude of ideas that made DS the testbed for some of the most cutting-edge techniques of parameterized algorithm design. For example, the initial study of parameterized subexponential algorithms for DS on planar graphs [1, 12, 27] resulted in the creation of bidimensionality theory characterizing a broad range of graph problems that admit efficient approximate schemes or fixed-parameter solutions on a broad range of graphs [13, 15, 19].

Another emerging technique in parameterized complexity is *kernelization*. A parameterized problem is said to admit a *polynomial kernel* if there is a polynomial time algorithm (the degree of polynomial is independent of the parameter k), called a *kernelization* algorithm, that reduces the input instance down to an instance with size bounded by a polynomial $p(k)$ in k, while preserving the answer. This reduced instance is called a $p(k)$ *kernel* for the problem. If $p(k) = O(k)$, then we call it a *linear kernel* (for a more formal definition, see Section 2). One of the first results on linear kernels is the celebrated work of Alber, Fellows, and Niedermeier on DS on planar graphs [2]. This work augmented significantly the interest in proving polynomial (or preferably linear) kernels for other parameterized problems. The result from [2], see also [7], has been extended to much more general graph classes. An important step in this direction was done by Alon and Gutner [3, 30] who obtained a kernel of size $O(k^h)$ for DS on H-minor free graphs, where the constant h depends on the excluded graph H. Later, Philip, Raman, and Sikdar [37] obtained a kernel of size $O(k^h)$ on $K_{i,j}$-free and d-degenerated graphs, where h depends on i, j and d respectively. Sizes of kernels in [3, 30, 37] are bounded by polynomials in k whose degrees depend on the size of the excluded minor H. Therefore, the challenge is to ask for polynomial kernels of size $f(h) \cdot k^{O(1)}$, where the function f depends *exclusively* on the graph class. In this direction, there are already results for more restricted graph classes. According to the meta-algorithmic results on kernels introduced in [6], DS has a kernel of size $f(g) \cdot k$ on graphs of genus g. Recently, an alternative meta-algorithmic framework, based on bidimensionality theory [13], was introduced in [26], implying the existence of a kernel of size $f(H) \cdot k$ for DS on graphs excluding an apex graph H as a minor. While apex-minor free graphs form much more general class of graphs than graphs of bounded genus, H-minor free graphs form much larger class than apex-minor free graphs. For example, the class of graphs excluding $H = K_7$, the complete graph on 7 vertices, as a minor, contains all apex graphs. Alon and Gutner posed as an open problem in [3, 30] whether one can obtain a linear kernel for DS on H-minor free graphs.

In this work we obtain a linear kernel for DS on graphs excluding some fixed graph H as a minor, which answer affirmatively the question of Alon and Gutner. Moreover, a non-trivial modification of the ideas for DS kernelization can be used to obtain a linear kernel for CDS, which is usually much more difficult problem to handle due to connectivity constrains. The extension of the results in [26] to the more general family of H-minor free graphs cannot be straightforward. Similar difficulties in transition of algorithmic techniques from apex-minor free to H-minor free graphs were observed in approximation [17] and parameterized algorithms [13, 21]. Intuitively, the explanation is that excluding an apex graph makes it possible to bound the tree-decomposability of the input graph by a *sublinear* function of the parameter which is not the case for more general classes of H-minor free graphs.

The main idea behind our algorithm is to identify and remove "irrelevant" vertices without changing the solution such that in the reduced graph one can select $\mathcal{O}(k)$ vertices whose removal leaves protrusions, that is, subgraphs of constant treewidth separated from the remaining vertices by a constant number of vertices. As far as we able to obtain such a graph, we can use the techniques from [26] to construct the linear kernel. But identifying "irrelevant" vertices is a non-trivial task and is the main technical contribution of this work. For this, we use the decomposition theorem of Robertson and Seymour [39] and its algorithmic variants [9, 16] to decompose a graph into a set of torsos connected via clique-sums. For each torso and its set of apex vertices we define the notion of irrelevant vertex whose removal does not change the problem. By performing such removal we are able to reduce the size of each torso but since the number of torsos can be $\Omega(n)$, this does not bring us directly to the desired constant-treewidth vertex removal subgraph. To overcome this obstacle, we have to implement the irrelevant vertex rule in a divide and conquer manner, and here the bidimensionality of DS comes into play.

Besides linear kernels for DS and CDS, an immediate byproduct of our "irrelevant vertex technique" is a radical simplification of the subexponential parameterized algorithm of Demaine et al. [13] for DS, and the first parameterized subexponential algorithm for CDS on H-minor free graphs.

We stress that the "irrelevant vertex technique" has already appeared in several paradigms of parameterized algorithm design. Its most notorious application was given by Robertson and Seymour in the Graph Minors series where they gave FPT-algorithms for the disjoint paths problem and minor checking problem [38], see also [9, 10, 28, 32, 33, 34] for further applications of this technique. However, to our knowledge, this is the first time this idea is used in the context of kernelization and the way we use it is completely different from all previous paradigms.

2 Definitions and Notations

In this section we give various definitions which we make use of in the paper. We refer to Diestel's book [18] for standard definitions from Graph Theory. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. For subset $V' \subseteq V(G)$, the subgraph $G' = G[V']$ of G is called a *subgraph induced by* V' if $E(G') = \{uv \in E(G) \mid u, v \in V'\}$. By $N_G(u)$ we denote (open) neighborhood of u in graph G that is the set of all vertices adjacent to u and by $N[u] = N(u) \cup \{u\}$. Similarly, for a subset $D \subseteq V$, we define $N_G[D] = \cup_{v \in D} N_G[v]$ and $N_G(D) = N_G[D] \setminus D$. We omit the subscripts when it is clear from the context.

We denote by K_h the complete graph on h vertices. For integer $r \geq 1$ and vertex subsets $P, Q \subseteq V(G)$, we say that a subset Q is *r-dominated* by P, if for every $v \in Q$ there is $u \in P$ such that the distance between u and v is at most r. For $r = 1$, we simply say that Q is dominated by P. We denote by $N^r_G(P)$ the set of vertices r-dominated by P.

Given an edge $e = xy$ of a graph G, the graph G/e is obtained from G by contracting the edge e, that is, the endpoints x and y are replaced by a new vertex v_{xy} which is adjacent to the old neighbors of x and y (except from x and y). A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. We denote it by $H \leq_c G$. A graph H is a *minor* of a graph G if H is the contraction of some subgraph of G and we denote it by $H \leq_m G$. We say that a graph G is *H-minor-free* when it does not contain H as a minor. We also say that a graph class $\mathcal{G}_H$ is *H-minor-free* (or, excludes H as a minor) when all its members are H-minor-free. An *apex graph* is a graph obtained from a planar graph G by adding a vertex and making it adjacent to some of the vertices of G. A graph class $\mathcal{G}_H$ is *apex-minor-free* if $\mathcal{G}_H$ excludes a fixed apex graph H as a minor.

We denote by $\mathbf{tw}(G)$ the treewidth of graph G. (See Appendix for the definition of treewidth.)

Kernels and Protrusions. A parameterized problem Π is a subset of $\Gamma^* \times \mathbb{N}$ for some finite alphabet Γ. An instance of a parameterized problem consists of (x, k), where k is called the parameter. We will assume that k is *given in unary* and hence $k \leq |x|^{\mathcal{O}(1)}$. A central notion in parameterized complexity is *fixed parameter tractability (FPT)* which means, for a given instance (x, k), solvability in time $f(k) \cdot p(|x|)$, where f is an arbitrary function of k and p is a polynomial in the input size [20]. The notion of *kernelization* is formally defined as follows.

A kernelization algorithm, or in short, a kernel for a parameterized problem $\Pi \subseteq \Gamma^* \times \mathbb{N}$ is an algorithm that given $(x, k) \in \Gamma^* \times \mathbb{N}$ outputs in time polynomial in $|x| + k$ a pair $(x', k') \in \Gamma^* \times \mathbb{N}$ such that (a) $(x, k) \in \Pi$ if and only if $(x', k') \in \Pi$ and (b) $|x'|, k' \leq g(k)$, where g is some computable function. The function g is referred to as the size of the kernel. If $g(k) = k^{\mathcal{O}(1)}$ or $g(k) = \mathcal{O}(k)$ then we say that Π admits a polynomial kernel and linear kernel respectively.

Given a graph G, we say that a set $X \subseteq V(G)$ is an *r-protrusion* of G if $\mathbf{tw}(G[X]) \leq r$ and the number of vertices in X with a neighbor in $V(G) \setminus X$ is at most r.

Graph Structure Theorem. The *torso* of a tree-decomposition $(\mathcal{X}, T)$ of a graph G is a graph L_t, $t \in V(T)$, obtained from $G[X_t]$ by adding edges uv such that u and v is in $X_t \cap X_{t'}$, where t and t' are nodes adjacent in T. Observe that it is possible that u and v may not be adjacent in G. To state the next theorem we need also the notion of a graph that can be h-nearly embedded in a surface. Due to space constraints, this definition is moved to Appendix. The following theorem is one of the most fundamental results in Graph Minors Theory of Robertson and Seymour, see also Section 12.4 in Diestel's book [18].

Theorem 1 ([39])**.** *For every graph H there exists an integer h, depending only on the size of H, such that every graph excluding H as a minor has a tree-decomposition whose torsos can be h-nearly embedded in a surface Σ in which H cannot be embedded.*

The main consequence of Theorem 1 we need for our purposes is that for every H there exist constants h and h' such that for every torso L of the decomposition from Theorem 1, there exists a set of vertices $A \subseteq V(L)$ of size at most h, called apices, such that the graph obtained from L after deleting the apices does not contain some apex graph H' of size h' as a minor. See, e.g. [29, Theorem 13].

Throughout the paper, given a graph G and vertex subsets Z and S, whenever we say that a subset Z *dominates all but (everything but)* S then we mean that $V(G) \setminus S \subseteq N[Z]$. Observe that a vertex of S can also be dominated by the set Z.

3 Kernel for Dominating Set

In this section we give a linear kernel for the Dominating Set problem. The kernelization algorithm has two phases. In the first phase we remove "irrelevant vertices" and obtain an equivalent graph with treewidth bounded by $\mathcal{O}(\sqrt{k})$. Then we apply the first phase in a recursive fashion to obtain a set D of size $\mathcal{O}(k)$ vertices such that its deletion leaves the graph of constant treewidth. We call such set D as *treewidth deletion set*. Then applying "protrusion rule" [6] together with the fact that Dominating Set has finite integer index, we get the desired linear kernel for Dominating Set.

Obtaining an equivalent graph of treewidth at most $\mathcal{O}(\sqrt{k})$. Let G be a graph excluding some fixed graph H as a minor. In this section we assume that we are given a tree-decomposition $(\mathcal{X}, T)$ of G as in Theorem 1, such that the torsos of the tree-decomposition can be h-nearly embedded in a surface Σ in which H cannot be embedded. Such a decomposition can be constructed in polynomial time [9, 16]. Let L_t be one such torso corresponding to some vertex $t \in V(T)$. Next we show how to obtain an equivalent graph G', in fact an induced subgraph of G obtained by deleting vertices from L_t, such that the treewidth of the subgraph corresponding to L_t in G' has treewidth $\mathcal{O}(\sqrt{k})$. We repeat this procedure for every torso corresponding to vertices in $V(T)$. Finally we obtain an equivalent graph G' such that it has a tree-decomposition $(\mathcal{X}', T)$ such that all its torsos are of treewidth $\mathcal{O}(\sqrt{k})$. Since the treewidth of a graph is at most the maximum treewidth of its torsos, see e.g. [13], this implies that the treewidth of G' is $\mathcal{O}(\sqrt{k})$.

Reducing the treewidth of a torso. We need to reduce the treewidth of a torso not only in the beginning of the procedure but also when we apply a recursive procedure to obtain $\mathcal{O}(k)$ sized treewidth deletion set. Thus we outline a generic procedure in exactly the way we will use it on our recursive algorithm later. Let G be a graph, L_t be one of its torsos, S be a dominating set of G, and $A = \{a_1, \ldots, a_h\}$ be the set of apices of L_t. Our objective is to apply a reduction rule that essentially "preserves" all dominating sets of size at most $|S|$ in G. Let $\mathcal{F}$ be the set of all functions from $A \rightarrow \{0, 1\}$. The set $\mathcal{F}$ essentially contains all possible guesses on the set of apices with respect to how they are dominated in each dominating set of size at most $|S|$ of G. With every dominating set D in G of size at most $|S|$ we associate a function $f \in \mathcal{F}$ such that for every $a_i \in A$, $f(a_i)$ is the distance from a_i to D. In other words, if $f(a_i) = 0$ then a_i is in D and if $f(a_i) = 1$ then a_i is dominated by some vertex in D.

To describe the reduction rule we need several definitions. For every subset $A' \subseteq A$, we select a vertex v of G as *representative* and denote by $v(A')$ if for all $a_j \in A'$ we have that $\{v, a_j\} \in E(G)$. Let R be the vertex subset formed by selecting for each subset of A a representative (if there is any – distinct subsets of S may have the same representative). We address R as a set of representative vertices for subsets of A. The size of R is at most $2^{|A|}$. For a given $f \in \mathcal{F}$, we also define the following:

- The set $W(f) = \Big(\bigcup_{\substack{1 \le i \le h \\ f(a_i)=0}} N_G(a_i) \Big) \setminus (A \cup S)$ of *dominated* vertices.
- We also need a "sanity" check to report if there is a feasible solution that can be obtained by "extending" f and whose size is not far from $|S|$. To perform this check, we use the factor 2-approximation algorithm for H-minor free graphs given in [14, 25], to compute a subset of $V(G) \setminus A$ of size at most $2(|S| + 2)$ which dominates $V(G) \setminus (A \cup W(f) \cup S)$ in graph $G \setminus A$. If the approximation algorithm returns such a set, then we say that f is *feasible*. Let us remark, that f *cannot* be feasible if the size of a minimum dominating set of $V(G) \setminus (A \cup W(f) \cup S)$ in $G \setminus A$ is at least $|S| + 2$. When f is feasible, we denote by $D(f)$ the corresponding vertex set of $G \setminus A$ of size at most $2(|S| + 2)$ dominating $V(G) \setminus (A \cup W(f) \cup S)$.
- A vertex $w \in W(f)$ is *irrelevant with respect to* f if $w \notin R$ and $(N_G(w) \setminus A) \subseteq W(f)$.

Let us remark that there always exists a feasible function $f \in \mathcal{F}$. Indeed, function f that assigns 0 to the vertices of $S \cap A$ and 1 to $A \setminus S$ is feasible because $S \setminus A$ is the desired dominating set for $V(G) \setminus (A \cup W(f) \cup S)$ of size at most $|S|$ and hence the approximation algorithm will always return $D(f)$ of size at most $2|S|$.

Now we are ready to state our irrelevant vertex rule.

Irrelevant Vertex Rule: If a vertex w is irrelevant with respect to every feasible $f \in \mathcal{F}$, then delete w from G.

We apply the irrelevant vertex rule as long as possible in G. Let the set of vertices deleted in this process be B. Let G' be the resulting graph, that is, $G' = G \setminus B$. The proof of the correctness of our algorithm is based on the following lemma.

Lemma 1. *Let S be a dominating set in a graph G, and $G' = G \setminus B$, where B are the vertices removed from G by applying Irrelevant Vertex Rule. Then (a) for every set Z in G of size at most $|S|$ that dominates everything but S, there is a set Z' of size at most $|Z|$ in G' such that Z' contains $Z \cap S$ and Z' dominates all the vertices of $N[Z]$ in G'; (b) for every set Z' in G' that dominates all but S there is a set Z in G of size at most $|Z'|$ such that Z contains $Z' \cap S$ and Z dominates $N[Z'] \cup B$.*

Proof. We prove the lemma for the case $B = \{w\}$. The case $|B| > 1$ follows by applying these arguments inductively.

We start with the first statement. Let Z be a subset in G that dominates everything but S. If $|Z| > |S|$, then we put $Z' = S$ and it clearly satisfies all the requirements. Thus we assume that $|Z| \le |S|$. Let $f \in \mathcal{F}$ be such that $a_i \in Z$ if and only if $f(a_i) = 0$. The function f is feasible as the set $Z \setminus A$ is a dominating set for $V(G) \setminus (A \cup W(f) \cup S)$ of size at most $|Z| \le |S|$ and hence the approximation algorithm will always return $D(f)$ of size at most $2|S|$. If $w \notin Z$, then $Z' = Z$ is a dominating set for vertices in $N[Z]$ that are in G'. Hence we assume that $w \in Z$. Let $A' \subseteq A$ be the set of apices $a_j \in A$ such that $a_j \in N_G(w)$. Because w is irrelevant, there is a vertex $v(A') \in R$ such that $v(A') \neq w$ and $N_G(w) \cap A \subseteq N_G(v(A')) \cap A$. Then we claim that $Z' = Z \setminus \{w\} \cup \{v(A')\}$ is a set of size at most $|Z|$ in G' that dominates $N[Z]$. Suppose that there is a vertex $u \in N[Z]$ contained in $V(G')$ and not dominated by Z'. Then u must be in the neighborhood of w. Since w is irrelevant, we have that w and its neighborhood is dominated for every feasible choice of function f. Thus if u is not in A, then since

f is feasible, we have that it is also dominated by some vertex in Z that is in A. Finally, for $u \in A$, because $N_G(w) \cap A \subseteq N_G(v(A')) \cap A$, we have that u is dominated by $v(A')$.

We proceed with the second statement of the lemma. Let Z' be a vertex subset of G' that dominates all but S. We show that Z' also dominates $N[Z'] \cup \{w\}$. Targeting a contradiction, let us assume that w is not dominated by a vertex in Z'. Let $A' = A \cap Z'$ and g be a function that assigns 0 to all vertices in A' and 1 to $A \setminus A'$. Clearly g is a feasible because set $(Z \setminus A) \cup \{w\}$ is of size at most $|S| + 1$ and it dominates $V(G) \setminus (A \cup W(g) \cup S)$. Since g is feasible and w is irrelevant, we have that $w \in W(g)$ Hence there exists a vertex in A' that is a neighbor of w. This concludes the proof. □

The following lemma provides the bounds on the treewidth of the torso without irrelevant vertices.

Lemma 2. *Let $L'_t = L_t \setminus B$. Then $\mathbf{tw}(L'_t) = \mathcal{O}(\sqrt{|S|})$.*

Proof. Let $L^*_t = L'_t \setminus A$. We first show that there exists a 2-dominating set of size $\mathcal{O}(|S|)$ for $L'_t \setminus A$. Towards this consider the following set

$$Q = \bigcup_{\substack{f \in \mathcal{F}, \\ f \text{ is feasible}}} D(f) \cup R \cup S.$$

The size of R is at most $2^{|A|} \leq 2^h$, the size of $\mathcal{F}$ is also at most 2^h, and the size of $D(f)$ is at most $2|S| + 2$. Thus $|Q| \leq 2^h(2|S| + 2) + 2^h + |S|$.

We first show that there exists a set Q' corresponding to Q that forms a 2-dominating set of size $\mathcal{O}(|S|)$ in $L'_t \setminus A$. Let $Q^* = Q \cap V(L^*_t)$ and let t' be a neighbor of t in the tree decomposition T, such that $(X_t \cap X_{t'}) \setminus B \neq \emptyset$ and such that there is no vertex from $(X_t \cap X_{t'}) \setminus B$ in Q^*. For every such t' we do the following: if there is a vertex from Q appearing in one of the bags corresponding to the subtree containing t' in $T \setminus \{t\}$, then we add an arbitrary vertex from $(X_t \cap X_{t'}) \setminus B$ to Q^*. Let the resulting set be Q'. Observe that for every vertex that we add this way, we can associate a distinct element from Q. This implies that the size of Q' is at most the size of Q. Hence $|Q'| \leq |Q| \leq 2^h(2|S|+2)+2^h+|S| = \mathcal{O}(|S|)$. Now we show that Q' forms the desired 2-dominating set of $L'_t \setminus A$. Let $V' = V(L'_t) \setminus A$ and let $C = (\cup_{t'} X_t \cap X_{t'}) \setminus (A \cup B)$, where t' is a neighbor of t in T. In other words, C is the set of vertices in L^*_t that also appears in other torsos. Every vertex $v \in (V' \setminus S)$ that is not dominated with respect to at least one feasible f is clearly dominated by a vertex from $D(f)$. If $v \notin C$ then there is a vertex of $D(f)$ in V' that dominates v and hence it is contained in Q'. So suppose that $v \in C$. If there is a vertex of $D(f)$ in V' that dominates v then again we are done. This implies that v is dominated by a vertex from outside the torso. But by our procedure, we have selected a vertex from $X'_t \cap X_t$ that contains v. However, $X'_t \cap X_t$ is a clique in torso and hence the selected vertex dominates v. So we can assume that v is dominated in every feasible f. Now the only reason v is not deleted from the graph is because there is some feasible $g \in \mathcal{F}$ such that $N_G(v) \setminus A \not\subseteq W(g)$. Hence there is a neighbor of v, say u, that is not dominated with respect to g. By earlier arguments, we know that there exists a vertex $w \in Q'$ that dominates u and hence vertex w 2-dominates v. This shows that Q' is the desired 2-dominating set for $L'_t \setminus A$.

To conclude, $L'_t \setminus A$ excludes an apex graph as a minor (see discussions after Theorem 1) and it has a 2-dominating set of size $\mathcal{O}(|S|)$. By the bidimensionality of 2-dominating set, we have that $\mathbf{tw}(L^*_t) = \mathcal{O}(\sqrt{|S|})$ [13, 24]. We add all the apices of A to all bags of the tree decomposition of $L'_t \setminus A$ of width $\mathcal{O}(\sqrt{|S|})$ and this increases its width by at most h. Hence $\mathbf{tw}(L'_t) = \mathcal{O}(\sqrt{|S|})$. □

Let us remark that Irrelevant Vertex Rule is based on the performance of a polynomial time approximation algorithm and thus by Lemmata 1 and 2, we obtain the following lemma.

Lemma 3. *There is a polynomial time algorithm that for a given graph G and a dominating set S of G, outputs an induced subgraph G' of G such that $S \subseteq V(G')$ and $\mathbf{tw}(G') = \mathcal{O}(\sqrt{|S|})$. Moreover, for every set Z in G of size at most $|S|$ that dominates everything but S, there is a set Z' in G' of size at most $|Z|$ such that $Z' \supseteq Z \cap S$ and Z' dominates all the vertices of $N[Z]$ in G'. Similarly for every set $Z' \subseteq V(G')$ that dominates in G' al vertices but S, there is set $Z \subseteq V(G)$ of size at most $|Z'|$ such that $Z \supseteq Z' \cap S$, Z dominates $N[Z']$ and also all the vertices in $V(G) \setminus V(G')$.*

Algorithm 1 DELETION-SET(G, Y)

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if |Y| ≤ 9d^2 then
    Return ∅.
else
    Apply Lemma 3 with G and S = Y and obtain an equivalent graph G′ as described in the statement of the
    Lemma 3 such that tw(G′) = O(√|Y|). Hence there exists α, a constant, such that tw(G′) ≤ α√|Y|.
    Compute an approximate tree decomposition (T, 𝒳 = {X_t}_{t∈V(T)}) of width at most d√|Y| for G′ using
    the factor β approximation algorithm for treewidth given in [22]. Here d = αβ.
    Find the partitioning of the vertex set V(G′) into V_1, V_2 and X (a bag corresponding to a node in T) as
    described in Lemma 4 with the weight function that assigns 1 to the vertices in S and 0 otherwise. Let
    Y_1 = (Y ∩ V_1) ∪ X and Y_2 = (Y ∩ V_2) ∪ X.
end if
Return (X ⋃ DELETION-SET(G′[V_1 ∪ X], Y_1) ⋃ DELETION-SET(G′[V_2], Y_2)).
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Before we proceed further, we show the power of Lemma 3 by deriving a simple subexponential time algorithm for DOMINATING SET on H-minor free graphs. It is one of the cornerstone result in [13] and is based on a non-trivial two-layer dynamic programming over clique-sum decomposition tree of a H-minor free graphs. Lemma 3 can be used to obtain much simpler algorithm. Given a graph G and a positive integer k we first apply factor 2-approximation algorithm given in [14, 25] for DOMINATING SET on G and obtain a set S. If the size of S is more than $2k$ then we return that G does not have a dominating set of size at most k. Otherwise, we apply Lemma 3 and obtain an equivalent graph G' such that $\mathbf{tw}(G') = \mathcal{O}(\sqrt{k})$. Now applying a constant factor approximation algorithm developed in [13] for computing the treewidth on G' we get a tree decomposition of width $\mathcal{O}(\sqrt{k})$. It is well known that checking whether a graph with treewidth t has a dominating set of size at most k in time $2^{\mathcal{O}(t)}n^{\mathcal{O}(1)}$ [1]. This together with the above bound on the treewidth, gives us an alternative proof of the following.

Theorem 2 ([14])**.** *Given an n-vertex graph G excluding a fixed graph H as a minor, one can check whether G has a dominating set of size at most k in time $2^{\mathcal{O}(\sqrt{k})}n^{\mathcal{O}(1)}$.*

Finding an equivalent graph with $\mathcal{O}(k)$ sized treewidth deletion set. Now we want to apply Lemma 3 recursively to obtain $\mathcal{O}(k)$-sized treewidth deletion set. That is, given a graph G excluding a fixed graph H as a minor and a positive integer k, in polynomial time we output a graph G' such that (a) G has a dominating set of size at most k if and only if G' has a dominating set of size at most k; and (b) it is possible to remove $\mathcal{O}(k)$ vertices from G' such that the resulting graph is of constant treewidth. We also need the following well known lemma, see e.g. [5], on separators in graphs of bounded treewidth.

Lemma 4. *Let G be a graph given with a tree-decomposition of width at most t and $w : V \to \mathbb{R}^+ \cup \{0\}$ be a weight function. Then in polynomial time we can find a bag X of the given tree-decomposition such that for every connected component $G[C]$ of $G \setminus X$, $w(C) \le w(V)/2$. Furthermore, the connected components $C_1, \ldots, C_\ell$ of $G \setminus X$ can be grouped into two sets V_1 and V_2 such that $\frac{w(V(G))-w(X)}{3} \le V_i \le \frac{2(w(V(G))-w(X))}{3}$, for $i \in \{1, 2\}$. We call this (V_1, X, V_2) separation.*

We proceed as follows. Given a graph G and a positive integer k we first apply factor 2-approximation algorithm given in [14, 25] for DOMINATING SET on G and obtain a set S. If the size of S is more than $2k$ then we return that G does not have a dominating set of size at most k. By Lemma 3, there is a constant α, such that the treewidth of a graph G' obtained from G by removing irrelevant vertices is at most $\alpha\sqrt{k}$. We apply the recursive procedure detailed in **Algorithm** 1 on (G, S) to obtain a treewidth deletion set of size $\mathcal{O}(k)$. Let D be the set of vertices returned by the algorithm described in **Algorithm** 1. Also let B be the set of irrelevant vertices deleted by repeated applications of Lemma 3 in the algorithm described in **Algorithm** 1 during the whole algorithm. Let G' be the graph obtained after deleting B from G. Now we show the following.

Lemma 5. *Let G, G', S, D and B be as above. Then G has a dominating set of size at most k if and only if G' has a dominating set of size at most k. Furthermore $\mathbf{tw}(G' \setminus D) = \mathcal{O}(1)$ and $|D| = \mathcal{O}(k)$.*

Proof. By using induction on $|S|$, we show the following property: *For any set Z in G that dominates everything but S, there is a set Z' in G' of size at most $|Z|$ such that Z' contains $Z \cap S$ and Z' dominates all the vertices of $N[Z]$ in G'. Similarly for any set Z' in G' that dominates all but S there is a set Z in G of size at most $|Z'|$ that contains $Z' \cap S$ and Z dominates $N[Z']$ and also all vertices in $G \setminus G'$.* If these conditions hold we say that G and G' are *S-equivalent*.

For $|S| \leq 9d^2$, then $G = G'$, and the statement holds. For inductive step, let us assume that $|S| > 9d^2$. We first apply Lemma 3 on G and S to obtain G^* such that G and G^* are *S-equivalent*. Then we find a partition of the vertex set $V(G^*)$ into V_1, V_2 and X such that $|S|/3 \leq |V_i \cap S| \leq 2|S|/3$ for $i \in \{1, 2\}$. Let $S_i = (S \cap V_i) \cup X$ for $i \in \{1, 2\}$. Since $d\sqrt{|Y|} + \frac{2|Y|}{3} < |Y|$ for $|Y| > 9d^2$ and $|X| \leq d\sqrt{|S|}$, we have that $|S_1| < |S|$ and $|S_2| < |S|$. Observe that no vertex in S can be irrelevant and hence $B = B_1 \cup B_2$, where $B_1 = B \cap V_1$ and $B_2 = B \cap V_2$.

By induction assumption, we have that $G^*[V_i \cup X]$ and $G^*[(V_i \cup X) \setminus B_i]$, $i \in \{1, 2\}$, are S_i-equivalent. Now we want to show that G^* and $\widehat{G} = G^* \setminus (B_1 \cup B_2)$ are S-equivalent. We first show that for a set Z in G^* that dominates everything but S there exists Z' in $\widehat{G}$ of size at most $|Z|$ that dominates all of $N[Z]$ in $\widehat{G}$ and contains $Z \cap S$. For $i \in \{1, 2\}$, we put $Z_i = Z \cap (V_i \cup X)$. Then Z_1 dominates all but S_1 in $G^*[V_1 \cup X]$ and Z_2 dominates all but S_2 in $G^*[V_2 \cup X]$. By induction assumption, there exists Z_i' in $G^*[(V_i \cup X) \setminus B_i]$ that dominates $N[Z_i]$ and contains $Z_i \cap S_i$, where $i \in \{1, 2\}$. We claim that $Z' = Z_1' \cup Z_2'$ contains $Z \cap S$ and Z' dominates $N[Z]$. First notice that since Z_i' contains $Z_i \cap S_i$, $|Z_i'| \leq |Z|$ and that $Z_1 \cap X = Z_2 \cap X$, we have that $|Z'| \leq |Z|$. Furthermore since $Z \cap S = (Z_1 \cap S) \cup (Z_2 \cap S)$, we have that Z' contains $Z \cup S$. Now we need to show that $N[Z']$ dominates $N[Z]$ in $\widehat{G}$ but that follows because $N[Z] = N[Z_1] \cup N[Z_2]$ and Z_1' dominates $N[Z_1]$ and Z_2' dominates $N[Z_2]$. This completes the proof in one direction.

Now we prove the reverse direction. We show that for a set Z' in $\widehat{G}$ that dominates everything but S there exists Z in G^* of size at most $|Z'|$, containing $Z' \cap S$, and dominating $N[Z']$ and B (the set of vertices in $V(G^*) \setminus \widehat{G}$). Let the intersections of Z' to $(V_1 \setminus B_1) \cup X$ and $(V_2 \setminus B_2) \cup X$ be Z_1' and Z_2' respectively. By induction hypothesis, for $i \in \{1, 2\}$ there is Z_i in $G^*[(V_i \cup X)]$ that dominates $N[Z_i']$ and all of B_i and contains $Z_i' \cap S_i$. As before we can show that $Z_1 \cup Z_2$ is the required set Z. This completes the proof of the property.

As far as we are done with the proof of the property, it is easy to show that G has a dominating set of size k if and only if G' has a dominating set of size k. Given a dominating set of size k, say Z, of G, we know that there exists a set Z' of the same size that dominates all the vertices of $N[Z]$ present in G'. Since G' is the induced subgraph of G and Z is the dominating set, we have that $N[Z] \cap V(G') = V(G')$ and hence Z' is the dominating set of size at most k for G'. Similarly given a dominating set Z' of G' of size at most k, we know that there exists a set Z in G of size at most k that dominates $N[Z'] \cup B = V(G)$. This completes the proof of the first part of the lemma.

Now we argue about the size and the properties of D. By our construction of D, it follows that after we remove D from the graph every connected component contains at most $9d^2$ vertices from S. Let us fix a connected component, say C. The graph which after partitioning gives rise to C contains at most $\mathcal{O}(d^2) = \mathcal{O}(1)$ vertices from S and hence the treewidth of this parent graph and hence its induced subgraph, C, is $\mathcal{O}(1)$. The size of D is estimated via the following recursive formula

$$\mu(|D|) \quad \leq \quad \max_{1/3 \leq \alpha \leq 2/3} \left\{ \mu\left(\alpha|S| + d\sqrt{|S|}\right) + \mu\left((1-\alpha)|S| + d\sqrt{|S|}\right) + d\sqrt{|S|} \right\}$$

Using simple induction one can show that the above solves to $\mathcal{O}(|S|)$. See for an example [25, Lemma 2]. Hence we conclude that $|D| = \mathcal{O}(|S|) = \mathcal{O}(k)$. This completes the proof of the lemma. $\square$

Final Kernel. Now we proceed with the proof of our main result for DOMINATING SET on graphs excluding a fixed graph H as a minor. For this we need the following lemmata.

Lemma 6 ([26, Lemma 3.4]). *For every fixed graph H and constant t there are constants ζ and r that satisfy the following. For any n-vertex graph G which excludes H as a minor and has a vertex set D of size k' such that $\mathbf{tw}(G \setminus D) \leq t$, then G has an r-protrusion of size at least $\zeta n/k'$.*

DOMINATING SET has finite integer index, and the following lemma is a special case of [26, Lemma 4.1], see also [6].

Lemma 7 ([6, 26])**.** *Let $\mathcal{G}_H$ be the class of graphs excluding a fixed graph H as a minor. Then there exists a constant c_r and an algorithm that given a graph $G \in \mathcal{G}_H$, an integer k and an r-protrusion X in G with $|X| > c_r$, runs in time $\mathcal{O}(|X|)$ and returns a graph $G^* \in \mathcal{G}_H$ and an integer k^* such that $|V(G^*)| < |V(G)|$, $k^* \leq k$, and G^* has a dominating set of size at most k^* if and only if G has a dominating set of size at most k.*

Theorem 3. *Let $\mathcal{G}_H$ be the class of graphs excluding a fixed graph H as a minor.* DOMINATING SET *has a linear kernel on $\mathcal{G}_H$.*

Proof. Given a graph G and a positive integer k we first apply factor 2-approximation algorithm given in [14, 25] for DOMINATING SET on G and obtain a set S. If the size of S is more than $2k$, then we return that G does not have a dominating set of size at most k. Otherwise, we apply Lemma 5 on G and S and obtain a graph G' such that G has a dominating set of size at most k if and only if G' has a dominating set of size at most k. We also obtain a treewidth deletion set D of size at most $\mathcal{O}(k)$, that is $\mathbf{tw}(G' \setminus D) \leq t$ for some fixed constant t.

By Lemma 6, G contains an r-protrusion of size at least $\zeta|V(G')|/tk$. The reduction algorithm exhaustively applies Lemma 7. Since an irreducible instance contains no r-protrusion of size at least c_r it follows that an irreducible instance (G', k) of DS must satisfy $\zeta|V(G')|/tk < c_r$. Thus $|V(G')|$ is at most $k \cdot tc_r/\zeta = \mathcal{O}(k)$.

Now we show that our kernelization procedure runs in polynomial time. Observe that we can find a protrusion by guessing the boundary which is of constant size. Once given a protrusion X, we can replace it with an equivalent instance in $\mathcal{O}(|X|)$ time using the Lemma 7. This concludes that the kernelization algorithm runs in polynomial time. □

4 Kernel for CONNECTED DOMINATING SET

In this section we give a linear kernel for CONNECTED DOMINATING SET (CDS). As in the kernelization algorithm for DS, the kernelization for CDS also has two phases. The main difference is the reduction rule, though here we also identify an irrelevant vertex and delete it, the correctness proof is much more involved and requires much more care. As for DS, we apply this reduction rule recursively and obtain a treewidth deletion set of size $\mathcal{O}(k)$. Then applying "protrusion rule" together with the fact that CDS has finite integer index, we obtain the desired linear kernel for CDS.

Reducing the treewidth of a torso. As with DS, we will reduce the treewidth of a torso not only in the beginning of the procedure but also when we apply a recursive procedure to obtain $\mathcal{O}(k)$ sized treewidth deletion set. Let G be a graph, S be a dominating set of G, L_t be one of its torsos, and $A = \{a_1, \ldots, a_h\}$ be the set of apices of L_t. Our objective is to define a reduction rule that essentially "preserves" all partially connected dominating sets of size at most $3|S|$. We define $\mathcal{F}$ to be the set of all functions from $A \to \{0, 1\}$ encoding all possible guesses on the set of apices with respect to how they are dominated in every connected dominating set of size at most $3|S|$ of G. More formally, with every connected dominating set D, we associate a function $f \in \mathcal{F}$, such that $f(a_i) = 0$ yields $a_i \in D$ and $f(a_i) = 1$ yields that $a_i \notin D$. For every subset $A' \subseteq A$, we find *a minimum sized Steiner tree* with A' being terminal nodes and denote it by $T(A')$. This can be done in time $2^{|A'|}n^{\mathcal{O}(1)} = 2^h n^{\mathcal{O}(1)}$ by making use of the algorithm from [4]. We call tree $T(A')$ *good* if the number of non-terminal vertices, that is, non-apex vertices, is at most $2h$. For each subset of $A' \subseteq A$ we select one good tree $T(A')$ (if there is such a tree) to represent A'. Let R be the set of vertices appearing in the representative trees corresponding to subsets of A.

The definition of a *feasible* function f and the set of *dominated* vertices $W(f)$ is exactly the same as in Section 3 but the definition of irrelevant vertex is significantly different.

- A vertex $w \in W(f)$ is called *fully dominated* if $w \notin R$ and $N_G(w) \setminus A \subseteq W(f)$. We denote the set of vertices *fully dominated with respect to* f by $Fdom(f)$. A vertex w is called *irrelevant with respect to* f, if w is fully dominated and $N_G^{3h}(w) \setminus A \subseteq W(f)$.

The irrelevant vertex rule for CDS is exactly the same as in Section 3 for DS but the proof of its correctness is much more complicated.

Irrelevant Vertex Rule: If a vertex w is irrelevant with respect to every feasible function $f \in \mathcal{F}$, then delete w from G.

Before we prove correctness of the reduction rule and its impact we prove a crucial lemma which will be used later.

Lemma 8. *Let G be a connected graph and $f \in \mathcal{F}$. If G contains a connected dominating set of size at most k, then G has a connected dominating set of size at most k with at most $2h$ fully dominated vertices from $FDom(f)$.*

Proof. Let Z be a connected dominating set of G of size at most k, $F = Z \cap FDom(f)$, and A' be the subset of apices A contained in Z. Let us observe that function f assigning 0 to all vertices of A' is feasible. Graph $G[Z \setminus F]$ is not necessary connected because vertices from F can be adjacent to some vertices from A', and removal of F can separate these vertices. We claim that in $G[Z \setminus F]$ every connected component of $G[Z \setminus F]$ contains a vertex from A'. Indeed, every connected component of $G[Z \setminus F]$ contains a neighbor of some vertex in F, and since F is fully dominated, we have that each such neighbor of F is also adjacent to some apex vertex in A'. Hence every connected component of $G[Z \setminus F]$ contains a vertex from A'. Furthermore since F is the set of fully dominated vertices, we have that $Z \setminus F$ is also a dominating set such that $G[Z \setminus F]$ has at most $|A'| \leq h$ connected components. Every dominating set with at most h connected components in a connected graph can be turned into a connected set by adding at most $2h$ vertices. Thus by adding at most $2h$ vertices, one can turn $Z \setminus F$ into a connected dominating set. $\square$

Given a set $Z \subseteq V(G)$, we associate a function $f \in \mathcal{F}$ that assigns 1 to vertices of $a \in A \cap Z$ and 0 to $A \setminus Z$. We call such function f *canonically* associated to Z. We need to define faithful and companion sets.

Definition 1. *Let G be a graph, G' be its induced subgraph and $S \subseteq V(G) \cap V(G')$.*

- *A set Z in G (or Z' in G') is called* semi faithful *if it dominates everything but S, and every connected component of $G[Z]$ (or $G[Z']$) contains at least one vertex of S.*
- *A set Z in G is called* faithful *if it is semi faithful and every connected component of $G[Z]$ contains at most $2h$ fully dominated vertices with respect to the function f canonically associated to Z.*
- *A set Z' in G' is called* companion *to a set Z in G, if Z' dominates all the vertices of $N[Z]$ in G'; $(Z \cap S) \subseteq Z'$ and for every connected component C' of $G'[Z']$ there exists a connected component C in $G[Z]$ such that $C \cap S \subseteq C' \cap S$.*
- *A set Z in G is called* companion *to a set Z' in G', if it dominates all the vertices of $N[Z'] \cup (V(G) \setminus V(G'))$ in G, $(Z' \cap S) \subseteq Z$ and every connected component C of $G[Z]$ there exists a connected component C' in $G'[Z']$ such that $C' \cap S \subseteq C \cap S$.*

As for DS, we apply the irrelevant vertex rule as long as possible. Let the set of vertices deleted in this process be B and let $G' = G \setminus B$ be the resulting graph. The following lemma is the analog of Lemma 1. It is the most important part of the proof—as far as Lemma 9 is settled, the remaining part of the proof follows almost the same lines as the proof in Section 3.

Lemma 9. [$\star$][1] *Let G be a graph, S be the dominating set of G and G' be the induced graph obtained after deleting B from G. Then for every faithful Z in G there is a companion set Z' of size at most $|Z|$ in G'. Similarly for every semi faithful set Z' in G' there is a companion set Z in G of size at most $|Z'|$. Furthermore one can find the desired G' in polynomial time.*

[1]The proofs marked with [$\star$] have been moved to the appendix due to space restrictions.

As for DS, it is possible to prove that after removing all irrelevant vertices, the treewidth of each torso in the reduced graph is $\mathcal{O}(\sqrt{|S|})$. The most important difference is that instead of 2-dominating set we construct a $(3h+1)$-dominating set in the proof.

Lemma 10. $[\star]$ *Let $L'_t = L_t \setminus B$ then* $\mathbf{tw}(L'_t) = \mathcal{O}(\sqrt{|S|})$.

Lemma 10 is used in the recursive phase to bound the treewidth. Now we apply Lemma 9 recursively to obtain $\mathcal{O}(k)$-sized treewidth deletion set. That is given a graph G excluding a fixed graph H as a minor and a positive integer k, in polynomial time we output a graph G' such that (a) G has a connected dominating set of size at most k if and only if G' has a connected dominating set of size at most k; and (b) G' has treewidth deletion set of size at most k.

We proceed as follows. Given a *connected* graph G and a positive integer k, we first apply factor 2-approximation algorithm given in [14, 25] for DS on G and obtain a dominating set S. If the size of S is more than $2k$ then we return that G does not have a connected dominating set of size at most k. If the size of S is at most $2k$, we proceed further. To prove Lemma 11, we need an additional property of S, namely that every dominating set *contains at least one vertex from* S. To ensure that S has this property, we choose a vertex v of minimum degree and add $N[v]$ to S. Since G excludes a fixed graph H as a minor there exists a constant c such that G is $p = c|V(H)|\sqrt{\log|V(H)|}$ degenerate [18]. This implies that the degree of v in G is at most p and hence $|S| \le 2k + p + 1 = \mathcal{O}(k)$. This property together with companionship of sets defined previously allow us to maintain connectivity during recursive applications of irrelevant vertex rule. The recursive procedure to obtain a treewidth deletion set of size $\mathcal{O}(k)$ is almost identical to the one detailed in **Algorithm** 1 on (G, S). The only difference is that in step 4 of **Algorithm** 1, instead of Lemma 3, we use Lemma 9. Let D be the set of vertices returned by this algorithm. and let B be the set of irrelevant vertices deleted by repeated applications of Lemma 9. Let G' be the graph obtained after deleting B from G. The following lemma holds.

Lemma 11. $[\star]$ *Let G, G', S, D and B be as above. Then G has a connected dominating set of size at most k if and only if G' has a connected dominating set of size at most k. Furthermore* $\mathbf{tw}(G' \setminus D) = \mathcal{O}(1)$ *and* $|D| = \mathcal{O}(k)$.

Finally, CDS has finite integer index [6] and the statement similar to Lemma 7 for CDS is a special case of [26, Lemma 4.1]. Now using Lemmata 6 and 11, we can show the following theorem along the lines of Theorem 3.

Theorem 4. *Let $\mathcal{G}_H$ be the class of graphs excluding a fixed graph H as a minor then* CONNECTED DOMINATING SET *has linear kernel on $\mathcal{G}_H$.*

Let us observe, that Theorem 4 combined with the standard dynamic programing on graphs of bounded treewidth implies that CDS on H-minor free graphs is solvable in time $2^{\mathcal{O}(\sqrt{k}\log k)}$. To our knowledge, this is the first subexponential parameterized algorithm for CDS on H-minor free graphs.

5 Conclusions

We conclude with several open questions. It is tempting to ask if the kernelization framework on apex-minor free graphs developed in [26] for contraction bidimensional problems with separation properties can be extended to minor free graphs. This question remains open even for r-domination with $r > 1$. Another natural question is if the linear kernel for DS can be obtained for more general classes. H-minor free graphs form a general class of sparse graph but DS is known to be FPT even on more general classes of sparse graphs like graphs locally excluding some graph as a minor, degenerated graphs, graphs of bounded expansions, and nowhere dense classes of graphs [9, 10, 35]. A word of caution is appropriate here: there are classes of sparse graphs where existence of a linear kernel for DS is highly unexpected. For example, an easy reduction from the result of Dell and van Melkebeek from [11] that d-HITTING SET has no kernel of size $k^{d-\varepsilon}$ for any $\varepsilon > 0$ unless coNP is in NP/poly, shows that DS has no kernel of size $k^{d-\varepsilon}$ on d-degenerate graphs. For CDS the situation is even worse, by the recent result of Cygan et al. [8], the problem does not have a polynomial kernel on d-degenerated graphs unless coNP is in NP/poly.

References

[1] J. ALBER, H. L. BODLAENDER, H. FERNAU, T. KLOKS, AND R. NIEDERMEIER, *Fixed parameter algorithms for dominating set and related problems on planar graphs*, Algorithmica, 33 (2002), pp. 461–493.

[2] J. ALBER, M. R. FELLOWS, AND R. NIEDERMEIER, *Polynomial-time data reduction for dominating sets*, J. ACM, 51 (2004), pp. 363–384.

[3] N. ALON AND S. GUTNER, *Kernels for the dominating set problem on graphs with an excluded minor*, Tech. Rep. TR08-066, ECCC, 2008.

[4] A. BJÖRKLUND, T. HUSFELDT, P. KASKI, AND M. KOIVISTO, *Fourier meets Möbious: Fast subset convolution*, in Proceedings of the 39th annual ACM Symposium on Theory of Computing (STOC 2007), New York, 2007, ACM Press, pp. 67–74.

[5] H. L. BODLAENDER, *A partial k-arboretum of graphs with bounded treewidth*, Theoret. Comput. Sci., 209 (1998), pp. 1–45.

[6] H. L. BODLAENDER, F. V. FOMIN, D. LOKSHTANOV, E. PENNINKX, S. SAURABH, AND D. M. THILIKOS, *(Meta) Kernelization*, in Proceedings of the 50th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2009), IEEE, 2009, pp. 629–638.

[7] J. CHEN, H. FERNAU, I. A. KANJ, AND G. XIA, *Parametric duality and kernelization: Lower bounds and upper bounds on kernel size*, SIAM J. Comput., 37 (2007), pp. 1077–1106.

[8] M. CYGAN, M. PILIPCZUK, M. PILIPCZUK, AND J. WOJTASZCZYK, *Kernelization hardness of connectivity problems in d-degenerate graphs*, in Proceedings of the 36th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2010), Lect. Notes Comp. Sc., Springer, 2010, p. to appear.

[9] A. DAWAR, M. GROHE, AND S. KREUTZER, *Locally excluding a minor*, in Proceedings of the 22nd IEEE Symposium on Logic in Computer Science (LICS 2007), IEEE Computer Society, 2007, pp. 270–279.

[10] A. DAWAR AND S. KREUTZER, *Domination problems in nowhere-dense classes*, in Proceedings of the IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2009), vol. 4 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2009, pp. 157–168.

[11] H. DELL AND D. VAN MELKEBEEK, *Satisfiability allows no nontrivial sparsification unless the polynomial-time hierarchy collapses*, in Proceedings of the 42nd ACM Symposium on Theory of Computing (STOC 2010), 2010, pp. 251–260.

[12] E. D. DEMAINE, F. V. FOMIN, M. HAJIAGHAYI, AND D. M. THILIKOS, *Fixed-parameter algorithms for (k, r)-center in planar graphs and map graphs*, ACM Trans. Algorithms, 1 (2005), pp. 33–47.

[13] ——, *Subexponential parameterized algorithms on bounded-genus graphs and H-minor-free graphs*, J. ACM, 52 (2005), pp. 866–893.

[14] E. D. DEMAINE AND M. HAJIAGHAYI, *Bidimensionality: new connections between FPT algorithms and PTASs*, in Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2005), New York, 2005, ACM-SIAM, pp. 590–601.

[15] ——, *The bidimensionality theory and its algorithmic applications*, The Computer Journal, 51 (2007), pp. 332–337.

[16] E. D. DEMAINE, M. T. HAJIAGHAYI, AND K.-I. KAWARABAYASHI, *Algorithmic graph minor theory: Decomposition, approximation, and coloring*, in Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2005), IEEE Computer Society, 2005, pp. 637–646.

[17] ——, *Approximation algorithms via structural results for apex-minor-free graphs*, in Proceedings of the 36th International Colloquium on Automata, Languages and Programming (ICALP 2009), vol. 5555 of Lect. Notes Comp. Sc., Springer, 2009, pp. 316–327.

[18] R. DIESTEL, *Graph theory*, vol. 173 of Graduate Texts in Mathematics, Springer-Verlag, Berlin, third ed., 2005.

[19] F. DORN, F. V. FOMIN, D. LOKSHTANOV, V. RAMAN, AND S. SAURABH, *Beyond bidimensionality: Parameterized subexponential algorithms on directed graphs*, in Proceedings of the 27th International Symposium on Theoretical Aspects of Computer Science (STACS 2010), vol. 5 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2010, pp. 251–262.

[20] R. G. DOWNEY AND M. R. FELLOWS, *Parameterized Complexity*, Springer, 1998.

[21] F. F. DRAGAN, F. V. FOMIN, AND P. A. GOLOVACH, *Spanners in sparse graphs*, in Proceedings of the 35th International Colloquium on Automata, Languages and Programming (ICALP 2008), vol. 5125 of Lect. Notes Comp. Sc., Springer, 2008, pp. 597–608.

[22] U. FEIGE, M. HAJIAGHAYI, AND J. R. LEE, *Improved approximation algorithms for minimum weight vertex separators*, SIAM J. Comput., 38 (2008), pp. 629–657.

[23] J. FLUM AND M. GROHE, *Parameterized Complexity Theory*, Texts in Theoretical Computer Science. An EATCS Series, Springer-Verlag, Berlin, 2006.

[24] F. V. FOMIN, P. A. GOLOVACH, AND D. M. THILIKOS, *Contraction bidimensionality: The accurate picture*, in Proceedings of the 17th Annual European Symposium on Algorithms (ESA 2009), vol. 5757 of Lect. Notes Comp. Sc., Springer, 2009, pp. 706–717.

[25] F. V. FOMIN, D. LOKSHTANOV, V. RAMAN, AND S. SAURABH, *Bidimensionality and EPTAS*, CoRR, abs/1005.5449 (2010).

[26] F. V. FOMIN, D. LOKSHTANOV, S. SAURABH, AND D. M. THILIKOS, *Bidimensionality and kernels*, in Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2010), ACM-SIAM, 2010, pp. 503–510.

[27] F. V. FOMIN AND D. M. THILIKOS, *Dominating sets in planar graphs: Branch-width and exponential speed-up*, SIAM J. Comput., 36 (2006), pp. 281–309.

[28] P. A. GOLOVACH, M. KAMINSKI, D. PAULUSMA, AND D. M. THILIKOS, *Induced packing of odd cycles in a planar graph*, in Proceedings of the 20th International Symposium on Algorithms and Computation (ISAAC 2009), vol. 5878 of Lecture Notes in Comput. Sci., Springer, Berlin, 2009, pp. 514–523.

[29] M. GROHE, *Local tree-width, excluded minors, and approximation algorithms*, Combinatorica, 23 (2003), pp. 613–632.

[30] S. GUTNER, *Polynomial kernels and faster algorithms for the dominating set problem on graphs with an excluded minor*, in Proceedings of the 4th Workshop on Parameterized and Exact Computation (IWPEC 2009), Lect. Notes Comp. Sc., Springer, 2009, pp. 246–257.

[31] T. W. HAYNES, S. T. HEDETNIEMI, AND P. J. SLATER, *Fundamentals of domination in graphs*, Marcel Dekker Inc., New York, 1998.

[32] K.-I. KAWARABAYASHI AND Y. KOBAYASHI, *The induced disjoint path problem*, in Proceedings of the 13th Conference on Integer Programming and Combinatorial Optimization (IPCO 2008), vol. 5035 of Lect. Notes Comp. Sc., Springer, Berlin, 2008, pp. 47–61.

[33] K.-I. KAWARABAYASHI AND B. REED, *Odd cycle packing*, in Proceedings of the 42nd ACM Symposium on Theory of Computing (STOC 2010), New York, NY, USA, 2010, ACM, pp. 695–704.

[34] Y. KOBAYASHI AND K.-I. KAWARABAYASHI, *Algorithms for finding an induced cycle in planar graphs and bounded genus graphs*, in Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2009), ACM-SIAM, 2009, pp. 1146–1155.

[35] J. NEŠETŘIL AND P. O. DE MENDEZ, *Grad and classes with bounded expansion II. Algorithmic aspects*, Eur. J. Comb., 29 (2008), pp. 777–791.

[36] R. NIEDERMEIER, *Invitation to fixed-parameter algorithms*, vol. 31 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2006.

[37] G. PHILIP, V. RAMAN, AND S. SIKDAR, *Solving dominating set in larger classes of graphs: FPT algorithms and polynomial kernels*, in Proceedings of the 17th Annual European Symposium on Algorithms (ESA 2009), vol. 5757 of Lect. Notes Comp. Sc., Springer, 2009, pp. 694–705.

[38] N. ROBERTSON AND P. D. SEYMOUR, *Graph minors. XIII. The disjoint paths problem*, J. Comb. Theory Series B, 63 (1995), pp. 65–110.

[39] ——, *Graph minors. XVI. Excluding a non-planar graph*, J. Combin. Theory Ser. B, 89 (2003), pp. 43–76.

6 Appendix

6.1 Definitions of treewidth, clique sums, and almost embeddable graphs

A *tree decomposition* of a graph G is a pair $(\mathcal{X}, T)$ where T is a tree and $\mathcal{X} = \{X_i \mid i \in V(T)\}$ is a collection of subsets of V such that:
1. $\bigcup_{i \in V(T)} X_i = V(G)$,
2. for each edge $xy \in E(G)$, $\{x, y\} \subseteq X_i$ for some $i \in V(T)$;
3. for each $x \in V(G)$ the set $\{i \mid x \in X_i\}$ induces a connected subtree of T.

The *width* of the tree decomposition is $\max_{i \in V(T)} |X_i| - 1$. The *treewidth* of a graph G is the minimum width over all tree decompositions of G. We denote by $\mathbf{tw}(G)$ the treewidth of graph G.

Definition 1 (CLIQUE-SUMS)**.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs, and $k \geq 0$ an integer. For $i = 1, 2$, let $W_i \subset V_i$, form a clique of size h and let G'_i be the graph obtained from G_i by removing a set of edges (possibly empty) from the clique $G_i[W_i]$. Let $F : W_1 \to W_2$ be a bijection between W_1 and W_2. We define the h-*clique-sum *of G_1 and G_2, denoted by $G_1 \oplus_{h,F} G_2$, or simply $G_1 \oplus G_2$ if there is no confusion, as the graph obtained by taking the union of G'_1 and G'_2 by identifying $w \in W_1$ with $F(w) \in W_2$, and by removing all the multiple edges. The image of the vertices of W_1 and W_2 in $G_1 \oplus G_2$ is called the* join *of the sum.*

Note that some edges of G_1 and G_2 are not edges of G, because it is possible that they were added by clique-sum operation. Such edges are called *virtual*.

We remark that $\oplus$ is not well defined; different choices of G'_i and the bijection F could give different clique-sums. A sequence of h-clique-sums, not necessarily unique, which result in a graph G, is called a *clique-sum decomposition* of G.

Definition 2 (h-nearly embeddable graphs)**.** *Let Σ be a surface with boundary cycles $C_1, \ldots, C_h$, i.e. each cycle C_i is the border of a disc in Σ. A graph G is h-*nearly embeddable *in Σ, if G has a subset X of size at most h, called* apices*, such that there are (possibly empty) subgraphs $G_0 = (V_0, E_0), \ldots, G_h = (V_h, E_h)$ of $G \setminus X$ such that*

- *$G \setminus X = G_0 \cup \cdots \cup G_h$,*
- *G_0 is embeddable in Σ, we fix an embedding of G_0,*
- *graphs $G_1, \ldots, G_h$ (called* vortices*) are pairwise disjoint,*
- *for $1 \leq \cdots \leq h$, let $U_i := \{u_{i_1}, \ldots, u_{i_{m_i}}\} = V_0 \cap V_i$, G_i has a path decomposition (B_{ij}), $1 \leq j \leq m_i$, of width at most h such that*
    - *for $1 \leq i \leq h$ and for $1 \leq j \leq m_i$ we have $u_j \in B_{ij}$*
    - *for $1 \leq i \leq h$, we have $V_0 \cap C_i = \{u_{i_1}, \ldots, u_{i_{m_i}}\}$ and the points $u_{i_1}, \ldots, u_{i_{m_i}}$ appear on C_i in this order (either if we walk clockwise or anti-clockwise).*

6.2 Proof of Lemma 9

Proof. It is sufficient to prove both statements for the case when $G' = G \setminus \{w\}$, where w is an irrelevant vertex and then to apply this proof inductively.

We start from the first statement. We show how to find the corresponding Z' when the size of $|Z| \geq 3|S|$. Let $G_1, \ldots, G_\ell$ be the connected components of G and $S_i = S \cap V(G_i)$ be the dominating set of G_i. Since G_i is connected and $G_i[S_i]$ has at most $|S_i|$ connected components we can make it connected by adding at most $2|S|$ vertices, at most 2 per component, from outside. Let S'_i be the set of vertices we selected this way. Clearly each of S'_i is a connected dominating set of G_i. Now we take $Z' = \cup S'_i$. Clearly $|Z'| \leq \sum_{i=1}^{\ell} |S'_i| \leq 3 \sum_{i=1}^{\ell} |S_i| \leq 3|S| \leq |Z|$. Since $S \subseteq Z'$ and each S_i is a connected dominating set of G_i, it is easy to see that Z' satisfies the desired requirements.

From now onwards we assume that $|Z| < 3|S|$. Suppose $w \notin Z$ then clearly taking Z itself as Z' does the work. So we assume that $w \in Z$. Let Z_w be the connected component of $G[Z]$ containing w and G_i be the connected component of G that conatins Z_w. By assumption we know that Z_w contains at most $2h$ fully dominated vertices. Let F be the set of fully dominated vertices contained in Z_w and A' be the subset of A (apices) contained in Z_w. Let the connected components of $G[Z_w \setminus A']$ be $E_1, \ldots, E_\ell$. Let E_i be the connected component that contains w. We claim that every vertex in E_i is fully dominated. Suppose not then there exists a vertex u that is *not fully dominated*. But since E_i is a connected component there exists a path P from w to u. Now since w is irrelevant, we have that the first $3h$ vertices on P are fully dominated. In fact, every path from w to any non fully dominated vertex will have first $3h$ vertices as fully dominated. But then it is a contradiction, as Z_w has at most $2h$ fully dominated vertices. This also implies that the size of E_i is bounded above by $2h$. Let $A^* \subseteq A'$ be the set of vertices in A' that has a neighbor in E_i. Now from Z_w we remove the vertices of E_i. Observe that every connected component contains a neighbor of some vertex in E_i, but the neighbors of E_i are precisely A^* and hence every connected component of $G[Z_w] \setminus E_i$ contains a vertex from A^*. Note that $A^* \cup E_i$ is connected and hence the minimum steiner tree $T(A^*)$ on A^* can only contain at most $|E_i|$ non-terminal vertices. Now since E_i only contains fully dominated vertices, we have that $Z_w \setminus E_i$ dominates $N[Z_w]$ and hence $Z' = (Z \setminus E_i) \cup V(T(A^*))$ dominates everything but S in G, does not contain w and has size at most $|Z|$. Now we need to show that for every connected component C' of $G[Z']$ there exists a connected component C in $G[Z]$ such that $C \cap S \subseteq C' \cap S$. Observe that by our process we never delete any vertices from S, only could decrease the number of connected components in $G[Z']$ and every vertex in S that is present in Z is also present in Z'. So for any connected component C' of $G[Z']$ we just associate a connected component C of $G[Z]$ such that $C' \cap C \neq \emptyset$. This association satisfies our final requirement for Z' being companion to Z.

Now we prove the second statement of the lemma. Let Z' in G' be a semi faithful set. We show that Z' itself is its companion set in G. To show that Z' is the desired set the only thing we need to prove is that Z' also dominates w. We prove it by contradiction. So assume that w is not dominated by a vertex in Z'. Let $g \in \mathcal{F}$ be the function canonically associated to Z'. Clearly g is a feasible as $(Z' \setminus A) \cup \{w\}$ is of size at most $|S| + 1$ and dominates $V(G) \setminus (A \cup W(f) \cup S)$. This implies that $w \in W(f)$, as a vertex is deleted only if it is irrelevant with respect to every feasible function $g \in \mathcal{F}$. Hence there exists a vertex in $A' = A \cap Z'$ that is a neighbor of w. This concludes the proof. $\square$

6.3 Proof of Lemma 10

Let $L'_t = L_t \setminus B$. The proof that $\mathbf{tw}(L'_t) = \mathcal{O}(\sqrt{|S|})$ is almost identical to the proof of Lemma 2. The most important difference is that instead of 2-dominating set we construct a $(3h+1)$-dominating set.

Proof. Let $L^*_t = L'_t \setminus A$. We first show that there exists a $(3h+1)$-dominating set of size $\mathcal{O}(|S|)$ for $L'_t \setminus A$. Towards this consider the following set

$$Q = \bigcup_{\substack{f \in \mathcal{F}, \\ f \text{ is feasible}}} D(f) \cup R \cup S.$$

We first show that there exists X' corresponding to X that forms $(3h+1)$-dominating set of size $\mathcal{O}(|S|)$ of $L'_t \setminus A$.

Let $Q^* = Q \cap V(L^*_t)$ be the intersection of X with the vertices in L^*_t. Let t' be a neighbor of t in T, such that $((X_t \cap X_{t'}) \setminus B) \neq \emptyset$ and there is no vertex from $(X_t \cap X_{t'}) \setminus B$ in Q^*. Furthermore if there is a vertex $q \in Q$ such that it appears in one of the bags corresponding to the subtree containing t' in $T \setminus \{t\}$ then we add an arbitrary vertex from $X_t \cap X_{t'} \setminus B$ to Q^*. We do this for every child t' of t. Let the resulting set be Q'. Observe that for every vertex that we add this way, we can associate a distinct element from Q. This implies that the size of Q' is at most the size of Q. Hence $|Q'| \leq |Q| \leq 2^h(2|S|+2) + (2h)2^h + |S| \leq \mathcal{O}(|S|)$. Now we show that Q' forms the desired $(3h+1)$-dominating set of $L'_t \setminus A$. Let $V' = V(L'_t) \setminus A$ and let $C = (\cup_{t'} X_t \cap X_{t'}) \setminus (A \cup B)$, where t' is a neighbor of t in

T , that is, C is the set of vertices in L_t^* that also appears in other torsos. Any vertex $v \in (V' \setminus S)$ that is not dominated with respect to at least one feasible f is clearly dominated by a vertex in $D(f)$. If $v \notin C$ then there is a vertex of $D(f)$ in V' that dominates v and hence it is contained in Q'. So suppose that $v \in C$. If there is a vertex of $D(f)$ in V' that dominates v then again we are done. This implies that the v is dominated by a vertex from outside. But by our procedure we have selected a vertex from $X_t' \cap X_t$ that contains v. Now we know that $X_t' \cap X_t$ is a clique and hence the selected vertex dominates v. So let us assume that v is dominated in every feasible f. Now the very reason v is not deleted from the graph is because there is some feasible $g \in \mathcal{F}$ such that v is not irrelevant with respect to g. Hence there is a vertex in $N_G^{3h}(v) \setminus A$, say u, that is not dominated with respect to g. By earlier argument we know that there exists a vertex $w \in Q'$ that dominates u and hence the vertex w $(3h+1)$-dominates v. This shows that Q' is the desired $(3h+1)$-dominating set for $L_t' \setminus A$.

The graph $L_t' \setminus A$ excludes an apex graph as a minor (see discussion after Theorem 1) and it has a $(3h+1)$-dominating set of size at most $\mathcal{O}(|S|)$, and hence we have that $\mathbf{tw}(L_t^*) = \mathcal{O}(\sqrt{|S|})$ [13, 24]. Now even if we add all the apices in A to all the bags corresponding to the tree-decomposition of $L_t' \setminus A$, the treewidth can only increase by h and hence $\mathbf{tw}(L_t') = \mathcal{O}(\sqrt{|S|})$. □

6.4 Proof of Lemma 11

Proof. We first show that G and G' are equivalent. In fact using induction on $|S|$, we prove a stronger result and show the following. *For any faithful set Z in G, there is a companion set Z' in G' of size at most $|Z|$. Similarly for any semi faithful set Z' in G' there is a companion set Z in G of size at most $|Z'|$.* If these conditions hold we say that G and G' are *faithfully S-equivalent*.

If the size of $|S| \leq 9d^2$ then G and G' is same and hence the assertion follows. So we assume that $|S| > 9d^2$. We first apply Lemma 9 on G and S and obtain G^* such that G and G^* are faithfully S^*-equivalent. Now we find a partitioning of the vertex set $V(G^*)$ into V_1, V_2 and X such that $|S^*|/3 \leq |V_i \cap S^*| \leq 2|S^*|/3$ for $i \in \{1,2\}$. Let $S_i = ((S \cap V_i) \cup X)$ for $i \in \{1,2\}$. Now since $d\sqrt{|Y|} + \frac{2|Y|}{3} < |Y|$ for all $|Y| > 9d^2$ we have that the size of $|S_1| < |S|$ and $|S_2| < |S|$. Observe that a vertex in S is never irrelevant and hence $B = B_1 \cup B_2$, where $B_1 = B \cap V_1$ and $B_2 = B \cap V_2$.

By induction hypothesis we have that $G^*[V_i \cup X]$ and $G^*[(V_i \cup X) \setminus B_i]$, $i \in \{1,2\}$, are faithfully S_i equivalent. Now we want to show that G^* and $\widehat{G} = G^* \setminus (B_1 \cup B_2)$ are faithfully S equivalent. We first show that for a faithful set Z in G^* there exists a companion Z' in $\widehat{G}$ of size at most $|Z|$. Let the intersections of Z to $(V_1 \cup X)$ and $(V_2 \cup X)$ be Z_1 and Z_2, respectively. First we show that Z_1 and Z_2 are faithful in $G^*[V_1 \cup X]$ and $G^*[V_2 \cup X]$ respectively. We show it for Z_1, the arguments for Z_2 is symmetric. Clearly Z_1 dominates all but S_1 in $G^*[V_1 \cup X]$. Look at the connected components of $G^*[Z_1]$. If a connected component C of $G^*[Z_1]$ is also a connected component in $G^*[Z]$ then clearly it contains a vertex from S_1. If C is not a connected component in $G^*[Z]$ then it must be a part of some connected component in $G^*[Z]$ such that its intersection with X is non-empty and hence C contains a vertex from $X \subseteq S_1$. Furthermore since the connected components of $G^*[Z]$ contains at most $2h$ fully dominated vertices with respect to the function f which is canonically associated to Z, we have that every connected components of $G^*[Z_1]$ also contains at most $2h$ fully dominated vertices with respect to the function g which is canonically associated to Z_1. This proves that Z_1 is faithful.

By induction hypothesis there exists a companion Z_i' in $G^*[(V_i \cup X) \setminus B_i]$, where $i \in \{1,2\}$, corresponding to Z_i. We claim that $Z' = Z_1' \cup Z_2'$ is a companion to Z in $\widehat{G}$. First notice that since Z_i' contains $Z_i \cap S_i$, $|Z_i'| \leq |Z_i|$ and that $Z_1 \cap X = Z_2 \cap X$ we have that $|Z'| \leq |Z|$. Furthermore since $Z \cap S = (Z_1 \cap S) \cup (Z_2 \cap S)$ we have that Z' contains $Z \cup S$. Now we need to show that $N[Z']$ dominates $N[Z]$ in $\widehat{G}$ but that follows since $N[Z] = N[Z_1] \cup N[Z_2]$ and Z_1' dominates $N[Z_1]$ and Z_2' dominates $N[Z_2]$. Finally we show that for every connected component C' of $\widehat{G}[Z']$ there exists a connected component C in $G^*[Z]$ such that $C \cap S \subseteq C' \cap S$. Let C' be a connected component of $\widehat{G}[Z']$. Now let its intersections to Z_1' and Z_2' be C_1' and C_2' respectively. By induction hypothesis we know that there exists a connected component C_1 in $G^*[Z_1]$ such that $C_1 \cap S_1 \subseteq C_1' \cap S_1$ and there exists a connected component C_2 in $G^*[Z_2]$ such that $C_2 \cap S_1 \subseteq C_2' \cap S_2$. For C take any component of $G^*[C_1 \cup C_2]$. This completes the proof in one direction.

Now we prove the reverse direction. We show that for a semi faithful set Z' in $\widehat{G}$ there exists a companion Z in G^* of size at most $|Z'|$. Let the intersections of Z' to $(V_1 \setminus B_1) \cup X$ and $(V_2 \setminus B_2) \cup X$ be Z'_1 and Z'_2 respectively. Now by induction hypothesis there exists a companion Z_i in $G^*[(V_i \cup X)]$, where $i \in \{1, 2\}$. As in the forward direction we can show that $Z_1 \cup Z_2$ is the required set Z.

Having shown the stronger property, it is now easy to show that G has a connected dominating set of size k if and only if G' has a connected dominating set of size k. By Lemma 8, we know that if there is a connected dominating set Q of G size k, then we have a connected dominating set Q' containing at most $2h$ fully dominated vertices with respect to $g \in \mathcal{F}$ that is canonically associated with Q. Let Z be such a connected dominating set of size k of G. We know that there exists a companion Z' of the same size that dominates all the vertices of $N[Z]$ present in G'. Since G' is the induced subgraph of G and Z is the dominating set we have that $N[Z] \cap V(G') = V(G')$ and hence Z' is the dominating set of size at most k for G'. Now we show that Z' is also connected. Observe that Z contains a vertex from S and every connected component of $G'[Z']$ contains a vertex from S. Now by the property that for every connected component C' of $G'[Z']$ there exists a connected component C in $G[Z]$ such that $C \cap S \subseteq C' \cap S$. This implies that Z' is also connected, as Z is connected and hence $Z \cap S \neq \emptyset$ is present in every connected component of $G'[Z']$. Similarly given a connected dominating set Z' of G' of size at most k we can show that its companion Z is a connected dominated set of G of size at most k. This completes the proof of the first part of the lemma.

Now we argue about the size and the properties of D. By our construction of D it is clear that after we remove D from the graph every connected component contains at most $9d^2$ vertices from S. Let us fix a connected component, say C. The graph which after partitioning gives rise to C contains at most $\mathcal{O}(d^2) = \mathcal{O}(1)$ vertices from S and hence the treewidth of this parent graph and hence its induced subgraph, C, is $\mathcal{O}(1)$. This implies that the set D is indeed the desired treewidth deletion set. The size of D is estimated via the following recursive formula

$$\mu(|D|) \leq \max_{1/3 \leq \alpha \leq 2/3} \left\{ \mu\left(\alpha|S| + d\sqrt{|S|}\right) + \mu\left((1-\alpha)|S| + d\sqrt{|S|}\right) + d\sqrt{|S|} \right\}$$

Using simple induction one can show that the above solves to $\mathcal{O}(|S|)$. See for an example [25, Lemma 2]. Hence we conclude that $|D| = \mathcal{O}(|S|) = \mathcal{O}(k)$. This completes the proof of the lemma. $\square$