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Domination games and treewidth

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Abstract

The r-domination search game on graphs is a game-theoretical approach to several graph and hypergraph parameters including treewidth and hypertree width. The task is to identify the minimum number of cops sufficient to catch the visible and very fast robber. To catch the robber, at least one of the cops should be within at most r edges from him. We give a constant factor approximation algorithm that for every fixed r and graph H, computes the minimum number of cops required to capture the robber in the r-domination game on graphs excluding H as a minor.

1 Introduction

Graph searching games are played on graphs (in this paper all graphs are undirected and simple), where a group of searchers (cops) tries to catch a fugitive (robber). In the model known as a node searching, the robber stands on a vertex of the graph and at any moment he can run (arbitrarily fast) to another vertex along a path in the graph. However he is not allowed to run through a vertex occupied by a cop. Each cop at any time either stands on a vertex or is in a helicopter (that is, is temporarily removed from the game). The aim of cops is to capture the robber by landing a cop via helicopter on a vertex occupied by the robber and the robber's objective is to avoid capture. There are two variants of the game, which were studied intensively depending on if cops posses complete information on the current location of the robber (i.e. the robber is visible to cops) [20] or when the cops have no such information (i.e. the robber is invisible) [13, 15, 16, 18]. It appeared that the visible case is strongly related to the fundamental graph parameter called treewidth, and that the invisible case is related to the pathwidth of a graph. We refer to [10] for further references on graph searching.

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In the domination or r-domination versions of graph searching, cops' aim is more modest: instead of capturing the robber their task is that during the game at least one of the cops will be at distance r or closer to the robber (the distance is in the standard shortest path metric of the graph). Another interpretation of the r-domination game is that cops have more power and can catch the robber not just by occupying his vertex but by entering the r-neighborhood of the robber's position. As in the case with classical search games, there are two versions of the game, one with visible [14] and the other with invisible robber [2, 9].

In this paper we study the r-domination search game with visible robber. This game is a natural generalization of the search game introduced by Seymour and Thomas [20], and thus for $r = 0$, $k + 1$ cops can capture the robber on a graph G if and only if the treewidth of G is at most k. For $r = 1$ the r-domination searching is a generalization of the Marshals and Robber game played on hypergraphs. The Marshals and Robber game is game-theoretic approach to hypertree-width, another intensively studied parameter within context of several applications [1, 11, 12]. Kreutzer and Ordyniak have shown in [14] that computing the minimum number of marshals required to win on a hypegraph can be reduced to the computations of the the 1-domination cop number of a specific graph. Thus the r-domination search game is an interesting model serving as a general game-theoretical model for a number of fundamental parameters.

However, there is a price one has to pay for such a generality—the computational complexity of the game changes drastically with even small changes of r, like from 0 to 1. For example, computing the treewidth of a graph, and thus the minimum number of cops for $r = 0$ is fixed parameter tractable [3], while for $r = 1$ the problem becomes $W[1]$-hard [14]. The main explanation of this behavior is that for $r \geq 1$ the problem is not closed under the operation of taking a graph minor, and thus most of the powerful techniques from Graph Minor Theory cannot be applied.

In this paper we give several algorithms computing the minimum number of cops required to win in the r-dominating search games for graphs excluding some fixed graph as a minor. For planar graphs, and more generally, for graphs excluding some fixed apex graph as a minor, we show for every fixed $r \geq 1$, the r-domination cop number of a graph G can be approximated within the constant multiplicative factor by the treewidth of G. Since there are constant factor approximation algorithms for the treewidth of such graphs, our results yields the approximation algorithms. While techniques from Graph Minor Theory do not seem to be applicable for $r \geq 1$, we use the recent results from [8] on contractions in graphs. This type of arguments cannot be extended further. For example, it is well known that the treewidth of an $n \times n$ grid is n. If we add one universal vertex v adjacent to all vertices of the grid, we obtain a graph of treewdith $n+1$. This graph also does not contain a complete graph on 6 vertices K_6 as a minor. However, for

$r = 1$, one cop placed on v is at distance at most one to every vertex of the graph, and one cop can always win. Thus on graphs excluding some fixes graph H as a minor, the r-domination cop number of a graph cannot be approximated by its treewidth of G. Our approximation algorithm computing the r-domination cop number of an H-minor free graph G is technical and its main idea is to find in polynomial time a specific subgraph of G and to construct from this subgraph of G another H-minor free graph, which treewidth up to constant factor sandwiches the cop number of G. It can be noticed that similar approach can be used for a number of graph or hypergraph parameters including fractional and generalized hypertree-width [7].

2 Definitions and preliminaries

We consider finite undirected graphs without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$, or simply by V and E if this does not create confusion. If $U \subseteq V(G)$ then the subgraph of G induced by U is denoted by $G[U]$. For a vertex v, the set of vertices which are adjacent to v is called the *(open) neighborhood* of v and denoted by $N_G(v)$. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. For $U \subseteq V(G)$, we put

$$N_G[U] = \bigcup_{v \in U} N_G[v].$$

The *distance* $\mathrm{dist}_G(u, v)$ between vertices u and v in a connected graph G is the number of edges in a shortest (u, v)-path in G. For a positive integer r, $N^r_G[v] = \{u \in V(G) \colon \mathrm{dist}_G(u, v) \leq r\}$ and for $U \subseteq V(G)$,

$$N^{(r)}_G[U] = \bigcup_{v \in U} N^{(r)}_G[v].$$

Whenever there is no ambiguity we omit the subscripts. If $U \subseteq V(G)$ (resp. $u \in V(G)$ or $E \subset E(G)$ or $e \in E(G)$) then $G - U$ (resp. $G - u$ or $G - E$ or $G - e$) is the graph obtained from G by the removal of vertices of U (resp. of vertex u or edges of E or of the edge e). For graphs G_1 and G_2, $G_1 \cap G_2$ ($G_1 \cup G_2$ respectively) is the graph with the vertex set $V(G_1) \cap V(G_2)$ and the edge set $E(G_1) \cap E(G_2)$ (the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$ respectively).

Cops and Robber game. We consider a generalization of the Helicopter Cops and Robber game introduced by Seymour and Thomas [20]. Let G be a connected undirected graph, and let r be a nonnegative integer. The distance r domination search game is played by two players: cop and robber.

The cop-player has a team of cops who attempt to capture the robber. The robber stands on a vertex of the graph, and can at any time run at great speed to any other vertex along a path of the graph. However, he is not permitted to run through a vertex at distance at least r from a vertex occupied by a cop. Each cop at any time either stands on a vertex or is in a helicopter (that is, is temporarily removed from the game). The aim of the cop-player is to capture the robber by landing a cop via helicopter on a vertex at distance at least r from the vertex occupied by the robber, and the robber's objective is to avoid capture. The robber can see the movements of helicopters and may run to a new vertex before the helicopter lands. We consider the variant of the game when the robber is visible. For an integer r and a graph G, we denote by $\mathbf{dc}_r(G)$ the minimum number of cops sufficient for the cops to win on graph G and call it the *r-domination cop number*.

Black White Domination Cops and Robber game. It is convenient for us to consider an annotated variant of the Domination Cop and Robber Game. In this variant the robber can only occupy vertices from a prescribed set and move along edges of the subgraph induced by this set. Let G be a graph, and let B (*black* vertices) and W (*white* vertices) be a partition of the set of vertices $V(G)$. We assume that $B \neq \emptyset$ (the set W can be empty). We call a graph with a given partition B and W the *black and white* graph. By $B(G)$ and $W(G)$ we denote the set of black vertices and the set of white vertices of G respectively.

Let G be a black and white graph, and let r be a nonnegative and k be positive integers. We define the *position* of the cops as a set of vertices $U \subset V(G)$, $|U| \leq k$, occupied by the cops (clearly, we can assume that each vertex is occupied by at most one cop). We denote by $\mathcal{U}_k$ the set of all possible position of the cops. The *position of the robber* is a vertex of $B(G)$ occupied by him. The *strategy of cops* is a function $C\colon \mathcal{U}_k \times B(G) \to \mathcal{U}$. Calls of this function correspond to moves of cops. If the cops have a position U and the robber has a position v, then the cops move to the position $U' = C(U, v)$: cops remain on the vertices of $U \cap U'$, the cops from $U \setminus U'$ are removed from the graph, and then cops are placed on vertices of $U' \setminus U$. Respectively, we define the *strategy of the robber* as a function $R\colon \mathcal{U}_k \times \mathcal{U}_k \times B(G) \to B(G)$ such that if $v' = R(U, U', v)$ then there is a (v, v')-path P in $G[B(G)]$ with the property $V(P) \cap N_G^{(r)}[U \cap U'] = \emptyset$. Calls of this function corresponds to moves of the robber. If the cops are moving from a position U to U' and the robber occupies v, then he moves from v to $v' = R(U, U', v)$.

The game is defined by the (possibly infinite) sequence of pairs from $\mathcal{U} \times B(G)$ $(U_0, v_0), (U_1, v_1), \ldots$, where $U_0 = \emptyset$, $U_i = C(U_{i-1}, v_{i-1})$ and $v_i = R(U_{i-1}, U_i, v_{i-1})$. This sequence is finite if there is $m \geq 1$ such that $v_m \in N_r(U_m)$. In this case we say that the cop-player wins, otherwise it is said that the robber-player wins.

A strategy of cops is called the *winning* strategy, if cop-players wins for any choice of a strategy by the robber-player. The *r-domination cop number* $\mathbf{dc}_r(G, B(G))$ is the minimum number of cops k such that they have a winning strategy of cops. For $W(G) = \emptyset$, we let $\mathbf{dc}_r(G) = \mathbf{dc}_r(G, V(G))$. The *winning* strategy for the robber is a strategy such that the robber wins against any strategy of cops. In what follows we usually give informal descriptions of strategies of the cops and the robber by describing their movements.

It is easy to make the following observation.

Proposition 1. *For any nonnegative integers r, r', $r \leq r'$, and any black and white graph G, $\mathbf{dc}_r(G, B(G)) \geq \mathbf{dc}_{r'}(G, B(G))$.*

Notice also the following.

Proposition 2. *Let G be a black and white graph, $X \subseteq W(G)$ and $N_G^{(r)}[X] \cap B(G) = \emptyset$. Then $\mathbf{dc}_r(G, B(G)) = \mathbf{dc}_r(G \setminus X, B(G))$.*

The complexity of the r-Domination Cops and Robbers problem was considered in [14]. This problem asks for given nonnegative integer r, positive integer k and a given connected graph G, whether $\mathbf{dc}_r(G) \leq k$.

Proposition 3 ([14])**.** *For $r \geq 1$, the r-*Domination Cops and Robbers *problem*

- NP*-hard,*
- W[2]*-hard when parameterized by k, and*
- *there is a constant c such that there is no polynomial time algorithm that approximates the r-domination cop number for n-vertex graphs within a multiplicative factor $c \cdot \log n$, unless* P $\neq$ NP*.*

Contractions and minors. Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e, i.e. the endpoints x and y are replaced by a new vertex v_{xy} which is adjacent to the old neighbors of x and y (except x and y). We say that x and y are contracted to v_{xy}, and we also sometimes say that x is contracted to y (or y to x). For a black and white graph G, it is assumed that if $x \in B(G)$ or $y \in B(G)$ then the obtained vertex v_{xy} is black and $v_{x,y}$ is white otherwise. A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G.

It can be observed that the r-domination cop number is a contraction-closed parameter.

Proposition 4. *Let H be a contraction of a connected black and white graph G, and let B' be the set of black vertices of H. For any $r \geq 0$, $\mathbf{dc}_r(H, B(H)) \leq \mathbf{dc}_r(G, B(G))$.*

It is said that a graph H is a *minor* of a graph G if H is the contraction of some subgraph of G.

We say that a graph G is *H-minor-free* when it does not contain H as a minor. We also say that a graph class $\mathcal{G}$ is *H-minor-free* (or, excludes H as a minor) when all its members are H-minor-free.

An *apex graph* is a graph obtained from a planar graph G by adding a vertex and making it adjacent to some of the vertices of G. A graph class $\mathcal{G}$ is *apex-minor-free* if $\mathcal{G}$ excludes a fixed apex graph H as a minor.

Grids and their triangulations. Let k and r be positive integers where $k, r \geq 2$. The $(k \times r)$*-grid* is the Cartesian product of two paths of lengths $k-1$ and $r-1$ respectively. A vertex of a $(k \times r)$-grid is a *corner* if it has degree 2. Thus each $(k \times r)$-grid has 4 corners. A vertex of a $(k \times r)$-grid is called *internal* if it has degree 4, otherwise it is called *external*.

A *partial triangulation* of a $(k \times r)$-grid is a planar graph obtained from a $(k \times r)$-grid (we call it the *underlying grid*) by adding edges. Let us note that there are many non-isomorphic partial triangulations of on underlying grid. For each partial triangulation of a $(k \times r)$-grid we use the terms *corner*, *internal* and *external* referring to the corners, the internal and the external vertices of the underlying grid.

We define Γ_k (see Figure 1)as the following (unique, up to isomorphism) triangulation of a plane embedding of the $(k \times k)$-grid. Let Γ be a plane embedding of the $(k \times k)$-grid such that all external vertices are on the boundary of the external face. We triangulate internal faces of the $(k \times k)$-grid such that all the internal vertices have degree 6 in the obtained graph and all non-corner external vertices have degree 4, and then one corner of degree two is joined by edges with all vertices of the external face (we call this corner *loaded*).

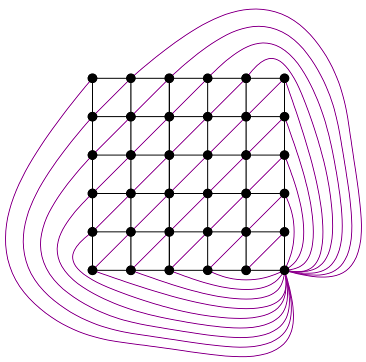

Figure 1: The graph Γ_6.

We need the following axiliar claim.

Lemma 1. *Let G be a black and white graph such that i) $W(G)$ is an independent set, ii) for any $v \in W(G)$, $N_G(v)$ induces a clique in G, and iii) $B(G)$ induces Γ_k for some $k > 1$. Then for any $r \geq 0$, $\mathbf{dc}_r(G, B(G)) \geq \frac{k-1}{2r+1}$.*

Proof. We prove that if $p < \frac{k-1}{2r+1}$ then the robber has a winning strategy on G against p cops. Let $B(G) = \{(i,j) | 0 \leq i \leq k-1, 0 \leq j \leq k-1\}$. It is assumed that the vertices are numbered in such a way that (i,j) and (i',j') are adjacent in the underlying grid for Γ_k if and only if $i = i'$ and $|j - j'| = 1$ or $|i - i'| = 1$ and $j = j'$. Denote by X_i the set of vertices $\{(i,j) | 0 \leq j \leq k-1\}$ for $i \in \{0, \ldots, k-1\}$, and let $Y_j = \{(i,j) | 0 \leq i \leq k-1\}$ for $j \in \{0, \ldots, k-1\}$. Let also $\mathcal{U}_p$ be the set of all subset of $V(G)$ with at most p elements (i.e. $\mathcal{U}_p$ is the set of all possible positions of p cops). Notice that for any $U \in \mathcal{U}_p$, there are $i(U), j(U) \in \{0, \ldots, k-1\}$ such that $N_G^{(r)}[U] \cap X_{i(U)} = \emptyset$ and $N_G^{(r)}[U] \cap Y_{j(U)} = \emptyset$. We define the robber's strategy R as follows: for any $U, U' \in \mathcal{U}_p$ and each $(i,j) \in B(G)$, $R(U, U', (i,j)) = (i(U'), j(U'))$. It remains to note that if $i = i(U)$ and $j = j(U)$ then $Z = X_i \cup Y_j \cup X_{i(U')} \cup Y_{j(U)}$ induces a connected subgraph in Γ_k and $N_{\Gamma_k}^{(r)}[U \cap U'] \cap Z = \emptyset$. Therefore R is a winning strategy for the robber. □

Treewidth. A *tree decomposition* of a graph G is a pair $(\mathcal{X}, T)$ where T is a tree with nodes $\{1, \ldots, m\}$ and $\mathcal{X} = \{X_i \mid i \in V(T)\}$ is a collection of subsets of $V(G)$ (called *bags*) such that:

1. $\bigcup_{i \in V(T)} X_i = V(G)$;

2. for each edge $\{x, y\} \in E(G)$, $\{x, y\} \subseteq X_i$ for some $i \in V(T)$, and

3. for each $x \in V(G)$ the set $\{i \mid x \in X_i\}$ induces a connected subtree of T.

The *width* of a tree decomposition $(\{X_i \mid i \in V(T)\}, T)$ is $\max_{i \in V(T)} \{|X_i| - 1\}$. The *treewidth* of a graph G denoted $\mathbf{tw}(G)$ is the minimum width over all tree decompositions of G.

It is well known that Seymour and Thomas [20] established a close connection between treewidth and graph searching.

Proposition 5 ([20]). *For any connected graph G, $\mathbf{dc}_0(G) = \mathbf{tw}(G) + 1$.*

Surfaces. A *surface* Σ is a compact 2-manifold without boundary (we always consider connected surfaces). Whenever we refer to a Σ-*embedded graph* G we consider a 2-cell embedding of G in Σ. To simplify notations, we do not distinguish between a vertex of G and the point of Σ used in the drawing to represent the vertex or between an edge and the line representing it. We also consider a graph G embedded in Σ as the union of the points corresponding to its vertices and edges. That way, a subgraph H of G can be seen as a graph H, where $H \subseteq G$. Recall that $\Delta \subseteq \Sigma$ is an open (resp. closed) disc if it is homeomorphic to $\{(x,y) : x^2 + y^2 < 1\}$ (resp. $\{(x,y) : x^2 + y^2 \leq 1\}$). The *Euler genus* of a non-orientable surface Σ is equal to the non-orientable genus $\tilde{g}(\Sigma)$ (or the crosscap number). The *Euler genus* of an orientable surface Σ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ. We refer to the book of Mohar and Thomassen [17] for more details on graphs embeddings. The *Euler genus* of a graph G (denoted by $\mathbf{eg}(G)$) is the minimum integer γ such that G can be embedded on a surface of the Euler genus γ.

3 The r-domination cop number for apex-minor-free graphs

We prove here that the r-domination cop number of an apex-minor-free graphs can be approximated by its treewidth.

Theorem 1. *Let r be a nonnegative integer and let H be an apex graph. Then for any connected graph G excluding H as a minor, it holds that $\mathbf{dc}_r(G) - 1 \leq \mathbf{tw}(G) \leq c_{H,r} \cdot \mathbf{dc}_r(G)$ where $c_{H,r}$ is a constant depending only on H and r.*

Proof. By Proposition 5, it is sufficient to prove this theorem for $r > 0$. The first inequality follows immediately from Propositions 1 and 5. The proof of the second inequality is based on the results established in [8].

Proposition 6 ([8])**.** *For every apex graph H, there is $c_H > 0$ such that every connected H-minor-free graph of treewidth at least $c_H \cdot k$ contains Γ_k as a contraction.*

By this proposition and Proposition 4, it remains to prove that there is a constant c_r (which depend only on r) such that $k \leq c_r \cdot \mathbf{dc}_r(\Gamma_k)$, but it follows immediately from Lemma 1. □

4 The r-domination cop number for H-minor-free graphs

The following theorem is the main result of this paper.

Theorem 2. *Let r be a positive integer and H be a graph. There is a polynomial time algorithm that given a connected graph G excluding H as a minor returns a $c_{H,r}$-factor approximation of $\mathbf{dc}_r(G)$, where $c_{H,r}$ is a constant depending only on H and r.*

The remaining part of this section is devoted to the proof of Theorem 2.

4.1 Graph minor theorem and preliminary results

The proof of Theorem 2 is based the Excluded Minor Theorem from the Graph Minor theory. Before we state it, we need some definitions.

Definition 1 (Clique-Sums)**.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs, and $k \geq 0$ an integer. For $i = 1, 2$, let $W_i \subseteq V_i$, form a clique of size h and let G'_i be the graph obtained from G_i by removing a set of edges (possibly empty) from the clique $G_i[W_i]$. Let $F : W_1 \to W_2$ be a bijection between W_1 and W_2. We define the h-*clique-sum *of G_1 and G_2, denoted by $G_1 \oplus_{h,F} G_2$, or simply $G_1 \oplus G_2$ if there is no confusion, as the graph obtained by taking the union of G'_1 and G'_2 by identifying $w \in W_1$ with $F(w) \in W_2$, and by removing all the multiple edges. The image of the vertices of W_1 and W_2 in $G_1 \oplus G_2$ is called the* join *of the sum.*

Note that some edges of G_1 and G_2 are not edges of G, since it is possible that they had edges which were removed by clique-sum operation. Such edges are called *virtual* edges of G. We remark that $\oplus$ is not well defined; different choices of G'_i and the bijection F could give different clique-sums. A sequence of h-clique-sums, not necessarily unique, which result in a graph G, is called a *clique-sum decomposition* of G.

Definition 2 (h-nearly embeddable graphs)**.** *Let Σ be a surface with cycles $C_1, \ldots, C_h$ such that each cycle C_i is the border of a open disc Δ_i in Σ. A graph G is h-*nearly embeddable *in Σ, if G has a subset Z of size at most h, called* apices, *such that there are (possibly empty) subgraphs $R_0, \ldots, R_h$ of $G \setminus Z$ such that*

- *i)* $G' = G \setminus Z = R_0 \cup R_1 \cup \cdots \cup R_h$
- *ii)* *G_0 is embeddable in Σ such that $V(R_0) \cap \bigcup_{i=1,\ldots,h} \Delta_i = \emptyset$, we fix an embedding of G_0,*
- *iii)* *graphs $R_1, \ldots, R_h$ (called* vertices*) are pairwise disjoint,*
- *iv)* *for $1 \leq i \leq h$, let $U_i := \{u_{i_1}, \ldots, u_{i_{m_i}}\} = V(R_0) \cap V(R_i)$, R_i has a path decomposition (B_{ij}), $1 \leq j \leq m_i$, of width at most h such that*
  - *a)* *for $1 \leq i \leq h$ and for $1 \leq j \leq m_i$ we have $u_j \in B_{ij}$,*
  - *b)* *for $1 \leq i \leq h$, we have $V(R_0) \cap C_i = \{u_{i_1}, \ldots, u_{i_{m_i}}\}$ and the points $u_{i_1}, \ldots, u_{i_{m_i}}$ appear on C_i in this order (either if we walk clockwise or anti-clockwise).*

The following proposition is known as the Excluded Minor Theorem [19] and is the cornerstone of Robertson and Seymour's Graph Minors theory.

Proposition 7 ([19])**.** *For every non-planar graph H, there exists an integer c_H, depending only on H, such that every graph excluding H as a minor can be obtained by c_H-clique-sums from graphs that can be c_H-nearly embedded in a surface Σ in which H cannot be embedded. Moreover, while applying each of the clique sums, at most three vertices from each summand other than apices and vertices in vortices are identified.*

Let G be a graph and let $\mathcal{C} = \{K_1, \dots, K_r\}$ be a collection of cliques of G. Then we define $\mathcal{K}(G, \mathcal{C})$ as the set containing every of graph that can be constructed from G by adding, for each $i = 1, \dots, r$, a new vertex v_i, joining it by edges with all vertices of K_i and removing some edges of K_i.

We already mentioned Proposition 6 which says that for every apex graph H, there is $c_H > 0$ such that every connected H-minor-free graph of treewidth at least $c_H \cdot k$ contains Γ_k as a contraction. The proof of this statement (and other results of [8]), was based on the following proposition (the proof is implicit in [8]).

Proposition 8 ([8])**.** *For any surface Σ, there is a constant $c >$ with the following property: Let G be a connected graph h-nearly embedded in Σ without apices, where $G = F_0 \cup F_1 \cup \cdots \cup F_h$ with vortices $F_1, \dots, F_h$. Suppose that the cycles $C_1, \dots, C_h$ are borders of non-intersecting open discs $\Delta_1, \dots, \Delta_h$ in Σ, and*

i) F_0 is embedded in Σ in such a way that $V(F_0) \cap \bigcup_{i=1,\dots,h} \Delta_i = \emptyset$,

ii) for $1 \le i \le h$, $V(F_0) \cap V(G_i) \subseteq C_i$.

Let also $\mathcal{K} = \{K_1, \dots, K_r\}$ be a collection of cliques in G and let $\hat{G} \in \mathcal{K}(G, \mathcal{K})$. Then if $\mathbf{tw}(G) \ge c \cdot k$, there is an open disk Δ in Σ with border a cycle C of F_0 where $\Delta \cap \bigcup_{i=1,\dots,h} \Delta_i = \emptyset$ and such that $\hat{G}$ can be contracted to Γ_k in a way that

a) each vertex v_i is contracted to some vertex of K_i for $1 \le i \le r$,

b) all vertices of G which do not lay on Δ are contracted to the loaded corner of Γ_k,

c) for each face of $G \cap \Delta$, the vertices on the boundary of it are contracted to vertices laying on the boundary of one face (triangle) of Γ_k.

Now we are ready to describe our approximation of the r-domination cop number for H-minor-free graphs. Let H be a graph. We assume that H is not planar (otherwise we can apply Theorem 1). Let G be a graph that does not contain H as a minor. Let $G_1 \oplus \cdots \oplus G_m$ be a c_H-clique-sum decomposition of G. Denote by Z_i the set of apices of G_i. For $i = 1, \dots, m$, we define $F(G_i)$ as the graph obtained if we consider $G_i - N_G^{(r)}[Z_i]$ and then we remove each virtual edge $\{x, y\}$ of G_i such that all (x, y)-paths in G whose internal vertices are not in $V(G_i)$ are intersected by $N_G^{(r)}[Z_i]$.

The proof of Theorem 2 is based on the following theorem.

Theorem 3. *Let r be a positive integer, let H be a graph and let G be a connected graph excluding H. Let also $k = \max\{\mathbf{tw}(F(G_i)) \mid i = 1, \dots, r\}$. Then, $\mathbf{dc}_r(G) - c_{H,r} \leq k \leq c_{H,r} \cdot \mathbf{dc}_r(G)$ where $c_{H,r}$ is a constant depending only on H and r.*

It is known that by the result of Demaine et al. [5] a clique-sum decomposition can be obtained in time $O(n^c)$ for some constant c which depends only from H (see also [4]). As far as we constructed summands G_i, the construction of graphs $F(G_i)$ can be done in polynomial time. Moreover, since the algorithm of Demaine et al. provides C_H-nearly embeddings of these graphs, it is possible to use it to construct a polynomial constant factor approximation algorithm for the computation of $\mathbf{tw}(F(G_i)$.

The remaining part of this section contains the proof of Theorem 3

4.2 Proof of the lower bound

We start with the proof of the first inequality.

Claim 1. $\mathbf{dc}_r(G) \leq k + 2c_H + 1$.

Proof. Let $p = k + 2c_H + 1$. We describe a winning strategy for p cops on H.

The clique-sum decomposition $G = G_1 \oplus G_2 \oplus \cdots \oplus G_m$ can be considered as a tree decomposition $(\mathcal{X}, T)$ of G for some tree T with nodes $\{1, 2, \dots, m\}$ with the bags $X_i = V(G_i)$, i.e. the vertex sets of the summands are the bags of this decomposition. The idea behind the winning strategy for cops is to "chase" the robber in the graph along $m + 1$ decompositions: one is induced by the clique-sum decomposition and others are tree decompositions of $F(G_i)$.

Let us note that the definition of $F(G_i)$ yields the following: if $x, y \in V(F(G_i)$, and there is a (x, y)-path in $G - N_G^{(r)}[Z_i]$ with all inner vertices not in $F(G_i)$, then $\{x, y\}$ is an edge of $F(G_i)$. (Indeed, if $\{x, y\}$ is an edge of G, then it is also an edge of $F(G_i)$. If $\{x, y\} \notin E(G)$ but such a path exits, then $\{x, y\}$ is a virtual edge in G_i and by the definition of $F(G_i)$, such an edge also is an edge of $F(G_i)$.)

For $i \in \{1, 2, \dots, m\}$, let $(\mathcal{X}_i, T^{(i)})$ be a tree decomposition of $F(G_i)$ of width at most k. We assume that trees T and $T^{(1)}, T^{(2)}, \dots, T^{(m)}$ are rooted trees with roots r and $r_1, r_2, \dots, r_m$ correspondingly. For a node x of $T^{(i)}$, we denote by $X_x^{(i)}$ the bag of the tree-decomposition of $F(G_i)$ which corresponds to x.

For a node $i \in V(T)$ and its parent j (in T), we define $S = V(G_i) \cap V(G_j)$. (If $i = r$ then we put $S = \emptyset$.) By the definition of the clique-sum, $|S| \leq$

c_H. Assume that at most c_H cops are already placed on the all vertices S. Assume also that the robber occupies some vertex of G_i or $G_{i'}$ where i' is a descendant of i in T. We put at most c_H cops on Z_i. Clearly, the robber now cannot stay on the vertices of $N_G^{(r)}[Z_i]$. Note also he can not go through the separator S in G since all vertices of S are occupied by the cops. Now cops start to "chase" the robber in $G_i - N_G^{(r)}[Z_i]$ along $T^{(i)}$. We put at most $k+1$ cops on the vertices of the bag $X_{r_i}^{(i)}$ in the tree-decomposition of $F(G_i)$. Assume now that all vertices of some bag $X_x^{(i)}$ for $x \in V(T^{(i)})$ are occupied by the cops, and that the robber can only occupy (or move to) vertices of $X_y^{(i)}$ where y is a child of x in $T^{(i)}$ or he can only occupy vertices of $X_{y'}^{(i)}$ where y' is a descendant of y in $T^{(i)}$. Then we remove cops from $X_x^{(i)} \setminus X_y^{(i)}$ and place cops on all vertices of $X_y^{(i)} \setminus X_x^{(i)}$. This maneuver can be done by making use of at most $k+1$ cops. We put $x = y$ and repeat this operation until the robber is "pushed" out of $V(G_i) \setminus N_G^{(r)}[Z_i]$.

Let p be a child of i in T such that the robber now can occupy only the vertices of G_p or $G_{p'}$ where p' is a descendant of p in T. Let $S' = V(G_i) \cap V(G_p)$. Since $|S'| \leq c_H$, we have that at most c_H cops can be moved to S' from S and, after that, all other cops can be removed from G.

We apply the described strategy of the cops starting from $i = r$ until the robber is captured in some leaf-node of T. For every node of T we have used at most C_H cops to occupy apices, at most c_H cops to occupy the vertices of the clique-sum, and at most $k+1$ cops to push the robber out of G_i. Thus in total at most $2c_H + k + 1$ cops have a winning strategy on G. □

4.3 Proof of the upper bound

Now our aim is to prove the second inequality.

Claim 2. *There is a constant $c_{H,r}$ such that $k \leq c_{H,r} \cdot \mathbf{dc}_r(G)$.*

Proof. Assume that $k = \mathbf{tw}(F(G_i))$ for some $1 \leq i \leq m$, and denote $F = F(G_i)$. Assume that F is connected (otherwise let F be a component of $F(G_i)$ with treewidth k). Consider a component of $G - N_G^{(r)}[Z_i]$ which contains vertices of $V(F)$, denote by $B(G)$ the set of its vertices and let $W(G) = V(G) \setminus B(G)$. Clearly, $\mathbf{dc}_r(G) \geq \mathbf{dc}_r(G, B(G))$. By Proposition 2, $\mathbf{dc}_r(G, B(G)) \geq \mathbf{dc}_r(G - Z_i, B(G))$. Also using this proposition we can assume that $G' = G - Z_i$ is connected (otherwise vertices of components of $G - Z_i$ which do not contain $B(G)$ can be removed, since they are at least $(r+1)$-distant from vertices of $B(G)$). Now we contract all edges $\{x, y\}$ of G' such that either $x, y \in W(G')$ or $x, y \in B(G') \setminus V(F)$. Denote the obtained graph by $\hat{G}$, and let $\hat{F}$ be the subgraph of $\hat{G}$ induced by $B(\hat{G})$. Note that $W(\hat{G})$ is an independent set of $\hat{G}$. By Proposition 4, $\mathbf{dc}_r(G', B(G)) \geq \mathbf{dc}_r(\hat{G}, B(\hat{G}))$. Hence $\mathbf{dc}_r(G) \geq \mathbf{dc}_r(\hat{G}, B(\hat{G}))$.

Recall that all summands in the clique-sum decomposition of G can be c_H-nearly embedded in some surface Σ in which H cannot be embedded in a such way that while applying each of the clique sums, at most three vertices from each summand other than apices and vertices in vortices are identified. Hence G_i can be embedded in Σ in this way, and we fix the embedding assuming that $G_i = R_0 \cup R_1 \cup \cdots \cup R_{C_H}$ with vortices $R_1, \ldots, R_{c_H}$. Suppose that cycles $C_1, \ldots, C_h$ are borders of non-intersecting open discs $\Delta_1, \ldots, \Delta_{c_H}$ in Σ such that

- R_0 is embedded in Σ in such a way that $V(R_0) \cap \bigcup_{j=1,\ldots,h} \Delta_j = \emptyset$,
- for $1 \leq j \leq c_H$, $V(R_0) \cap V(R_j) \subseteq C_j$.

The graph F is a subgraph of G_i. Therefore, the c_H-nearly embedding of G_i induces c_H-nearly embedding of F such that

i) $F = F_0 \cup F_1 \cup \cdots \cup F_{C_H}$ and $F_j = R_j \cap F$ for $0 \leq j \leq c_H$ (some intersections may be empty),

ii) $F_1, \ldots, F_{c_H}$ are vortices,

iii) F_0 is embedded in Σ in such a way that $V(F_0) \cap \bigcup_{j=1,\ldots,h} \Delta_j = \emptyset$,

iv) for $1 \leq j \leq c_H$, $V(F_0) \cap V(F_j) \subseteq C_j$.

Recall that other summands in the clique-sum decomposition of G are joined to G_i by means of clique-sum operations. It follows that there is a collection of cliques $\mathcal{C} = \{K_1, \ldots, K_r\}$ in F such that $\hat{F}$ is a graph in $\mathcal{K}(G, \mathcal{C})$. By Proposition 8, there exists a constant $c > 0$, depending only on Σ, such that if $\mathbf{tw}(F) \geq c \cdot p$, then there is a open disk Δ in Σ with border a cycle C of F_0, where $\Delta \cap \bigcup_{j=1,\ldots,C_H} \Delta_j = \emptyset$ and such that $\hat{F}$ can be contracted to Γ_p in a way that

a) each vertex v_j is contracted to some vertex of K_j for $1 \leq j \leq r$,

b) all vertices of F which do not lay on Δ and all vertices laying on C are contracted to the loaded corner of Γ_p,

c) for each face of $F \cap \Delta$, vertices which lay on the boundary of it are contracted to vertices laying on the boundary of one face (triangle) of Γ_p.

Let $p = \lfloor \frac{\mathbf{tw}(F)}{c} \rfloor = \lfloor \frac{k}{c} \rfloor$. We consider $\hat{G}$ and contact in it the edges that are contracted in $\hat{F}$ in order to construct Γ_p. Denote the obtained black and white graph by Q. Now the set $B(Q)$ induces Γ_p and $W(Q)$ is independent. By Proposition 4, $\mathbf{dc}_r(\hat{G}, B(\hat{G})) \geq \mathbf{dc}_r(Q, B(Q))$.

Recall that embedding of F is induced by the embedding of G_i. Particularly, $\Delta \cap G_i$ is a plane graph embedded in the disk Δ and the boundary of the disk is a cycle in F. It follows that after all contractions each vertex of $W(Q)$ is adjacent to a clique in $B(Q)$. Therefore it is possible to apply Lemma 1 and conclude that $\mathbf{dc}_r(Q, B(Q)) \geq \frac{p-1}{2r+1}$ and $\mathbf{dc}_r(G) \geq \frac{p-1}{2r+1}$.

It remains to note that $(2r+1)\mathbf{dc}_r(G) - 1 \geq p \geq \frac{k}{c} - 1$ and let $c_{H,r} = c(2r+1)$. $\square$

5 Conclusion

Anything to conclude?

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