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Contractions of planar graphs in polynomial time*

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Abstract. We prove that for every graph H, there exists a polynomial-time algorithm deciding if a planar graph can be contracted to H. We introduce contractions and topological minors of embedded (plane) graphs and show that a plane graph H is an embedded contraction of a plane graph G, if and only if, the dual of H is an embedded topological minor of the dual of G. We show how to reduce finding embedded topological minors in plane graphs to solving an instance of the disjoint paths problem. Finally, we extend the result to graphs embeddable in an arbitrary surface.

Keywords: planar graph, dual graph, contraction, topological minor

1 Introduction

An *edge contraction* of an edge e in a graph is the graph obtained by removing e, identifying its two endpoints, and eliminating parallel edges that may appear. Some basic properties of contractions are collected in [21]. Formally, for an edge e with endpoints u and w, the contraction of e, denoted by G/e, is the graph with vertex set $V(G/e) = V(G) \setminus \{u, w\} \cup \{v_{uw}\}$ and edge set

$$\begin{aligned} E(G/e) = E \setminus \{\ \{x, y\} \in E\ :\ x \in \{u, w\},\ y \in V\ \} \\ \cup \{\ \{v_{uw}, x\}\ :\ \{x, u\} \in E\ \vee\ \{x, w\} \in E\ \}. \end{aligned}$$

A graph H is a *contraction* of a graph G (or G is *contractible* to H) if H can be obtained from G by a sequence of edge contractions.

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1.1 Previous work.

The problem of checking whether a graph is a contraction of another has already attracted some attention. In this subsection we briefly survey known results.

Stars and triangle-free patterns. Perhaps the first systematic study of contractions was undertaken by Brouwer and Veldman [2]. Here are two main theorems from that paper.

Theorem 1 (Theorem 3 in [2]). *A graph G is contractible to $K_{1,m}$ if and only if G is connected and contains an independent set S of m vertices such that $G - S$ is connected.*

In particular, a graph is contractible to P_3 if and only if it is connected and is neither a cycle nor a complete graph. The theorem also allows to detect, in polynomial time, if a graph is contractible to $K_{1,m}$. It suffices to enumerate over all sets S with m independent vertices and check if the graph $G - S$ is connected. This gives an $|V(G)|^{\mathcal{O}(m)}$ algorithm, which is polynomial for every fixed m.

Theorem 2 (Theorem 9 in [2]). *If H is a connected triangle-free graph other than a star, then contractibility to H is* NP*-complete.*

Hence, checking if a graph is contractible to P_4 or C_4 is NP-complete. More generally, it is NP-complete for every bipartite graph with at least one connected component that is not a star.

Patterns up to 5 vertices. The research direction initiated by Brouwer and Veldman was continued by Levin, Paulusma, and Woeginger [10], [11]. Here is the main result established in these two papers.

Theorem 3 (Theorem 3 in [10]). *Let H be a connected graph on at most 5 vertices. If H has a dominating vertex, then contractibility to H can be decided in polynomial time. If H does not have a dominating vertex, then contractibility to H is* NP*-complete.*

However, the existence of a dominating vertex in the pattern H is not enough to ensure that contractibility to H can be decided in polynomial time. A pattern on 69 vertices for which contractibility to H is NP-complete was exhibited in [9].

When the pattern is part of input. Looking at contractions to fixed pattern graphs is justified by the following theorems proved by Matoušek and Thomas in [13].

Theorem 4 (Theorem 4.1 in [13]). *The problem of deciding, given two input graphs G and H, whether G is contractible to H is* NP*-complete even if we impose one of the following restrictions on G and H:*

(i) *H and G are trees of bounded diameter,*
(ii) *H and G are trees all whose vertices but one have degree at most 5.*

Theorem 5 (Theorem 4.3 in [13]). *For every fixed k, the problem of deciding, given two input graphs G and H, whether G is contractible to H is NP-complete even if we restrict G to partial k-trees and H to k-connected graphs.*

The authors also proved a positive result.

Theorem 6 (Theorem 5.14 in [13]). *For every fixed Δ, k, there exists an $\mathcal{O}(|V(H)|^{k+1} \cdot |V(G)|)$ algorithm to decide, given two input graphs G and H, whether G is contractible to H, when the maximum degree of H is at most Δ and G is a partial k-tree.*

Cyclicity. The *cyclicity* of a graph G is defined as the largest integer k for which G is contractible to a cycle on k vertices. Exact values for some graphs and lower and upper bounds for some classes of graphs are given by Hammack in [8]. He also presented a polynomial-time algorithm for computing cyclicity of a planar graph.

Non-recursive classes closed under taking of contractions. Another type of containment relation close to contractions – induced minors – was studied by Matoušek, Nešetřil, and Thomas in [12]. A graph is an *induced minor* of another if the first is a contraction of an induced subgraph of the latter. The authors of [12] prove (Theorem 1.8) that there exists a class closed under taking of induced minors which is non-recursive (i.e. there is no algorithm to test the membership in this class). Clearly, a class of graphs closed under taking of induced minors is also closed under taking of contractions (and induced subgraphs) so we restate their result in the following way.

Theorem 7 (Theorem 1.18 in [12]). *There exists a non-recursive class of graphs closed under taking of contractions (and induced subgraphs).*

Wagner's Conjecture for contractions. The statement usually referred to as *Wagner's Conjecture* (although Klaus Wagner insisted he had never posed it, as explained in [4], p.355) is that for any infinite sequence $G_0, G_1, \ldots$ of graphs, there is a pair i, j such that $i < j$ and G_i is a minor of G_j. The proof of Wagner's Conjecture is one of the highlights of the Graph Minors project [20].

A contraction version of Wagner's Conjecture was considered by Demaine, Hajiaghayi, and Kawarabayashi in [3]. They disproved this version showing the following.

Theorem 8 (Theorem 31 in [3]). *There is an infinite sequence $G_0, G_1, \ldots$ of connected graphs such that, for every pair i, j ($i \neq j$), G_i is not a contraction of G_j.*

However, the authors also proved that the conjecture holds when the graphs in the sequence are required to be trees, or triangulated planar graphs, or 2-connected embedded outerplanar graphs.

1.2 Our contribution.

A graph is a *minor* of another if the first is a contraction of a subgraph of the latter. Graph Minors is a celebrated project by Robertson and Seymour that is considered to be an important part of modern Graph Theory. One of the algorithmic consequences of Graph Minors is that, for every graph H, there exists a cubic-time algorithm deciding whether the input graph contains H as a minor [18]; and another, for every class of graphs closed under taking minors, there exists a cubic-time algorithm deciding whether the input graph belongs to this class [20].

While graph minors are well-studied both from combinatorial and algorithmic point of view, relatively little is known about graph contractions which are rather close to graph minors. Algorithmically, they are much less tractable compared to minors. As mentioned in the previous subsection, there are graphs for which it is NP-complete to decide if the input graph is contractible to them; and there are non-recursive classes of graphs, that are closed under taking of contractions, where is there is no algorithm deciding whether an input graph belongs to this class.

In this work we show, for a large class of inputs – graphs embeddable on surfaces, how to decide in polynomial time if a fixed graph is a contraction of the input. We focus on the case of planar graphs – graphs embeddable in the plane (or, equivalently, on the sphere). All the essential ingredients of the solution are already present when the input is constrained to be planar. In Section 5 we show how to extend the algorithm from graphs embeddable in the plane to graphs embeddable in an arbitrary surface.

The key idea is to introduce embedded versions of contractions and topological minors for plane graphs. Those embedded containment relations differ from usual contractions and topological minors in respecting the embedding. We show that a plane graph H is an embedded contraction of a plane graph G, if the dual of H is an embedded topological minor of the dual of G.

To use this duality algorithmically, we need to show that embedded topological minors can also be found in polynomial time. This is done by reducing the problem of finding an embedded topological minor to solving an instance of the disjoint path problem that, in turn, can be solved in cubic time due to the main algorithmic result of Graph Minors [19].

2 Definitions

Basics. We consider both simple graphs and multigraphs. We do not allow any of them to have loops. When there is no ambiguity, we say "a graph" and mean a simple graph or multigraph. We say "a multigraph" when we want to stress that multiple edges are allowed and "a simple graph" if they are not allowed. For a (multi)graph G, let $V(G)$ be its vertex set and $E(G)$ its edge (multi)set. Plane graphs are always assumed to be drawn on the unit sphere and their edges are arbitrary polygonal arcs (not necessarily straight line segments). For notation not defined here, we refer the reader to the monograph [4].

The dual of a plane graph G will be denoted by G^*. Notice that there is a one-to-one correspondence between the edges of G and the edges of G^*. We keep the convention that e^* is the edge of G^* corresponding to edge e of G.

A graph H is a *subdivision* of a graph G, when H can be obtained from G by subdividing its edges (i.e., replacing edges by paths). A graph H is a *topological minor* of a graph G if H is a subdivision of a subgraph of G. Vertices of degree ≥ 3 in a subdivision are called *branch vertices*.

In this paper we consider the algorithmic problem of contracting an input graph G to a fixed graph H. Below we will assume that both H and G are connected. This can be done without loss of generality. If G and H are not connected, we consider contracting different connected components of G to different connected components of H. Since H is fixed, this will only contribute to a constant (in $|V(G)|$) factor in the computational complexity of the algorithm.

Embeddings. In this work, we only need to distinguish between essentially different embeddings of a planar graph. This motivates the following definition.

Two plane graphs G and H are *combinatorially equivalent* ($G \simeq H$) if there exists a homeomorphism of the unit sphere (in which they are embedded) which transforms one into the other. The relation of being combinatorially equivalent is reflexive, symmetric and transitive, and thus an equivalence relation. Let $\mathcal{G}$ be the class of all plane graphs isomorphic to a planar graph G and let us consider the quotient set $\mathcal{G}/\simeq$. The equivalence classes (i.e., the elements of the quotient set) can be thought of as *embeddings*. In fact, we will work with embeddings but for simplicity, we will pick a plane graph representative for each embedding.

Homotopic edges and thin graphs. Two edges of a plane graph are *homotopic edges* if they together bound a 2-face. Following [1], a *thin graph* is a plane multigraph without homotopic pairs of edges. In other words, if there are two parallel edges e, f between a pair of vertices in a thin graph, each of the two open regions defined by the union of e and f must contain at least one vertex. It turns out that thin plane multigraphs cannot have more edges than simple plane graphs.

Lemma 1 (Lemma 5 in [1]). *If G is a thin graph, then $|E(G)| \leq 3|V(G)| - 6$.*

Embedded containment relations. An *embedded contraction* of an edge e of a plane graph G is a plane graph G' that is obtained by homeomorphically mapping the endpoints of e in G to a single vertex without any edge crossings and recursively removing one of two homotopic edges, if a graph has such a pair. Notice that there are many embedded contractions of an edge of a plane graph G but they are all combinatorially equivalent.

An *embedded dissolution* of a vertex v of degree 2 in a plane graph G is an embedded contraction of one of the two edges v is incident with in G.

Let G and H be two plane graphs. We say that H is an embedded contraction of G ($H \leq_{ec} G$), if H is combinatorially equivalent to a graph that can be obtained from G by a series of embedded contractions. We say that H is an *embedded topological minor* of G ($H \leq_{etm} G$), if H is combinatorially equivalent to a graph that can be obtained from G by a series of vertex and edge deletions, and embedded dissolution of vertices of degree 2.

3 Contractions vs topological minors

Lemma 2. *Let H and G be two thin graphs and H^*, G^* their respective duals.*

$$H \leq_{ec} G \iff H^* \leq_{etm} G^*$$

Proof. Let G be a thin graph and e an edge of G. Let $G_{/e}$ be an embedded contraction of e in G. Notice that ${G_{/e}}^*$ is isomorphic to a plane graph obtained from G^* by deleting e^* and recursively applying embedded dissolutions of vertices of degree 2. (Homotopic faces in a plane graph correspond to vertices of degree 2 in its dual.) Let us also note that $G_{/e}$ is a plane graph with no homotopic edges.

If H can be obtained from G by a series of embedded contractions, then H^* can be obtained from G^* by a series of edge deletions and embedded dissolutions of vertices of degree 2. Hence, if H is an embedded contraction of G, then H^* is an embedded topological minor of G^*. This proves the forward implication.

For the backward implication, suppose that H^* is an embedded topological minor of G^*; that is H^* can be obtained from G^* by a sequence of vertex deletions, edge deletions, and embedded dissolutions of vertices of degree 2.

Let us notice that removing a vertex v in a thin plane graph can be simulated by removing all but two edges incident to v, then applying an embedded dissolution to v and removing the new edge. (No vertices of degree 1 will be created since the graph was thin and can be made thin after recursively applying embedded dissolution to vertices of degree 2.) Hence, a sequence of vertex deletions, edge deletions, and embedded dissolutions of vertices of degree 2 can be replaced by a sequence of edge deletions and embedded dissolutions of vertices of degree 2. The sequence can be rearranged and split into groups – every group consists of an edge removal and appropriate embedded dissolutions of vertices of degree 2. (When a graph has no homotopic edges, all vertices of its dual are of degree ≥ 3.)

Each group of operations in a plane graph corresponds to an embedded edge contraction in its dual. The sequence of operations that transform G^* into H^* corresponds to a sequence of embedded edge contractions that brings G into H. Hence, the backward implication. □

A simple planar graph H is a *pattern* of a planar multigraph H', if $V(H) = V(H')$ and two vertices are adjacent in H if and only if they are adjacent in H'. In other words, a pattern of the multigraph is the graph obtained by replacing multiple with single edges. Let $\mathcal{C}(H)$ be a maximal set of thin plane multigraphs whose pattern is H such that they are all combinatorially different.

Lemma 3. *For every planar graph H, the set $\mathcal{C}(H)$ is finite.*

Proof. Combinatorially different embeddings of a planar multigraph H are determined by cyclic orders of neighbors on vertices. There might be infinitely many embeddings of a planar multigraph. However, we are confined to thin plane graphs only and each will have at most $3|V(H)| - 6$ edges by Lemma 1. Hence, the number of possible different cyclic orderings is finite. $\square$

Theorem 9. *Let H and G be simple planar graphs and $\widetilde{G}$ a plane graph isomorphic to G. Then,*

$$H <_c G \iff \exists\, \widetilde{H} \in \mathcal{C}(H) \text{ such that } \widetilde{H} <_{ec} \widetilde{G}.$$

Proof. For the backward implication, let H be the pattern of some $\widetilde{H} \in \mathcal{C}(H)$. ($H$ is a simple graph.) Let us notice that if $\widetilde{H}$ is combinatorially equivalent to an embedded contraction of $\widetilde{G}$, then $\widetilde{G}$ (and its abstract graph G) are isomorphic to a contraction of H.

For the forward implication, let us assume that $H <_c G$. There exists a sequence of edge contractions that brings G into H. Let us apply the same sequence as a sequence of embedded contractions to $\widetilde{G}$ and call the resulting graph $\widetilde{T}$. From the definition of embedded contraction, $\widetilde{T}$ is thin. Notice that its pattern is H. From the choice of $\mathcal{C}(H)$, there exists $\widetilde{H} \in \mathcal{C}(H)$ that is combinatorially equivalent to $\widetilde{T}$. $\square$

A direct consequence of Lemma 2 and Theorem 9 is the following corollary.

Corollary 1. *Let H and G be planar graphs and $\widetilde{G}$ a plane graph isomorphic to G. Then,*

$$H <_c G \iff \exists\, \widetilde{H} \in \mathcal{C}(H) \text{ such that } \widetilde{H}^* <_{etm} \widetilde{G}^*.$$

4 Embedded topological minors and the algorithm

In this section, we reduce the problem of finding an embedded topological minor to the the problem of finding a collection of disjoint paths in a graph. Here is a result from Graph Minors we will need later.

Theorem 10 ([19]). *There exists an algorithm that given a graph G and k pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices of G decides whether there are k vertex-disjoint paths $P_1, \ldots, P_k$ in G such that P_i joins s_i and t_i, for all $i = 1, \ldots, k$, and if so, finds them. The algorithm runs in time $\mathcal{O}(|V(G)|^3)$.*

This result can be used to determine whether a graph H is a topological minor of the input graph. The idea is to choose a set of $|V(H)|$ branch vertices in G and turn it into an instance of disjoint paths problem with $|E(H)|$ paths, each edge of H should correspond to one path. The disjoint path algorithm from Theorem 10 needs to be run for every choice of branch vertices. The running time of the algorithm deciding whether a graph H is a topological minor of the input graph G is then $\mathcal{O}(|V(G)|^{|V(H)|})$. Note that the running time is indeed polynomial as we assume H to be a fixed graph. Whether there is an algorithm for this problem that runs in time $f(|V(H)|)\cdot|V(G)|^{\mathcal{O}(1)}$ is one of the major open problems in the theory of parameterized complexity, even when G is assumed to be planar. The reduction from embedded topological minors to disjoint paths is more complicated since we have to take into account the cyclic order of paths incident with the vertex. Below we show how this can be done.

Theorem 11. *For every plane graph H, there exists a polynomial-time algorithm that given a plane graph G decides if H is an embedded topological minor of G, and if so, finds the subgraph which is a subdivision of H.*

Proof. A k-star is a connected bipartite graph whose one part has one vertex (the *center*) and the other part has k vertices (the *leaves*). A star is a graph that is a k-star for some k. A *labelled star* is a subgraph of G that is a star and whose center is labelled with a vertex from $V(H)$ and whose leaves are labelled with different edges from $E(H)$ that are incident with v in H. A labelled star Q is said to be *compatible with* $v \in V(H)$ if it is a $\deg(v)$-star, its center is labelled with v, and the cyclic ordering of the labels on the leaves of Q is the same as the cyclic ordering of the edges incident with v in H.

Let us fix an ordering $v_1,\ldots,v_{|V(H)|}$ of $V(H)$ and an ordering $e_1,\ldots,e_{|E(H)|}$ of $E(H)$. A *branching* is a $|V(H)|$-tuple $(Q_{v_i} : i = 1,\ldots,|V(H)|)$ such that Q_{v_i} for $i = 1,\ldots,|V(H)|$ is a labelled star compatible with v_i. A *good branching* is one in which no two centers of stars coincide. Let $\mathcal{Q}$ be the set of all different good branchings. Notice that $|\mathcal{Q}|$ is bounded by $|V(G)|^{\mathcal{O}(|V(H)|)}$.

For a branching from $\mathcal{Q}$ we define an instance of the disjoint path problem. We start with $|E(H)|$ pairs of terminals and later will be possibly removing some. For every $j = 1,\ldots,|E(H)|$, let $\{s_j,t_j\}$ is the two vertices of G that are labelled with e_j in the branching. We then remove from the set of pairs such $\{s_j,t_j\}$ that s_j and t_j are adjacent. We then remove all centers of stars from G.

Claim. H is an embedded topological minor of G if and only if there exists a branching from $\mathcal{Q}$ that defines a feasible disjoint path instance.

If H is an embedded topological minor of G, then the branching is given by the set of stars centered at the branch vertices of H whose edges incident with the center are those that belong to the model of H in G.

If there is a branching in $\mathcal{Q}$ that defines a feasible disjoint path instance, then the union of the disjoint paths and the stars in the branching give a model of an embedded topological minor of H in G.

Now we are ready to present an algorithm that for a fixed graph H decides if a plane graph G contains H as an embedded topological minor of G. First the algorithm constructs the set $\mathcal{Q}$. Then, for every branching from $\mathcal{Q}$ the algorithm constructs an instance of the disjoint paths problem and tests its feasibility.

The correctness of the algorithm is a direct consequence of the Claim. To see that the running time of the algorithm is polynomial in $|V(G)|$, notice that – as mentioned before – the cardinality of $\mathcal{Q}$ is bounded by $|V(G)|^{\mathcal{O}(|V(H)|)}$; building an instance of the disjoint paths problem out of a branching can be done in polynomial time; and testing feasibility of those instances can also be done in polynomial time by Theorem 10. □

Theorem 12. *For every graph H, there exists a polynomial-time algorithm that given a planar graph G decides whether H is a contraction of G, and if so finds a series of contractions transforming G into H.*

Proof. We can assume that both G and H are connected; otherwise, we consider contractions of different connected components of G to different connected components of H. We can also assume that H is planar since G can never be contracted to a non-planar graph.

First we embed G in the plane using the linear-time algorithm from [14]. Let $\widetilde{G}$ be this plane graph isomorphic to G and $\widetilde{G}^*$ its dual. For every graph H from $\mathcal{C}(H)$, test if $\widetilde{H}^*$ is an embedded topological minor of $\widetilde{G}^*$, using the algorithm from Theorem 11.

The correctness of the algorithm follows from Corollary 1 and Lemma 3. The fact that the algorithm runs in polynomial time follows from Lemma 3 and Theorem 11. □

5 Bounded genus graphs

In this section, we show how to extend our result from Theorem 12 to graphs on surfaces other then the plane. For terminology and notions related to graphs on surfaces, we refer the reader to the standard monograph [**?**].

Thin graphs on surfaces. We fix a surface Σ of Euler genus g and consider graphs embeddable in this surface. First, we notice that it is possible to extend the definition of thin graphs to graphs embedded in other surfaces. A *thin graph* is a multigraph embeddable in Σ without homotopic pairs of edges. Then, we observe that the proof of Lemma 1 in [1] uses the Euler's formula only. Since all faces of a thin graph are incident with at least 3 edges, one can derive the following counterpart of Lemma 1 for graphs of genus g.

Lemma 4. *If G is a thin graph embeddable on a surface of Euler genus g, then*

$$|E(G)| \leq 3 \cdot (|V(G)| + g) - 6.$$

It is not difficult to see that this leads to the following equivalent of Lemma 3.

Lemma 5. *For every graph H embeddable on a surface of Euler genus g, the set $\mathcal{C}(H)$ is finite.*

Containment relations on surfaces. The same definitions of embedded contractions and embedded topological minors that we provided for the plane stay valid for graphs embeddable in Σ. The main reason is that surfaces are locally homeomorphic to the plane and our definitions are also local. Also the dual graph of a graph embedded in Σ is well defined. It is easy to check that the proofs of Lemma 2 and Theorem 9 hold in case of any surface, not only the plane. A direct consequence of this is that Corollary 1 has a version for surfaces of higher genus.

Corollary 2. *Let H and G be graphs embeddable on a surface of Euler genus g and $\widetilde{G}$ a graph embedded on a surface of genus g and isomorphic to G. Then,*

$$H <_c G \iff \exists\, \widetilde{H} \in \mathcal{C}(H) \text{ such that } \widetilde{H}^* <_{etm} \widetilde{G}^*.$$

Algorithm. It is also possible to adapt the proof of Theorem 11 to graphs embedded in Σ. Combining these together, we can prove the following theorem.

Theorem 13. *For every integer $g \geq 0$ and a graph H, there exists a polynomial-time algorithm that given a graph embeddable on a surface of Euler genus g decides whether H is a contraction of G, and if so finds a series of contractions transforming G into H.*

6 Discussion

We conclude with a number of remarks and a conjecture.

Solution via dual. We prove our result by investigating what operation in the dual graph G^* corresponds to contractions in G. We want to mention that the same approach proved to be successful in studying maximum cuts in planar graphs. A maximum cut of a graph is its maximum bipartite subgraph. Orlova and Dorfman [17] and independently Hadlock [7] noticed that a (maximum) bipartite subgraph in G is an (maximum) Eulerian subgraph in G^*. Maximum Eulerian subgraphs can be found in polynomial time, therefore they proved that the maximum cut problem can be solved in polynomial-time in planar graphs.

Non-recursive classes of planar graphs. We prove in this paper that for every graph H, there exists an algorithm that given a planar input graph G decides whether H is a contraction of G. To complement this result we would like to note that there are classes of planar graphs closed under taking of contractions that are non-recursive. A closer look at the proof of Theorem 7 in [12] reveals that the graphs in the non-recursive class from the theorem are in fact planar (and even have no K_5^- minor). We state it more formally.

Corollary 3. *There exists a non-recursive class of planar graphs closed under taking contractions.*

Cyclicity in bounded genus graphs. Cycles have a unique embedding into the plane (up to combinatorial equivalence) and the dual of a cycle on k vertices is the multigraph with two vertices and k parallel edges. Therefore computing cyclicity of a plane graph is equivalent to solving the maximum flow problem in the dual for every pair of vertices (as the source and sink).

Hammack, also using the maximum flow problem, showed how to compute cyclicity of a planar graph [8]. Our result for bounded genus graphs allow to extend this result to graphs embeddable in an arbitrary suface.

Complexity of topological minor checking. The total running time of our algorithm is $\mathcal{O}(|V(G)|^{|V(H)|})$. (We will focus on planar graphs but the discussion also holds for classes of graphs of bounded genus.) The computational complexity heavily depends on the complexity of computing topological minors. As mentioned before, the best algorithm for deciding whether a fixed graph H is a topological minor of the input graph G runs in time $\mathcal{O}(|V(G)|^{|V(H)|})$. If we consider the problem from the parameterized complexity point of view, that is asking for an $f(|V(H)|) \cdot |V(G)|^{O(1)}$ step algorithm (i.e. classify it in the complexity class FPT when parameterized by the size of H), this is not satisfying. (We refer to [5, 16, 6] for more information on parameterized complexity.)

Whether topological minor checking belongs to the class FPT is not known even when the input graph is restricted to be planar. It is conceivable that an FPT algorithm for this problem would also give an FPT algorithm for embedded topological minor, and consequently, for contractions in surface embeddable graphs.

However, we believe that the existence of such an algorithm is rather unlikely and the problem is $\mathsf{W}[1]$-hard. (W-hardness is a technical notion in the theory of parameterized complexity that makes is rather impossible that a problem belongs in FPT.) We state this as a conjecture.

Conjecture. *For a graph H, the problem of deciding whether H is a topological minor of a (planar) input graph G is $\mathsf{W}[1]$-hard, when parameterized by $|V(H)|$.*

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