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Containment relations in split graphs

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ABSTRACT

A graph containment problem is to decide whether one graph can be modified into some other graph by using a number of specified graph operations. We consider edge deletions, edge contractions, vertex deletions and vertex dissolutions as possible graph operations permitted. By allowing any combination of these four operations we capture the following ten problems: testing on (induced) minors, (induced) topological minors, (induced) subgraphs, (induced) spanning subgraphs, dissolutions and contractions. A split graph is a graph whose vertex set can be partitioned into a clique and an independent set. Our results combined with existing results settle the parameterized complexity of all ten problems for split graphs.

1. Introduction

There are several natural and elementary algorithmic problems to test whether the structure of some graph H shows up as a *pattern* within the structure of another graph G. Before we give a survey of existing work and present our results, we first state our terminology.

Terminology. We consider undirected finite graphs with no loops and with no multiple edges. We denote the vertex set and edge set of a graph G by V_G and E_G, respectively. If no confusion is possible, we may omit subscripts. We refer the reader to Diestel [7] for any undefined graph terminology.

Let $G = (V, E)$ be a graph. We write $G[U]$ to denote the subgraph of G *induced* by $U \subseteq V$, i.e., the graph on vertex set U and an edge between any two vertices if and only if there is an edge between them in G. We say that U is a *clique* if there is an edge in G between any two vertices of U, and U is an *independent set* if there is no edge in G between any two vertices of U. Two sets $U, U' \subseteq V$ are called *adjacent* if there exist vertices $u \in U$ and $u' \in U'$ such that $uu' \in E$. A vertex v is a *neighbor* of u if $uv \in E$. The *degree* of a vertex u is its number of neighbors.

Let $e = uv$ be an edge in a graph G. The *edge contraction* of e removes u and v from G, and replaces them by a new vertex adjacent to precisely those vertices to which u or v were adjacent. If one of the two vertices, say u, has exactly two neighbors which in addition are nonadjacent, then we call this operation the *vertex dissolution* of u.

The total number of different combinations of the graph operations vertex deletion, edge deletion, edge contraction and vertex dissolution is 16. However, four of these combinations are not possible, because a vertex dissolution is a special case of an edge contraction. Hence, if we allow edge contractions, then we must also allow vertex dissolutions. Ten of the 12

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Table 1
Known containment relations in terms of the graph operations.

Containment relation	VD	ED	EC	VDi	Decision problem
Minor	Yes	Yes	Yes	Yes	MINOR
Induced minor	Yes	No	Yes	Yes	INDUCED MINOR
Topological minor	Yes	Yes	No	Yes	TOPOLOGICAL MINOR
Induced topological minor	Yes	No	No	Yes	INDUCED TOPOLOGICAL MINOR
Contraction	No	No	Yes	Yes	CONTRACTIBILITY
Dissolution	No	No	No	Yes	DISSOLUTION
Subgraph	Yes	Yes	No	No	SUBGRAPH ISOMORPHISM
Induced subgraph	Yes	No	No	No	INDUCED SUBGRAPH ISOMORPHISM
Spanning subgraph	No	Yes	No	No	SPANNING SUBGRAPH ISOMORPHISM
Isomorphism	No	No	No	No	GRAPH ISOMORPHISM

remaining combinations lead to known graph containment relations. This is shown in Table 1 where VD, ED, EC, and VDi stand for ''vertex deletions'', ''edge deletions'', ''edge contractions'', and ''vertex dissolutions'', respectively. For instance, we say that a graph H is an induced minor of a graph G if H can be obtained from G by a sequence of graph operations that allow vertex deletions, vertex dissolutions and edge contractions, but no edge deletions. The corresponding decision problem, in which G and H form the ordered input pair (G, H), is called INDUCED MINOR. The other rows in Table 1 should be interpreted similarly. The remaining two combinations ''no yes yes yes'', and ''no yes no yes'', which are not in Table 1, are equivalent to minors and topological minors, respectively, if we allow an extra operation that removes isolated vertices. Finally, we note that a graph G is called a *subdivision* of a graph H if and only if H is a dissolution of G. In that case G can be obtained from H by a sequence of *edge subdivisions*; this operation removes an edge $e = uv$ from G and introduces a new vertex that is (only) adjacent to u and v.

Existing and new results. The problems in Table 1 except GRAPH ISOMORPHISM are easily seen to be NP-complete, as has been observed by Matoušek and Thomas [21] for all these problems except SPANNING SUBGRAPH ISOMORPHISM, which is not included in their analysis; this problem is NP-complete, because it contains as a special case the NP-complete problem that tests whether a graph has a Hamiltonian cycle. Therefore Matoušek and Thomas [21] put some restrictions on the ordered input pair (G, H). In particular, they showed that all problems from Table 1 except GRAPH ISOMORPHISM, DISSOLUTION and SPANNING SUBGRAPH ISOMORPHISM stay NP-complete when G is a partial 2-tree.

Another natural direction is to fix the graph H in an ordered input pair (G, H) and consider only the graph G to be part of the input. We indicate this by adding ''H-'' to the names of the decision problems. For any fixed H, the problems H-SUBGRAPH ISOMORPHISM, H-INDUCED SUBGRAPH ISOMORPHISM, H-SPANNING SUBGRAPH ISOMORPHISM, and H-GRAPH ISOMORPHISM can be solved in polynomial time by brute force. It is easy to see that H-DISSOLUTION is fixed-parameter tractable, when the order of H is the parameter; for completeness we show this in our paper. A celebrated result by Robertson and Seymour [23] states that H-MINOR and H-TOPOLOGICAL MINOR can be solved in cubic time and polynomial time, respectively, for every fixed graph H. The latter result has recently been improved by Grohe et al. [16] who showed that TOPOLOGICAL MINOR is fixed-parameter tractable, when the order of H is the parameter.

The computational complexity classification of the problems H-INDUCED MINOR, H-INDUCED TOPOLOGICAL MINOR and H-CONTRACTIBILITY is still open, although many partial results are known. Fellows et al. [9] give both polynomial-time solvable and NP-complete cases for the H-INDUCED MINOR problem. Lévêque et al. [18] do the same for the H-INDUCED TOPOLOGICAL MINOR problem. Polynomial-time solvable and NP-complete cases for the H-CONTRACTIBILITY problem can be found in a series of papers starting by Brouwer and Veldman [5] and followed by Levin et al. [19,20] and van't Hof et al. [24]. Because some of the open cases are notoriously difficult, special graph classes have been studied. Fellows et al. [9] showed that for every fixed graph H, the H-INDUCED MINOR problem can be solved in polynomial time on planar graphs. Also, the H-CONTRACTIBILITY problem is polynomial-time solvable for every fixed H on this graph class [17]. Fiala et al. [10] show that for every fixed H, the H-INDUCED TOPOLOGICAL MINOR problem can be solved in polynomial time on claw-free graphs.

A graph G is a *split graph* if G has a *split partition*, which is a partition of its vertex set into a clique C_G and an independent set I_G. Split graphs were introduced by Foldes and Hammer [12] in 1977 and have been extensively studied since then; see e.g. the monographs of Brandstädt et al. [4], or Golumbic [15]. Belmonte et al. [2] showed that for every fixed graph H, the H-CONTRACTIBILITY can be solved in polynomial time for split graphs.

We determine the parameterized complexity of the problems in Table 1 for split graphs. In particular, we answer a question of Belmonte et al. [2] regarding the parameterized complexity of the CONTRACTIBILITY problem for split graphs. Combining our work with previously known results leads to the following three theorems which we prove in Sections 3–5, respectively.

Theorem 1. *The problems* GRAPH ISOMORPHISM *and* DISSOLUTION *are* GRAPH-ISOMORPHISM-*complete for ordered pairs* (G, H) *where* G *and* H *are split graphs. The other 8 problems in Table 1 are* NP-*complete for such input pairs.*

Theorem 2. *For any fixed graph* H, *all problems in Table 1 can be solved in polynomial time for ordered pairs* (G, H) *where* G *is a split graph.*

Theorem 3. *The problems* Induced Minor, Induced Topological Minor, Contractibility *and* Induced Subgraph Isomorphism *are* W[1]*-complete for ordered pairs* (G, H) *where* G *and* H *are split graphs, and* $|V_H|$ *is the parameter. The other* 6 *problems from* Table 1 *are fixed-parameter tractable for ordered pairs* (G, H) *where* G *is a split graph, and* $|V_H|$ *is the parameter.*

2. Preliminaries

For some of our proofs the following global structure is useful. Let G and H be two graphs. An *H-witness structure W* is a vertex partition of a (not necessarily proper) subgraph of G into $|V_H|$ (nonempty) sets $W(x)$ called (*H-witness*) *bags*, such that

(i) each $W(x)$ induces a connected subgraph of G,
(ii) for all $x, y \in V_H$ with $x \neq y$, bags $W(x)$ and $W(y)$ are adjacent in G if x and y are adjacent in H.

In addition, we may require the following additional conditions:

(iii) for all $x, y \in V_H$ with $x \neq y$, bags $W(x)$ and $W(y)$ are adjacent in G *only* if x and y are adjacent in H,
(iv) every vertex of G belongs to some bag.

By contracting all bags to singletons we observe that H is a minor, induced minor, or contraction of G if and only if G has an H-witness structure such that conditions (i)–(ii), (i)–(iii), or (i)–(iv) hold, respectively. We note that G may have more than one H-witness structure with respect to the same containment relation.

Let G be a graph. The *incidence graph* of G is the bipartite graph with partition classes V and E and edges ue if and only if u is an end vertex of e.

Let G be a split graph with split partition (C_G, I_G). If C_G is a maximal clique, then we call the split partition (C_G, I_G) *maximal*. This means that there is no vertex in I_G that is adjacent to all vertices in C_G. For our purposes, maximal split partitions are very useful. Note that a (maximal) split partition does not have to be unique. If G has an H-witness structure for some graph H with split partition (C_H, I_H), then we call the bags corresponding to the vertices in C_H and I_H *clique bags* and *independent bags*, respectively. We observe that split graphs are closed under edge contractions, vertex deletions, and vertex dissolutions; they are not – in general – closed under edge deletions and edge subdivisions.

We finish this section by giving a short introduction into the theory of parameterized complexity. Here, we consider the problem input as a pair (I, k), where I is the main part and k is the parameter. A problem is *fixed-parameter tractable* if an instance (I, k) can be solved in time $f(k)n^c$, where f denotes a computable function, $n = |I|$ is the size of I, and c a constant independent of k. The class FPT is the class of all fixed-parameter tractable decision problems. Similar to the theory of NP-completeness, parameterized complexity offers a completeness theory that allows the accumulation of strong theoretical evidence that some parameterized problems are not fixed-parameter tractable. This completeness theory is based on a hierarchy of complexity classes W[1], W[2], ..., XP. For more background on these classes we refer to the monographs of Downey and Fellows [8], Flum and Grohe [11], and Niedermeier [22]. We only mention the following. The complexity class XP consists of parameterized decision problems Π such that for each instance (I, k) it can be decided in $f(k)|I|^{g(k)}$ time whether $(I, k) \in \Pi$, where f and g are computable functions depending only on the *parameter* k, and $|I|$ denotes the size of I. So XP consists of parameterized decision problems which can be solved in polynomial time if the parameter is a constant. Furthermore, the aforementioned classes form a chain $\mathsf{FPT} \subseteq \mathsf{W[1]} \subseteq \mathsf{W[2]} \subseteq \cdots \subseteq \mathsf{XP}$ where all inclusions are conjectured to be proper; only $\mathsf{FPT} \neq \mathsf{XP}$ is known [8,11].

A well-known technique to show that a parameterized problem Π is fixed-parameter tractable is to find a *reduction to a problem kernel*, i.e., to replace an instance (I, k) of Π with an instance (I', k') of Π (called a *problem kernel*) such that

(i) $k' \leq k$ and $|I'| \leq g(k)$ for some computable function g;
(ii) the reduction from (I, k) to (I', k') is computable in polynomial time;
(iii) (I, k) is a Yes-instance of Π if and only if (I', k') is a Yes-instance of Π.

The upper bound $g(k)$ is called the *kernel size*. A kernel is *polynomial* if the kernel size is polynomial in k. It is well known that a parameterized problem is fixed-parameter tractable if and only if it has a kernel (cf. [22]).

3. The proof of Theorem 1

The Graph Isomorphism problem stays Graph Isomorphism-complete when restricted to split graphs (cf. the survey of Booth and Colbourn [3]). One can use this result to show that the Dissolution problem is also Graph Isomorphism-complete for split graphs as follows. Subdividing an edge of a split graph results in a graph that is not a split graph unless the edges of the original split graph form a star. Now let G and H be two split graphs that form an instance of Graph Isomorphism. Then we may assume without loss of generality that the edges of G do not form a star; otherwise we can check if G is isomorphic to H in polynomial time. Hence, from the above observation, G is isomorphic to H if and only if G contains H as a dissolution.

Belmonte et al. [2] show that Contractibility is NP-complete for ordered pairs (G, H) where G is a split graph and H is a threshold graph; threshold graphs form a subclass of the class of split graphs. The problems Spanning Subgraph Isomorphism, Subgraph Isomorphism, Minor, Topological Minor are NP-complete for split graphs as shown in Theorem 4.

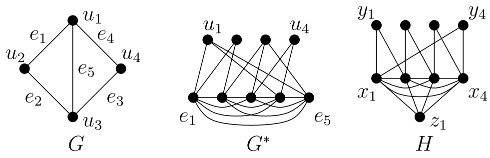

Fig. 1. An example of a graph H and a graph G^* constructed from a graph G.

The problems Induced Minor, Induced Topological Minor and Induced Subgraph Isomorphism are NP-complete for split graphs as shown in Theorem 5. The result for Induced Subgraph Isomorphism has already been proven by Damaschke [6] but also follows directly from our reduction for the other two problems.

Theorem 4. *The problems* Spanning Subgraph Isomorphism, Subgraph Isomorphism, Minor, *and* Topological Minor *are* NP-*complete for ordered pairs* (G, H) *where G and H are split graphs.*

Proof. We give a reduction from the Hamiltonian Cycle problem, which asks whether a graph has a *Hamiltonian cycle*, i.e., a spanning subgraph that is a cycle. This problem is well known to be NP-complete (cf. [13]).

Let $G = (V, E)$ be a graph on n vertices. We may assume without loss of generality that $|E| \geq n+1$. Let I be the incidence graph of G. From I we construct a graph G^* by adding an edge between any two vertices in E. Note that G^* is a split graph with $C_{G^*} = E$ and $I_{G^*} = V$. Recall that $|E| \geq n+1$. Then we can define a split graph H by $C_H = \{x_1, \ldots, x_n, z_1, \ldots, z_{|E|-n}\}$ and $I_H = \{y_1, \ldots, y_n\}$ such that y_i is (only) adjacent to x_i and x_{i+1} for $i = 1, \ldots, n-1$ and y_n is (only) adjacent to x_n and x_1. An example of G^* and H is shown in Fig. 1.

We only need to show that the following five statements are equivalent.

(1) G has a Hamiltonian cycle;
(2) G^* contains H as a spanning subgraph;
(3) G^* contains H as a subgraph;
(4) G^* contains H as a topological minor;
(5) G^* contains H as a minor.

"(1) ⇒ (2)" Suppose that G has a Hamiltonian cycle $u_1u_2 \cdots u_nu_1$. Let $e_i = u_iu_{i+1}$ for $i = 1, \ldots, n-1$ and $e_n = u_nu_1$. Then the spanning subgraph of G^* that has as edges all edges of C_{G^*} together with edges u_1e_1, u_1e_n and u_ie_i, u_ie_{i-1} for $i = 2, \ldots, n$ is isomorphic to H.
"(2) ⇒ (3)", "(3) ⇒ (4)" and "(4) ⇒ (5)" are true by definition.
"(5) ⇒ (1)" Suppose that G^* contains H as a minor. Let $\mathcal{W}$ be an H-witness structure of G^*. Because G^* and H have the same number of vertices, all bags in $\mathcal{W}$ consist of exactly one vertex. No clique bag can consist of a vertex of I_{G^*}, because such a vertex has degree at most $n-1$, whereas every clique bag is adjacent to $|E|-1 \geq n$ other clique bags. Hence, every clique bag consists of a vertex of C_{G^*}, and consequently, every independent bag consists of a vertex of I_{G^*}. Let e_{x_i} be the vertex of C_{G^*} that forms clique bag $W(x_i)$ for $i = 1, \ldots, n$, and let u_{y_i} be the vertex of I_{G^*} that forms independent bag $W(y_i)$ for $i = 1, \ldots, n$. Because $x_1y_1x_2y_2 \cdots x_ny_nx_1$ is a $2n$-vertex cycle in H, we find that $e_{x_1}u_{y_1}e_{x_2}u_{y_2} \cdots e_{x_n}u_{y_n}e_{x_1}$ is a $2n$-vertex cycle in G^*. Then, from the definition of G^*, we find that $u_{y_1}u_{y_2} \cdots u_{y_n}u_{y_1}$ is an n-vertex cycle in G, i.e., a cycle that contains all vertices of G. Hence, we have found a Hamiltonian cycle in G. This completes the proof of Theorem 4. □

Theorem 5. *The problems* Induced Subgraph Isomorphism, Induced Minor, *and* Induced Topological Minor *are* NP-*complete for ordered pairs* (G, H) *where G and H are split graphs.*

Proof. We give a reduction from the Clique problem, which asks whether a graph has a clique of size at least k. This problem is NP-complete (cf. [13]).

Let $G = (V, E)$ be a graph, and let k be some integer. For our purposes, we require that $k \geq 6$. From G we construct a split graph G^* as follows. Let I be the incidence graph of G. We change the subgraph of $I[V]$ into a complete graph by adding an edge between any two vertices in V. For each $e \in E$ we add a new vertex $\bar{e}$ to I that is adjacent to all vertices in V except to the two end-vertices of e. We let $\bar{E}$ denote the set of all vertices $\bar{e}$. This completes the construction of G^*. We observe that G^* is a split graph with $C_{G^*} = V$ and $I_{G^*} = E \cup \bar{E}$; also see Fig. 2.

We let H be the split graph with $C_H = \{x_1, \ldots, x_k\}$ and $I_H = \{y_{ij} \mid 1 \leq i < j \leq k\} \cup \{\bar{y}_{ij} \mid 1 \leq i < j \leq k\}$, such that every y_{ij} is (only) adjacent to x_i and x_j, and every $\bar{y}_{ij}$ is only adjacent to $C_H \setminus \{x_i, x_j\}$. See Fig. 2 for an example of H; for clarity we chose to depict the case $k = 5$, although we assume that $k \geq 6$ in our proof.

We only need to show that the following four statements are equivalent.

(1) G has a clique of size (at least) k;
(2) G^* has H as an induced subgraph;
(3) G^* has H as an induced topological minor;
(4) G^* has H as an induced minor.

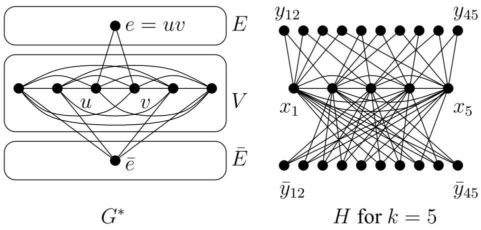

Fig. 2. The construction of the graph G^* and the graph H for $k = 5$.

"(1) ⇒ (2)" Suppose G has a clique $K = \{u_1, \ldots, u_k\}$ of size k. Then the subgraph of G^* induced by

$$K \cup \{e \mid e = u_iu_j \text{ for some } i, j\} \cup \{\bar{e} \mid e = u_iu_j \text{ for some } i, j\}$$

is isomorphic to H.
"(2) ⇒ (3)" and "(3) ⇒ (4)" are true by definition.
"(4) ⇒ (1)" Suppose that G^* has H as an induced minor. Then there exists an H-witness structure $\mathcal{W}$ of G^* that satisfies conditions (i)–(iii). We start by proving the following claim.

Claim 1. *For every y_{ij} there exists some $e \in E$ such that $W(y_{ij}) = \{e\}$ and $W(\bar{y}_{ij}) = \{\bar{e}\}$.*

We prove Claim 1 as follows. Consider a pair i, j with $1 \leq i < j \leq k$. We first consider bag $W(y_{ij})$ and then bag $W(\bar{y}_{ij})$.

In order to obtain a contradiction, suppose that $W(y_{ij})$ does not consist of exactly one vertex from E. Because E is independent, $W(y_{ij})$ cannot be a subset of E of at least two vertices. This means that $W(y_{ij})$ must contain a vertex $w \in \bar{E} \cup V$. By construction, $W(y_{ij})$ is adjacent to exactly two clique bags, namely $W(x_i)$ and $W(x_j)$. Because $k \geq 6$, we then find that $W(y_{ij})$ is not adjacent to at least four clique bags. Because these bags are mutually adjacent and $E \cup \bar{E}$ is independent, at least three of these four clique bags must contain a vertex from V. If $w \in V$ then w is adjacent to a vertex in all three of them, because V is a clique in G^*. If $w \in \bar{E}$, then w is adjacent to a vertex in at least one of them. This is not possible. We conclude that $W(y_{ij}) = \{e\}$ for some $e \in E$, as desired.

To complete the proof of Claim 1, we must show that $W(\bar{y}_{ij}) = \{\bar{e}\}$. Suppose that $W(\bar{y}_{ij}) \neq \{\bar{e}\}$. We first consider the case in which $W(\bar{y}_{ij})$ contains a vertex in V. By construction, $W(\bar{y}_{ij})$ is adjacent to all but two clique bags, namely $W(x_i)$ and $W(x_j)$. Because V is a clique, $W(x_i)$ and $W(x_j)$ cannot contain a vertex from V. Then $W(x_i)$ and $W(x_j)$ only contain vertices from $E \cup \bar{E}$. Because $E \cup \bar{E}$ is independent, we then find that $W(x_i)$ and $W(x_j)$ are not adjacent. However, this cannot happen, because $W(x_i)$ and $W(x_j)$ are clique bags. We conclude that $W(\bar{y}_{ij})$ contains no vertices from V. Because $E \cup \bar{E}$ is independent, this means that $W(\bar{y}_{ij})$ consists of exactly one vertex $w \in E \cup \bar{E}$.

Suppose that $w \in E$. Because $k \geq 6$, $W(\bar{y}_{ij})$ is adjacent to at least four bags. This is not possible, because w has degree 2. Hence, $w \notin E$. This implies that $w \in \bar{E}$. Let $e = uv$. Because $W(y_{ij}) = \{e\}$, we find that one end-vertex of e, say u, belongs to $W(x_i)$, whereas the other one, v, belongs to $W(x_j)$. Because $\bar{e}$ is the only vertex in $\bar{E}$ that is adjacent to neither u nor v, we find that $w = \bar{e}$, as desired. This finishes the proof of Claim 1.

Due to Claim 1, we may write $W(y_{ij}) = \{e_{ij}\}$ and $W(\bar{y}_{ij}) = \{\bar{e}_{ij}\}$ for $1 \leq i < j \leq k$, where each e_{ij} is a vertex of G^* that corresponds to an edge in G. We also need the following claim.

Claim 2. *For all $1 \leq i < j \leq k$, the bags $W(x_i)$ and $W(x_j)$ each contain exactly one vertex of V, which is an end-vertex of e_{ij}, and possibly one or more vertices of $E \cup \bar{E}$.*

We prove Claim 2 as follows. Let $1 \leq i < j \leq k$, and let $e_{ij} = uv$. Because e_{ij} is only adjacent to u and v in G^*, and $W(y_{ij}) = \{e_{ij}\}$ is (only) adjacent to $W(x_i)$ and $W(x_j)$, we may without loss of generality assume that $u \in W(x_i)$ and $v \in W(x_j)$. In order to obtain a contradiction, suppose that one of these bags, say $W(x_i)$, contains some other vertex $w \in V$. Then $W(x_i)$ and $W(\bar{y}_{ij}) = \{\bar{e}_{ij}\}$ are adjacent, because w and $\bar{e}_{ij}$ are adjacent. However, this is not possible because by construction $\bar{y}_{ij}$ is only adjacent to $C_H \setminus \{x_i, x_j\}$. Hence we have proven Claim 2.

Due to Claim 2, there are k vertices $u_1, \ldots, u_k$ in $(W(x_1) \cup \cdots \cup W(x_k)) \setminus (E \cup \bar{E})$ that belong to V. Claim 2 also tells us that any two vertices u_i, u_j are adjacent in G. Hence, we have found a clique in G of size k. This completes the proof of Theorem 5. □

4. The proof of Theorem 2

Recall that for any fixed graph H, the problems H-Subgraph Isomorphism, H-Induced Subgraph Isomorphism, H-Spanning Subgraph Isomorphism, and H-Graph Isomorphism can be solved for general graphs in polynomial time by brute force, and that Robertson and Seymour [23] showed that for any fixed graph H, the problems H-Minor and H-Topological Minor can be solved in cubic time and polynomial time, respectively, for general graphs. Also recall that Belmonte et al. [2] showed that for any fixed graph H, the H-Contractibility problem can be solved in polynomial time for split graphs. It is easy to see that H-Dissolution is fixed-parameter tractable for parameter $|V_H|$, as we show in Section 5. Hence, the remaining cases are the classifications of H-Induced Minor and H-Induced Topological Minor. We prove these cases in Theorem 6 and Corollary 1, respectively. Theorem 6 also contains a proof for the H-Contractibility problem restricted to split graphs; our proof has been found independently and uses different arguments than the proof of Belmonte et al. [2] for this problem and graph class. We first prove the following lemma.

Lemma 1. *Let G and H be two split graphs with maximal split partitions (C_G, I_G) and (C_H, I_H), respectively. Let $\mathcal{W}$ be an H-witness structure of G that satisfies conditions* (i)–(iii) *or* (i)–(iv). *Then G has an H-witness structure that satisfies conditions* (i)–(iii) *or* (i)–(iv), *respectively, and in which every independent bag consists of exactly one vertex of I_G.*

Proof. Suppose that some independent bag W of $\mathcal{W}$ does not consist of exactly one vertex of I_G. Because I_G is independent and $G[W]$ is connected, W contains a vertex from C_G. Because C_G is a clique and independent bags may not be adjacent, this means that all other independent bags contain no vertices from C_G. Consequently, these bags consist of single vertices from I_G. Because (C_H, I_H) is a maximal split partition, there is a clique bag W' not adjacent to W.

By combining the three facts that C_G is a clique, W contains a vertex from C_G, and W' is not adjacent to W, we find that W' does not contain a vertex from C_G. Hence W' can only contain vertices from I_G. Because I_G is independent and $G[W']$ is connected, this means that W' consists of a single vertex from I_G.

Recall that all independent bags other than W contain no vertices from C_G. Hence, they must consist of (single) vertices from I_G. Consequently, they are not adjacent to W', because W' consists of a single vertex from I_G as well. By definition, all independent bags not equal to W are not adjacent to W either. All clique bags other than W' must contain at least one vertex from C_G; otherwise they are not adjacent to W', as W' consists of a single vertex from I_G. Because W contains a vertex from C_G, this means that W is also adjacent to all clique bags not equal to W'. Hence, W and W' are adjacent to exactly the same bags of $\mathcal{W}$, and we can swap them. In this way we have obtained an H-witness structure $\mathcal{W}^*$ of G that satisfies conditions (i)–(iii) and in which every independent bag consists of exactly one vertex of I_G. Because we did not remove any vertices from any bag of $\mathcal{W}$, we find that $\mathcal{W}^*$ also satisfies condition (iv) if $\mathcal{W}$ satisfies this condition. This proves Lemma 1. □

Let G be a graph that has an H-witness structure $\mathcal{W}$ satisfying conditions (i)–(iii) for some graph H. We define the *order* of $\mathcal{W}$ to be the number of vertices in the union of all the witness bags of $\mathcal{W}$. Note that the order of $\mathcal{W}$ is at most $|V_G|$, with equality if and only if $\mathcal{W}$ satisfies condition (iv) as well. If G has another H-witness structure $\mathcal{W}'$ satisfying conditions (i)–(iii), then we call $\mathcal{W}'$ a *substructure* of $\mathcal{W}$ if $W'(x) \subseteq W(x)$ for all $x \in V_H$.

Lemma 2. *Let G and H be two split graphs with maximal split partitions (C_G, I_G) and (C_H, I_H), respectively. Let $\mathcal{W}$ be an H-witness structure of G satisfying conditions* (i)–(iii), *in which every independent bag consists of exactly one vertex of I_G. Then G has an H-witness structure $\mathcal{W}'$ satisfying conditions* (i)–(iii) *that has order at most $(|C_H|+1)(|I_H|+1)$ and that is a substructure of $\mathcal{W}$.*

Proof. We write $C_H = \{x_1, \ldots, x_p\}$ and $I_H = \{y_1, \ldots, y_q\}$. By our assumption on $\mathcal{W}$, there exist q vertices $u_1, \ldots, u_q$ of I_G such that $W(y_j) = \{u_j\}$ for $j = 1, \ldots, q$. We call these vertices the *u-vertices*.

For $j = 1, \ldots, q$, let $\mathcal{W}_j$ be the set of adjacent witness bags of $W(y_j)$. Note that every $\mathcal{W}_j$ is a subset of the clique bags of $\mathcal{W}$. By definition, each u_j has at least one neighbor in every bag of $\mathcal{W}_j$. For each u_j we pick one such neighbor t_{ij} from every bag $W(x_i)$ of $\mathcal{W}_j$. We call such a vertex a *t-vertex*. Note that $t_{hj} = t_{ij}$ is possible for two vertices u_h and u_i that are adjacent to the same witness bag $W(x_i)$.

Because two vertices from I_G are not adjacent, there exists at most one clique bag W^* that contains no vertex from C_G. Moreover, if this clique bag W^* exists, then it consists of exactly one vertex u^* from I_G due to the same reason. We call this vertex a u-vertex as well. By definition, u^* has at least one neighbor in every other clique bag. Then, in the case that u^* exists, we choose one neighbor t_i^* of u^* from each other clique bag $W(x_i)$ and call it a t-vertex as well.

By definition, vertices in I_G are only adjacent to vertices in C_G. Hence, every t-vertex belongs to C_G. This means that the adjacency relations between the clique bags that are not equal to the bag W^* (if W^* exists) are preserved, when we remove all vertices that are neither t-vertices nor equal to u^* from the clique bags. By our choice of t-vertices, also the adjacency relations between the clique bags and the independent bags, and between W^* and the other clique bags, are then preserved as well. This leads to an H-witness structure $\mathcal{W}'$ of G that satisfies conditions (i)–(iii). Because for each of the at most $q+1$ u-vertices we picked at most p t-vertices, $\mathcal{W}'$ has order at most $q+1+p(q+1) = (p+1)(q+1) = (|C_H|+1)(|I_H|+1)$. Because we did not move any vertices from one witness bag to some other, $\mathcal{W}'$ is a substructure of $\mathcal{W}$. This completes the proof of Lemma 2. □

Theorem 6. *For any fixed graph H, the problems H-*Induced Minor *and H-*Contractibility *can be solved in polynomial time for split graphs.*

Proof. Let G be a split graph on n vertices with maximal split partition (C_G, I_G). The class of split graphs is closed under contractions and vertex deletions. Hence, if H is not a split graph, then H cannot be an induced minor or a contraction of G. Assume that H is a split graph with maximal split partition (C_H, I_H) where $C_H = \{x_1, \ldots, x_p\}$ and $I_H = \{y_1, \ldots, y_q\}$.

We start with the H-Induced Minor problem. Lemmas 1 and 2 tell us that we only have to guess a set of at most $(p+1)(q+1)$ vertices in G and check if this set gives us an H-witness structure of G satisfying conditions (i)–(iii). Because H is fixed, $(p+1)(q+1)$ is a constant. Consequently, our guessed set has constant size, and the check can be done in polynomial time. If the answer is negative, we guess a different set until we have considered them all. There are at most $n^{(p+1)(q+1)}$ different sets, which is a polynomial number, because $(p+1)(q+1)$ is a constant. We conclude that our algorithm runs in polynomial-time.

We now consider the H-Contractibility problem and apply the following algorithm. First we guess a set S of at most $(p+1)(q+1)$ vertices in G. Second we consider every possibility of placing the vertices of S in witness bags to obtain an H-witness structure of G satisfying conditions (i)–(iii) such that every independent bag consists of exactly one vertex of I_G. If this is not possible, we try a different set S. Now suppose that we have obtained such an H-witness structure $\mathcal{W}$ of G. We note that $\mathcal{W}$ does not have to satisfy condition (iv). Hence the set $R = V_G \setminus S$ may be nonempty.

We perform the following two steps. In the first step, we consider the vertices of $C_G \cap R$ one by one and place them in the first clique bag that does not create any edges between vertices of two bags that may not be adjacent. In the second step, we consider every vertex of $I_G \cap R$ and place it in a clique bag that contains one of its neighbors. It is possible that one of these steps fails; note that the second step can fail because a vertex of $I_G \cap R$ may be an isolated vertex of G. Hence, in the case that we fail in one of these steps, we try to form another H-witness structure of G using the vertices of S as above, or a different set S. Otherwise we have found an H-witness structure of G satisfying conditions (i)–(iv).

Because H is fixed, we find that $(p+1)(q+1)$ is a constant. Consequently, every guessed set S has constant size, and we can process it in constant time. Also, the two steps afterwards can be performed in polynomial time for each set S. Because the total number of different sets S is at most $n^{(p+1)(q+1)}$, we conclude that our algorithm runs in polynomial time.

In order to prove the correctness of our algorithm, we must show that our algorithm will detect an H-witness structure of G if G contains H as a contraction. We show this below.

Suppose that G contains H as a contraction. This means that G has an H-witness structure that satisfies conditions (i)–(iv). Then, by Lemma 1, we find that G has an H-witness structure $\mathcal{W}$ satisfying conditions (i)–(iv), in which every independent bag consists of exactly one vertex of I_G. Because $\mathcal{W}$ satisfies conditions (i)–(iii), we find that G has an H-witness structure $\mathcal{W}'$ satisfying conditions (i)–(iii) that has order at most $(p+1)(q+1)$ and that is a substructure of $\mathcal{W}$, due to Lemma 2. Because $\mathcal{W}'$ is a substructure of $\mathcal{W}$, every independent bag of $\mathcal{W}'$ consists of a single vertex from I_G. At some point during its execution, our algorithm will detect the corresponding set S and the possibility of placing its vertices in bags such that $\mathcal{W}'$ is obtained. It will then find an H-witness structure of G that satisfies conditions (i)–(iv) by considering the vertices in $R = V_G \setminus S$ in the way we described. Note that this H-witness structure may not be equal to $\mathcal{W}$, because the vertices in R may be placed in different clique bags than $\mathcal{W}$; the existence of $\mathcal{W}$ guarantees however that they can be placed. This completes the proof of Theorem 6. □

We let P_k denote the path on k vertices. A graph is called *P_k-free* if it contains no induced subgraph isomorphic to P_k. We observe that split graphs are P_5-free. For induced topological minors we can show the following.

Theorem 7. *For any fixed integer $k \geq 1$ and any fixed graph H, the H-Induced Topological Minor problem can be solved in polynomial time for P_k-free graphs.*

Proof. Let G be a P_k-free graph on n vertices. Because G is P_k-free, every edge of H corresponds to a path on at most $k-1$ vertices in any induced subgraph G' of G that is isomorphic to a subdivision of H. Taking into account isolated vertices of H as well, we then find that G' has at most $\ell = |V_H| + (k-1)|E_H|$ vertices. We guess the vertices of G' and check if they induce a subdivision of H. The latter can be done in constant time, because the number of vertices of G' is bounded by ℓ, which is a constant as H and k are assumed to be fixed. Because the total number of guesses is at most n^ℓ, which is a polynomial number for the same reason, this means that our algorithm runs in polynomial time. □

Corollary 1. *For any fixed H, the H-Induced Topological Minor problem can be solved in polynomial time for split graphs.*

5. The proof of Theorem 3

The problems Spanning Subgraph Isomorphism and Graph Isomorphism are trivially fixed-parameter tractable for ordered pairs (G, H) where G and H are arbitrary graphs, and $|V_H|$ is the parameter. Recall that Robertson and Seymour [23] and Grohe et al. [16] showed that Minor and Topological Minor, respectively, are fixed-parameter tractable for such pairs and parameter. We show in Theorem 8 that Minor, Subgraph Isomorphism, and Topological Minor have relatively small kernels for ordered pairs (G, H) where G is a split graph, and $|V_H|$ is the parameter. In Theorem 9 we show that the Dissolution problem has a kernel of polynomial size even for general graphs. We show in Theorem 10 that Induced Minor, Induced Topological Minor and Induced Subgraph are W[1]-hard for ordered pairs (G, H) where G and H are split graphs, and $|V_H|$ is the parameter. In Theorem 11 we show the same result for the Contractibility problem.

For the first result in this section, we use the following terminology. Two non-adjacent vertices in a graph G are called *twins* if they share the same neighbors. An independent set of vertices in G is a *twin set* if any two vertices from this set are twins.

Theorem 8. *The problems* Minor, Topological Minor *and* Subgraph Isomorphism *have a kernel of size $|V_H| + |V_H|2^{|V_H|}$ for ordered pairs (G, H) where G is a split graph.*

Proof. Let G be a split graph and H be an arbitrary graph on k vertices. Suppose that $|C_G| \geq k$. Then $G[C_G]$ includes H as a subgraph, and consequently, as a topological minor and as a minor.

Suppose that $|C_G| < k$. Let $T \subseteq I_G$ be a twin set. If $|T|$ has more than k vertices we remove $|T| - k$ vertices from G for the following reason. If G contains a subgraph H' isomorphic to H, then H' contains at most k twin vertices from T, as H has at most k vertices. If G contains H as a minor, then we may assume that each bag in a corresponding H-witness structure contains at most one vertex from T; we can remove any extra vertex of T from a bag without violating conditions (i) and (ii). If G contains H as a topological minor, then G contains H as a minor as well. Hence, we may apply the above reduction rule in this case as well. We apply it on each of the at most $2^{|C_G|}$ twin sets in I_G after detecting these twin sets in polynomial time. Then, for all three problems, the resulting graph has at most $|C_G| + k2^{|C_G|} < k + k2^k$ vertices. □

A path in a graph G between two vertices u and v that each have degree not equal to two in G is a 2-*path* if all vertices on the path except u and v have degree two in G. When $u = v$ we speak of a 2-*cycle* instead. A connected component of a graph G is called a *cycle component* of G if it is a cycle. We make the following observation, which immediately follows from our assumption that we only consider finite graphs.

Observation 1. *Every vertex of a graph that contains no cycle components and no isolated vertices lies on a* 2*-path or a* 2*-cycle.*

A graph is *empty* if it is isomorphic to the empty graph $(\emptyset, \emptyset)$. The Dissolution problem allows a polynomial kernel for general graphs.

Theorem 9. *The* Dissolution *problem has a kernel of size* $2|V_H|^2$.

Proof. If G and H have a different number of isolated vertices, then H is not a dissolution of G. Otherwise, we remove all isolated vertices from both graphs. Hence, from now on, we assume that G and H do not contain any isolated vertices.

We write H as the disjoint union $H = H_1 \cup H_2$ where H_1 consists of all cycle components of H, and H_2 consists of all other components of H; note that H_1 or H_2 can be empty. We write $G = G_1 \cup G_2$, accordingly. Because a subdivision of a cycle component is a cycle, G contains H as a dissolution if and only if G_1 contains H_1 as a dissolution and G_2 contains H_2 as a dissolution.

Due to the above, we first check if the number of cycle components in G_1 is equal to the number of cycle components in H_1. If not, then H_1 is not a dissolution of G_1. Otherwise, let r be the maximum number of vertices in any cycle component of H_1; if H_1 is empty, then we let $r = 0$. Suppose that $r \geq 1$. In every cycle component C of G_1 on more than r vertices, we can dissolve $|V_C| - r$ vertices. This leads to a graph G'_1 on at most $r|V_{H_1}| \leq |V_{H_1}|^2$ vertices. Suppose that $r = 0$. Then H_1 and G_1 are empty, and we let G'_1 be the empty graph as well. Hence, G_1 contains H_1 as a dissolution if and only if G'_1 contains H_1 as a dissolution.

We now check if G_2 and H_2 have the same number of vertices of degree not equal to two. If not, then H_2 is not a dissolution of G_2. Otherwise, we do as follows. Let p be the total number of 2-paths and 2-cycles in H_2. Let q be the maximum of vertices in any 2-path or 2-cycle in H_2. If H_2 is empty, then $p = 0$ and we let $r = 0$ as well. If H_2 is not empty, then we use Observation 1 to find that $p \geq 1$, and consequently, $r \geq 1$.

We check if the total number of 2-paths and 2-cycles in G_2 is larger than p. If so, then H_2 is not a dissolution of G_2. Suppose that the total number of 2-paths and 2-cycles in G_2 is not larger than p. If G_2 has a 2-path or 2-cycle containing $s > q$ vertices, we dissolve $s - q$ of such vertices. Applying this rule exhaustively results in a graph G'_2 such that G_2 contains H_2 as a dissolution if and only if G'_2 contains H_2 as a dissolution. By Observation 1 and our reduction rule we find that G'_2 has at most $pq \leq |V_{H_2}|^2$ vertices.

Summarizing, H is not a dissolution of G if H_1 or H_2 is not a dissolution of G_1 or G_2, respectively. Otherwise, we have found a graph $G' = G'_1 \cup G'_2$ with at most $|V_{G_1}| + |V_{G_2}| \leq |V_{H_1}|^2 + |V_{H_2}|^2 \leq |V_H|^2$ vertices, such that H is a dissolution of G if and only if H is a dissolution of G'. This completes the proof of Theorem 9. □

In the remainder of this section we show that we get W[1]-hardness when the containment is required to be induced.

Theorem 10. *The problems* Induced Minor, Induced Topological Minor *and* Induced Subgraph *are* W[1]*-hard for ordered pairs* (G, H) *where G and H are split graphs, and* $|V_H|$ *is the parameter.*

Proof. We use the same reduction as in the proof of Theorem 5, namely from the Clique problem, which asks whether a graph has a clique of size at least k. This problem is W[1]-complete when parameterized by k (cf. [8]). □

We let $K_1 \bowtie G$ denote the graph obtained from a graph G after adding a new vertex and making it adjacent to all vertices of G.

Lemma 3 (*[24]*)**.** *Let H and G be two graphs. Then G has H as an induced minor if and only if* $K_1 \bowtie G$ *is* $(K_1 \bowtie H)$*-contractible.*

We observe that $K_1 \bowtie G$ is a split graph if G is a split graph. Then, combining Lemma 3 with Theorem 10 yields the following result.

Theorem 11. *The* Contractibility *problem is* W[1]*-hard for ordered pairs* (G, H) *where G and H are split graphs, and* $|V_H|$ *is the parameter.*

6. Conclusions

In Theorems 1–3 we settle the computational complexity of finding (induced) minors, (induced) topological minors, (induced) subgraphs, (induced) spanning subgraphs, dissolutions and contractions in split graphs. Our research was motivated by a similar study for a superclass of split graphs, namely the class of *chordal* graphs, i.e., graphs that contain no cycle on four or more vertices as an induced subgraph. Very recently, Belmonte et al. [1] showed that for any fixed graph H, the problems H-Contractibility and H-Induced Minor are polynomial-time solvable on chordal graphs. On the contrary, we observed in another recent paper [14] that for any fixed graph, the problems H-Subgraph Isomorphism, H-Minor, H-Topological Minor can be solved in linear time for chordal graphs.

Because split graphs form a subclass of the class of chordal graphs, the W[1]-hardness results in Theorem 3 carry over to chordal graphs. Hence, the aforementioned polynomial-time results of Belmonte et al. [1] can be considered as best possible for chordal graphs, and it is natural to consider other subclasses of chordal graphs such as interval graphs and proper interval graphs. A graph is an *interval graph* if intervals of the real line can be associated with its vertices in such a way that two vertices are adjacent if and only if their corresponding intervals overlap. An interval graph is *proper interval* if it has an interval representation, in which no interval is properly contained in any other interval. It may be interesting to undertake a study on containment relations for these two classes of graphs. In particular, we pose the following two open problems.

1. Is Contractibility W[1]-hard for ordered pairs (G, H) where G and H are intervals, and $|V_H|$ is the parameter?
2. Is Contractibility fixed-parameter tractable for ordered pairs (G, H) where G and H are proper intervals, and $|V_H|$ is the parameter?

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