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Treewidth for graphs with small chordality[☆]

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Abstract

A graph G is k-chordal, if it does not contain chordless cycles of length larger than k. The chordality lc of a graph G is the minimum k for which G is k-chordal. The degeneracy or the width of a graph is the maximum min-degree of any of its subgraphs. Our results are the following:

(1) The problem of treewidth remains NP-complete when restricted to graphs with small maximum degree.
(2) An upper bound is given for the treewidth of a graph as a function of its maximum degree and chordality. A consequence of this result is that when maximum degree and chordality are fixed constants, then there is a linear algorithm for treewidth and a polynomial algorithm for pathwidth.
(3) For any constant $s \geqslant 1$, it is shown that any $(s+2)$-chordal graph with bounded width contains an $\frac{1}{2}$-separator of size $O(n^{(s-1)/s})$, computable in $O(n^{3-(1/s)})$ time. Our results extent the many applications of the separator theorems in [1, 32, 33] to the class of k-chordal graphs.

Several natural classes of graphs have small chordality. Weakly chordal graphs and cocomparability graphs are 4-chordal. We investigate the complexity of treewidth and pathwidth on these classes when an additional degree restriction is used. We present an application of our separator theorem on approximating the maximum independent set on k-chordal graphs with small width.

Keywords: Algorithms; NP-complete problems; Separators in Graphs; Treewidth; Width

1. Introduction

In this paper, we study a relatively new graph parameter: the chordality of a graph. (All graphs are assumed to be undirected, finite, and simple.) We call a graph *k-chordal*, if it does not contain a chordless cycle of length larger than k. The *chordality* of a graph G is defined as the minimum k for which G is k-chordal. In this paper

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we investigate the complexity of treewidth and pathwidth in relation to this parameter, the maximum degree, and the width of G. We also present a separator theorem for graphs with bounded chordality.

The class of k-chordal graphs contains as subclasses many known natural classes of graphs, even for small values of k. Clearly 3-chordal graphs are exactly the chordal graphs. Also, as we mention in Section 6, the classes of the weakly chordal graphs and the cocomparability graphs are subclasses of the class of 4-chordal graphs.

The notions of treewidth and pathwidth appear to play an important role in the analysis of the complexity of several problems in graph theory. They were introduced by Robertson and Seymour in their series of fundamental papers on graphs minors (see [37, 39]). Roughly spoken, the *treewidth* of a graph is the minimum k such that G can be decomposed into a "tree structure" of pieces with at most $k+1$ vertices. (For the precise definition, see Section 2.) A series of recent results show that many NP-complete problems become polynomial or even linear time solvable, or belong to NC, when restricted to graphs with small treewidth (see [5, 7]). Much research has been done on the problem of determining the treewidth and the pathwidth of a graph, and finding optimal tree or path decompositions with optimal treewidth or pathwidth. These problems are NP-complete even if we restrict the input graph to bipartite graphs [26] or the complements of bipartite graphs [6]. Moreover, pathwidth remains NP-complete on chordal graphs [22], planar graphs [36] and graphs with bounded maximum degree [36]. In Section 3, we prove that treewidth is also NP-complete on graphs with bounded maximum degree.

Treewidth can be computed in polynomial time on chordal graphs, cographs [13], circular arc graphs [40], chordal bipartite graphs [28], permutation graphs [12], circle graphs [25] and distance hereditary graphs [3]. Bodlaender presented in [8] a linear time algorithm that finds an optimal tree decomposition for a graph with bounded treewidth. Also Bodlaender and Hagerup in [10] provide (near) optimal parallel algorithms for constructing minimum-width tree decompositions of graphs with bounded treewidth. In Section 4, we prove that if a k-chordal graph has maximum degree bounded by a value Δ, then there is a function $f(k,\Delta)$ that is an upper bound for treewidth. A consequence of our result is that, for k-chordal graphs with bounded maximum degree, there is a linear time algorithm for computing treewidth and a polynomial time algorithm for computing pathwidth.

In Section 5, we present a connection between the parameters of treewidth and width for k-chordal graphs. The *degeneracy* of a graph $G=(V,E)$ is defined to be the maximum min-degree of any of the subgraphs of G (see also [20, 30, 34]). In [35], it is proved that the degeneracy of a graph is equal to its width, a graph parameter that is also known as linkage (see also [20, 24]). A layout L of a graph $G=(V,E)$ is a bijective function, mapping its vertices to numbers $\{1,2,\ldots,|V|\}$. The *width* of a layout L of G is the maximum back-degree of any vertex in L (the *back-degree* of a vertex $v \in L$ is defined to be the number of vertices that are adjacent to v and are preceeding it in L). The width of G is the minimum width over all possible layouts of G. Width has been studied in the context of Constraint Satisfaction, as it is known that for constraint graphs of bounded width, it is possible to apply backtrack free search, the classical method to solve the Constraint Satisfaction Problem (see [20]). Also, width appears in many combinatorial results. For instance, in [2], improved time bounds are presented for algorithms that count and find cycles, when the input graph is considered to have small width. Finally, there exist several parameters characterizing the sparsity of a graph that are related to width. It is known that graphs with small arboricity or genus have small width. Moreover, any graph in a graph class with an excluded minor has bounded width.

In this paper we also use the parameter width$_s$. When $s = n - 1$ or $s = 1$, width$_s$ is equivalent with treewidth and width, respectively. The parameter of width$_s$ was defined in [17]. In [17], Dendris, Kirousis, and Thilikos examine various versions of fugitive search games on graphs and present their connections with the parameters of treewidth, pathwidth, and width. They show that the width$_s$ of a graph with chordality at most $s + 2$, equals its treewidth. Using their result (Theorem 5 in our paper), we prove a connection between width and treewidth that leads to a separator theorem for $(s + 2)$-chordal graphs with small width.

Given a vertex weighted graph $G = (V, E)$, a set $S \subseteq V$ is an $\frac{1}{2}$-separator of G iff the vertex set of each connected component of $G[V - S]$ (the subgraph of G induced by the vertices in $V - S$) has total vertex weight no more than the half of the total vertex cost of V. Separator theorems have appeared to play an important role for algorithmic graph theory as these (in combination with a divide and conquer strategy) lead to a series of important complexity results. In [32], Lipton and Tarjan were the first to prove a separator theorem for the class of planar graphs. Several applications of this separator theorem are presented in [31], including approximation algorithms for NP-complete problems, nonserial dynamic programming, time-space trade offs' study, lower bounds on boolean circuit size, embedding of data structures, and maximum matching (see also [31] for an application on sparse Gaussian elimination and [29] for results relating small separators with layouts of graphs in a model of VLSI). In [1], Alon et al. provide a considerably more general result and extent the previous applications. They prove a separator theorem for any graph not containing a specific graph as a minor.

Our separator theorem guarantees the existence of a separator of size $O(kn^{(s-1)/s})$ in a $(s+2)$-chordal graph with width at most k. Moreover our results lead to an $O(n^{3-(1/s)})$ time algorithm that computes such a separator. As any graph not containing a specific graph as a minor has *bounded* width, our separator theorem gives an extension of the results in [1, 32, 33] when the chordality is bounded. Finally we present an application of our separator theorem for the problem of approximating the independent set problem on $(s + 2)$-chordal graphs with small width.

As there are sparse graphs with small width that do not contain small separators (see Lemma 3) we feel that the requirements of small width and small chordality help to approach a characterization of the concept of "usefully sparse", a question posed from Lipton and Tarjan in [33] about the existence of separator theorems for non planar sparse graphs.

2. Definitions and preliminary results

In this section some definitions and results will be presented, which are useful in later sections.

Let $G=(V,E)$ be a finite undirected graph without multiple edges or loops. For a set of vertices $V' \subseteq V$, the subgraph of G, induced by V' is denoted by $G[V']$. The vertex and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

The notions of treewidth and pathwidth were introduced by Robertson and Seymour in [39, 37].

Definition. A *tree decomposition* of $G=(V,E)$ is defined to be a pair $(\{X_i : i \in I\}, T)$, where $\{X_i : i \in I\}$ is a collection of subsets of V (we call these subsets the *nodes* of the decomposition) and $T=(I,F)$ is a tree having the index set I as set of vertices, such that the following conditions are satisfied:
1. $\bigcup_{i \in I} X_i = V$.
2. $\forall \{u,w\} \in E,\ \exists i \in I : u,w \in X_i$.
3. $\forall i,j,k \in I$: if j is on a path in T from i to k then $X_i \cap X_k \subseteq X_j$.

The *treewidth of a tree decomposition* $(\{X_i : i \in I\}, T)$ is defined to be equal to $\max_{i \in I} |X_i| - 1$.

The *treewidth of* G is defined to be the minimum treewidth over all tree decompositions of G.

Definition. A *path decomposition* of $G=(V,E)$ is defined to be a sequence $\{X_i : i = 1,\ldots,r\}$ of subsets of V (we call these subsets the *nodes* of the decomposition) such that the following conditions are satisfied:
1. $\bigcup_{i=1}^{r} X_i = V$.
2. $\forall \{u,w\} \in E,\ \exists i : u,w \in X_i$.
3. $\forall i,j,k$, if $1 \leqslant i \leqslant j \leqslant k \leqslant r$, then $X_i \cap X_k \subseteq X_j$.

The *pathwidth of a path decomposition* $\{X_i : i = 1,\ldots,r\}$ is defined to be equal to $\max_{1 \leqslant i \leqslant r} |X_i| - 1$.

The *pathwidth of* G is defined to be the minimum pathwidth over all path decompositions of G.

The problem of computing the treewidth of a given graph has been proved to be NP-complete by Arnborg et al. in [6]. More precisely, they proved that treewidth is NP-complete even when restricted to the class of cobipartite graphs (i.e. the complements of bipartite graphs).

Bodlaender proved the following result about the fixed parameter complexity of treewidth (see [8]).

Theorem 1. *For any fixed integer k, there is a linear time algorithm, that tests whether a given graph $G=(V,E)$ has treewidth at most k, and if so, outputs a tree decomposition of G with treewidth at most k.*

A graph H is a minor of a graph G if H can be obtained from a subgraph of G by a number of edge contractions. (A contraction of an edge $\{u,v\}$ replaces the vertices u and v by a new vertex that is adjacent to all vertices that were adjacent to v or u.)

We use the following well known result mentioned by Robertson and Seymour in [38].

Lemma 2. *Let H be a minor of G. Then treewidth$(H) \leqslant$ treewidth(G).*

The parameter width$_s(G)$ introduced below characterizes treewidth in terms of layouts when $s = n - 1$ (see [17]).

Let $L = (v_1, \ldots, v_n)$ be a layout of the vertices in G.

Definition. The *width$_s$* of a vertex $v \in L$ is the number of vertices preceeding v in L that are connected with v via a path of vertices not preceeding v which has length at most s.

The *width$_s$* of a layout of G is the maximum width$_s$ over all vertices in L.

The *width$_s$* of a graph G is the minimum width$_s$ over all possible layouts of G.

For $s = 1$, Definition 2 gives the width of a graph. We mention that, for chordal graphs, width is equal to treewidth (i.e., one less than the maximum clique size), which is polynomially computable and has an NC approximation algorithm for constant approximation factors $< \frac{1}{2}$ (see [4]). It is known that there is an algorithm that, given a graph $G = (V,E)$ checks whether there is a layout of G with width at most k and, if so, constructs it in $O(|E|)$ time. It can be easily proved that width is bounded for classes of graphs with an excluded minor, i.e. graphs with no minor isomorphic to a given graph H (see [14]). However the class of graphs with bounded width is larger: there are graphs with small width containing arbitrary large minors, as is shown in the following lemma.

Lemma 3. *For any $k \geqslant 2$ and any graph H, there is a graph G such that width$(G) \leqslant k$ and H is a minor of G.*

Proof. Suppose H has h vertices. Take a clique with h vertices and subdivide all its edges (replace each edge e with a path of length 2 having the same endpoints as e). It is easy to see that the graph G obtained has width$(G) = 2 \leqslant k$ and contains H as a minor. □

Lick and White in [30] proved the following extremal result about width (see also [24]).

Theorem 4. *Let G be a graph with n vertices, e edges and width$(G) \leqslant k$. Then $e \leqslant \binom{k}{2} + k(n-k)$.*

Clearly $\text{width}_{s_1}(G) \leqslant \text{width}_{s_2}(G)$ when $s_1 \leqslant s_2$. Using this fact, we can see that the above extremal result holds also for width_s for any s greater than 1. It follows that if $\text{width}_s(G) \leqslant k$, then $|E(G)| = O(kn)$ for any s, $1 \leqslant s \leqslant n-1$.

A cycle $C = (v_1, \ldots, v_l, v_1)$ in a graph $G = (V, E)$ is *chordless* if it does not contain any chords (i.e. $\forall v_i, v_j$, $l-1 > |i-j| > 1$ $\{v_i, v_j\} \notin E$). We denote as $\text{lc}(G)$ the length of the longest chordless cycle in G and call this parameter the *chordality* of a graph (in the case that G is a tree we take $\text{lc}(G) = 2$). A graph G is *k-chordal* if $\text{lc}(G) \leqslant k$.

The decision version of the chordality problem is the following:

Chordality

Instance: Graph $G = (V, E)$, integer $k \leqslant |V|$.

Question: Is the chordality of G at least k?

By an easy reduction (subdivide every edge) to Longest Cycle or Hamiltonian Circuit, one proves that Chordality is NP-complete.

Using the notation above we mention the following result proved in [17].

Theorem 5. *Let G be a graph such that $lc(G) \leqslant s + 2$. The treewidth of G equals its width_s.*

Theorem 6. *The problem of computing the width_s of a graph is* NP-*complete when* $s > 1$.

Proof. We mentioned in the introduction that treewidth is NP-complete also when restricted to the class of cobipartite graphs. As for any graph G in this class $\text{lc}(G) = 4$, the result follows from Theorem 5 and the NP-completeness of treewidth. □

Definition. Let $G = (V, E)$ be a graph and $w : V \to Q^+$ a function assigning a positive rational weight to each vertex in V. We call the sum of the weights over all the vertices of a set $V' \subseteq V$ the *total weight* of V' and denote it as $w(V')$.

A set $S \subseteq V$ is an $\frac{1}{2}$-separator of the function w in G with size k iff the sum of weights of the vertices in each of its connected components of $G[V - S]$ is no more than $\frac{1}{2}w(V)$. Also, a set $S \subseteq V$ is a $\frac{2}{3}$-separator of the function w in G with size k iff there exist a partition A, B, S of V such that: (i) no edge connects a vertex in A with a vertex in B, (ii) neither A nor B has total vertex cost exceeding $\frac{2}{3}w(V)$, (iii) S contains no more than k vertices.

It seems to be useful to have results that tell how to find separators of small size in graphs, as these have several applications in combination with a divide-and-conquer strategy. For such theorems and applications see e.g. [1,32,33]. A well known separator result is the following (see [1, 9, 39]

Theorem 7. *Let $G = (V, E)$ be a graph and $w : V \to Q^+$ a function assigning a positive rational weight to each vertex in V. Then any tree decomposition $(\{X_i : i \in I\}, T)$ of*

G with treewidth no more than k contains a node that is an $\frac{1}{2}$-separator of w in G. Moreover, there exist an algorithm that finds in $O(k^2 n)$ time a set S with $|S| \leqslant k$ that is an $\frac{1}{2}$-separator (a $\frac{2}{3}$-separator) of w in G.

3. Treewidth is NP-complete for graphs with bounded max-degree

In this section we prove that treewidth is also NP-complete when restricted to graphs with maximum degree at most 9.

The decision version of the treewidth problem is the following:

TREEWIDTH

Instance: Graph $G=(V,E)$, integer $k \leqslant |V|-1$.

Question: Is the treewidth of G at most k?

Definition. We call a graph (n_1,m_1,n_2,m_2)-*bigrid* if it can be constructed in the following way:

Take two grids G_1 and G_2 with sizes n_1,m_1 and n_2,m_2 respectively. Extend each grid G_i, $i=1,2$ in the following way:

Let $S_i=\{v_1^i,\dots,v_{n_i}^i\} \subseteq V(G_i)$ be the vertices of a side of G_i containing n_i vertices. Add a vertex set $S_i'=\{u_1^i,\dots,u_{n_i}^i\}$ and connect v_j^i with u_j^i for $j=1,\dots,n_i$. We call the two graphs obtained, the *pruned grids* of the construction and we denote them as G_1' and G_2'.

The construction is completed by adding an arbitrary collection of edges, each between a vertex in S_1' and a vertex in S_2'.

We call the transformation below *a q-clique-grid transformation* from a cobipartite graph G to a bigrid graph G'.

Let $G=(V,E)$ be a cobipartite graph where V_1, V_2 induce disjoint cliques and $|V_1|+|V_2|=|V|$. Let $n_1=|V_1|$ and $n_2=|V_2|$. Now we take a (n_1,q,n_2,q)-bigrid $G'=(V',E')$ in the following way: each vertex in S_i' represents a vertex in V_i and each edge $e=\{v_k^1,v_l^2\}$, $v_k^1\in S_1'$, $v_l^2\in S_2'$ represents the edge in E which has as endpoints the corresponding to u_k^1 and u_l^2 vertices of V.

We now need the following, rather technical lemmas.

Lemma 8. *Consider a tree decomposition $(\{X_j,\ j\in J\},T)$ of a graph $G=(V,E)$. Then for any clique K of G, $\exists j\in J: V(K)\subseteq X_j$.*

For a proof of this lemma see e.g., [13].

Lemma 9. *Let G be a grid with sizes n,q and $(\{X_j : j\in J\},T)$ be a tree decomposition of G with width at most k. Then, if $q \geqslant 2k+3$, there is a node X_j in the decomposition that contains at least one vertex of each of the q rows of G.*

Proof. Let $w: V \to Q^+$ be a function such that $\forall v \in V$, $w(v)=1$. As treewidth$(G) \leqslant k$, from Theorem 7 it follows that there must exist a node X_j in the decomposition tree that is an $\frac{1}{2}$-separator of G. Clearly X_j has common vertices with at most $k+1$ columns in G. So there are at least $q-(k+1)$ columns in G not meeting X_j. Suppose that all of the vertices of these columns belong in the same component of $G[V-X_j]$. As X_j is an $\frac{1}{2}$-separator of w in G, this component must contain at most $\frac{1}{2}(nq)$ vertices. Therefore $n(q-(k+1)) \leqslant \frac{1}{2}(nq)$ which gives $q \geqslant 2k+2$, a contradiction. Hence, there are two columns in different components of $G[V-X_j]$. Now any row contains a vertex of each of the two components, and hence X_j must contain a vertex of each row, by definition of tree decomposition. □

The following lemma asserts that treewidth is an invariant of the q-clique-grid transformation when q is sufficiently large.

Lemma 10. *Let G be a cobipartite graph. Let G' be the graph we obtain from G if we apply a q-clique-grid transformation on G with $q \geqslant 2k+3$. Then treewidth $(G) \leqslant k$, if and only if treewidth $(G') \leqslant k$.*

Proof. Suppose that treewidth$(G) \leqslant k$. We will prove that treewidth$(G') \leqslant k$. Notice first that G contains two cliques C_i of n_i vertices each $(i=1,2)$. By Lemma 8, G must have a tree decomposition that has a node X_j^i containing C_i. An easy construction shows that any (n_i,m_i)-grid has a tree decomposition of treewidth at most n_i, that contains all the vertices of some side of n_i vertices in one of its nodes. Using this fact, we can build a tree decomposition of each pruned grid G_i', $i=1,2$ in G' that has width at most n_i and has a node containing S_i'. Now, if we identify the vertices of each clique C_i in G with the corresponding set S_i' in each pruned grid, we can see that, composing the tree decompositions of G, G_1' and G_2', the graph G_{merge} thus obtained has a tree decomposition of width $\leqslant \max\{n_1, n_2, \text{treewidth}(G)\} \leqslant k$. As G' is a subgraph of G_{merge} we have the required result.

Suppose now that treewidth$(G') \leqslant k$. Fix a tree-decomposition $(\{X_i \mid i \in I\}, T)$ of G' of treewidth at most k. Let G'' be obtained from G' by adding edges between all pairs of vertices v,w for which there is at least one node $i \in I$ with $v,w \in X_i$. Clearly, $(\{X_i \mid i \in I\}, T)$ is also a tree-decomposition of G''. Let G''' be the graph, obtained by contracting all rows in both grids with the corresponding vertex u_i. Note that G is a subgraph of G''': edges between vertices in the different cliques clearly are present. We must verify all edges in the cliques are present in G''': when v,w belong to the same clique in G, then two rows in one of the grids in G'' have been contracted to v and w, respectively. As there is a node that contains a vertex of each row of this grid (Lemma 9), it follows that there is an edge between a vertex of v's row and a vertex of w's row in G'', hence v and w are adjacent in G'''. So, G is a minor of G'''. The result now follows by Lemma 2. □

Theorem 11. *Treewidth remains NP-complete when restricted to graphs with maximum degree 9.*

Proof. In the NP-completeness proof of Arnborg et al. [6], a transformation from the cutwidth problem on general graphs to the treewidth problem on cobipartite graphs is given. Cutwidth is NP-complete when restricted to graphs with maximum degree at most three [36]. Applying the transformation from [6] to graphs of maximum degree three yields cobipartite graphs where any vertex is adjacent to at most eight vertices in the clique to which it does not belong. Hence, treewidth is NP-complete for the latter type of cobipartite graphs. When we apply a q-clique-grid transformation on such a cobipartite graph, we obtain a graph of degree at most 9. Applying such a transformation with a properly chosen value of q (e.g., take $q = 2k+3$, where k is the desired treewidth), yields the result. □

4. Graphs with bounded $\Delta(G)$ and lc(G)

In the previous section we proved that treewidth is NP-complete for graphs with maximum degree at least 9. Similarly, pathwidth is NP-complete for graphs of maximum degree 3, (more specifically, this holds due to the results in [36] and the fact that pathwidth is equivalent to the vertex separation number). In this section we show that if both max-degree and the length of the chordless cycles are bounded, then the treewidth is bounded by a constant, and hence computable in linear time. It also follows that there is a polynomial time algorithm for pathwidth in this case.

For graphs G, let $D(G)$ denote the quantity $\sum_{u \in V(G)} (\Delta - \deg(u))$, and let $\Delta(G)$ denote the maximum degree over all the vertices of G.

Lemma 12. *Let k, Δ, s be fixed constants. Let $\mathcal{D}$ be the class of connected graphs such that for any $G \in \mathcal{D}$ holds that*
1. $2 \leqslant \Delta(G) \leqslant \Delta$
2. *there exists a vertex $v \in V(G)$ such that* $\deg(v) \leqslant k \leqslant \Delta$ *and for any vertex $u \in V(G)$ there is a path between u and v of length at most s.*

Then $D(G) \leqslant k(\Delta-1)^s + \Delta - k$.

Proof. First observe that if $|V(G)| = n$ and $|E(G)| = e$, $D(G) = n\Delta - 2e$. Consider fixed values of k, Δ and s. If $\Delta = 2$, then any graph $G \in \mathcal{D}$ is a line and $D(G) = 2$. Also, if $s = 0$ then G contains only vertex v and $D(G) = \Delta$. We now examine the case that $\Delta > 2$ and $s > 0$.

Note that $\mathcal{D}$ is a finite set. Consider a graph G in $\mathcal{D}$ such that $D(G)$ is maximal. We will prove first that G is a tree. Assume that G contains a cycle. Let T be a breadth first spanning tree of G with root v, and let e be the edge of the cycle not in $E(T)$. Now let G' be the graph obtained by deleting e from G. Clearly $G' \in \mathcal{D}$ and $D(G') = D(G) + 2$, a contradiction.

We claim now that each vertex in $V(G)-\{v\}$ that is not a leaf has degree Δ and that $\deg(v)=k$. Suppose w is not a leaf, and either $v=w$ and $\deg(v)<k$, or $v\neq w$ and $\deg(v)<\Delta$. We construct a graph G' by adding a new vertex and connecting it with w. Now $G'\in\mathscr{D}$ and $D(G')=D(G)+\Delta-2$, a contradiction. Finally, for each leaf u, the unique path between u and v in G must have length s because otherwise we can add a vertex w to G connected with u and the thus obtained graph G' also belongs to $\mathscr{D}$ and $D(G')=D(G)+\Delta-2$, a contradiction.

Now observe that there is only one possibility left for G. It is easy to see that $e=k+k(\Delta-1)+\cdots+k(\Delta-1)^{s-1}=\frac{k}{\Delta-2}((\Delta-1)^s-1)$ and, as G is a tree, we have that $D(G)=(e+1)\Delta-2e=e(\Delta-2)+\Delta=k(\Delta-1)^s+\Delta-k$. □

Definition. Let $G=(V,E)$ be a graph and let $A,B\subset V$, $A\cap B=\emptyset$. We define the *degree of A in B*, denoted by *$deg(A,B)$*, as the number of vertices in B that are connected with vertices in A.

Theorem 13. *Let G be a graph with $\Delta(G)=\Delta\geqslant 2$ and $width(G)\leqslant k$. Let $s\geqslant 1$. Then $width_s(G)\leqslant k(\Delta-1)^{s-1}+\Delta-k$.*

Proof. We examine the nontrivial case where $s>1$. As width$(G)\leqslant k$, there is a layout L such that width$(L)\leqslant k$. Consider L' as the layout of V obtained by reversing L. Let v be any vertex in L'. It is sufficient to prove that the width$_s$ of v cannot be more than $k(\Delta-1)^{s-1}$. Let A be the set of vertices not preceding v in L' that are connected with v via paths of vertices not preceding v and of length at most $s-1$. Also let B be the set of vertices preceding v. Clearly width$_s(v)=\deg(A,B)$. We can now see that $\deg(A,B)\leqslant D(G[A])=\sum_{u\in V(G[A])}(\Delta-\deg(u))$ (the degree of u is taken with respect to $G[A]$). Also notice that $G[A]$ is a connected graph with $\Delta(G)\leqslant\Delta$, vertex $v\in G[A]$ has degree at most k and is connected with any vertex in $V(G[A])$ with a path of length at most $s-1$. Now by Lemma 12 we have the required result. □

The following result can easily be derived from Theorems 5 and 13.

Theorem 14. *Let G be a graph such that $lc(G)\leqslant s+2$, $width(G)\leqslant k$ and $\Delta(G)\leqslant\Delta$. Let $s\geqslant 1$. Then $treewidth(G)\leqslant k(\Delta-1)^{s-1}+\Delta-k$.*

As width$(G)\leqslant\Delta(G)$, we have the following corollaries.

Corollary 15. *Let G be a graph such that $lc(G)\leqslant s+2$ and $\Delta(G)\leqslant\Delta$. Let $s\geqslant 1$. Then $treewidth(G)\leqslant\Delta(\Delta-1)^{s-1}$.*

Corollary 16. *Let $s\geqslant 1$, and Δ be fixed constants. Let $\mathscr{G}$ be the class of graphs with $lc(G)\leqslant s+2$, and $\Delta(G)\leqslant\Delta$. Then there exist:*

1. *A linear time algorithm that computes the treewidth of graphs in $\mathscr{G}$.*
2. *A polynomial time algorithm that computes the pathwidth of graphs in $\mathscr{G}$.*

3. *A* $O(\log^2 n)$ *time parallel algorithm that computes the treewidth of graphs in* $\mathscr{G}$ *and that uses* $O(n/\log^2 n)$ *processors on an EREW PRAM.*

Each of the above algorithms outputs the corresponding tree or path decomposition of minimum treewidth or pathwidth.

Proof. Theorems 1 and 14 imply the first result. The second result follows from the result in [11] stating that for graphs with bounded treewidth, there is a polynomial time algorithm for pathwidth. The third result is obtained by combining the parallel algorithm given in [10] with Theorem 14. □

We have to point out that the upper bound of Theorem 14 can be improved in the following way: first, we notice that if α is a positive integer, then treewidth$(G) \leqslant$ treewidth (G^α) (where $G^\alpha = (V(G), \{\{v,u\} \mid$ there exist a path of no more than α vertices in G connecting v and $u\}))$. It is easy to see that if G is $(s+2)$-chordal, then $G^{\lfloor (s+2)/2 \rfloor}$ is chordal. As for chordal graphs treewidth is equivalent to width, we have that treewidth$(G) \leqslant$ width$(G^{\lfloor (s+2)/2 \rfloor})$. Also, using a variant of Lemma 12 we can see that if $\Delta \leqslant \Delta(G)$, then $\Delta(G^\alpha) \leqslant [\Delta((\Delta-1)^\alpha - 1)]/(\Delta - 2)$. Now, if we take into account that width$(G) \leqslant \Delta(G)$ and set $\alpha = \lfloor (s+2)/2 \rfloor$, we can conclude that the upper bound of Theorem 14 can be replaced by $[\Delta((\Delta-1)^{\lfloor (s+2)/2 \rfloor} - 1)]/(\Delta - 2)$ which is an improvement for any graph with chordality no less than 5. We need to mention that a (more complicated) proof of the same improvement has been proposed by Ton Kloks.

5. A separator theorem for k-chordal graphs with small width

In this chapter we prove a separator theorem for s-chordal graphs with small width. We first give the following lemma about the high degree vertices of a graph with small width.

Lemma 17. *Let G be a graph where width$(G) \leqslant k$ and let V_d be the set of vertices that have degree $\geqslant d \geqslant k$. Then $|V_d| \leqslant 2kn/d$.*

Proof. Each vertex in V_d has degree at least d. Therefore, we have that $d|V_d| \leqslant \sum_{v \in V_d} \deg(v) \leqslant \sum_{v \in V} \deg(v) = 2|E|$. By Theorem 4 we have that $d|V_d| \leqslant k^2 - k + 2kn - 2k^2 \leqslant 2kn$ which completes the proof of the lemma. □

When the width of a graph is given, the following theorem provides an upper bound to width$_s$.

Theorem 18. *Let G be a graph G with width$(G) \leqslant k$. Then width$_s(G) \leqslant (k+1)(2n)^{(s-1)/s}$.*

Proof. By Lemma 17 we have that if $d = \alpha n$, then there are at most $2k/\alpha$ vertices with degree $\geqslant \alpha n$ in G (α is a value to be chosen later). Let L be a layout of width at most k and V_{rich} be the set of vertices with degree at least αn. Notice that any vertex in L that is not in V_{rich} is adjacent to at most αn vertices in V_{rich}.

We take a layout L' of G such that the vertices in V_{rich} are the $|V_{\text{rich}}|$ first vertices. We arrange the rest of the vertices (we call them *poor vertices*) following the reversed order of their arrangement in L. Clearly the width$_s$ of each of the first $2k/\alpha$ vertices in L is at most $2k/\alpha$.

Notice that any poor vertex v can be adjacent to at most k vertices not preceding it in L'. Following the notation of Theorem 13, we define A as the set of vertices not preceding v in L' that are connected with v via paths of vertices not preceding v of length at most $s-1$. Also, let B be the set of vertices preceding v. Clearly width$_s(v) =$ deg $(A,B) \leqslant D(G[A]) = \sum_{u \in V(G[A])}(\Delta - \deg(u))$ (the degree of u is taken with respect to $G[A]$). If we observe that $\deg(v) \leqslant k$ in $G[A]$ and all the vertices in $V(G[A])$ have degree less than αn, then from Lemma 12, it follows that width$_s(v) \leqslant k(\alpha n - 2)^{s-1} + \alpha n - 1 - k \leqslant (k+1)(\alpha n)^{s-1}$. So width$_s(L') \leqslant \max\{\frac{2k}{\alpha}, (k+1)(\alpha n)^{s-1}\}$. Now, if we choose $\alpha = 2^{(1/s)} n^{-(s-1)/s}$, we have that width$_s(G) \leqslant (k+1)(2n)^{(s-1)/s}$. □

Theorem 19. *If lc(G) $\leqslant s+2$ and width(G) $\leqslant k$, then treewidth(G) $\leqslant (k+1)(2n)^{(s-1)/s}$. Moreover, the corresponding tree decomposition can be found in* $O(kn^{3-(1/s)})$ *time.*

Proof. Recall that, as width$(G) \leqslant k$, a layout L of G with width at most k can be constructed in $O(kn)$ time. Thus, if lc$(G) \leqslant s+2$, we can find (in $O(kn)$ time) a layout L as in Theorem 18 that has width$_s(L) \leqslant (k+1)(2n)^{(s-1)/s}$. Now, from Theorem 5, we have that there is a tree decomposition of G that has treewidth $\leqslant (k+1)(2n)^{(s-1)/s}$. According to the proof of Theorem 5 in [17], there is an algorithm that, given a layout of the vertices in G with width$_s$ at most k, outputs a tree decomposition of G with treewidth at most k in $O(kn^2)$ time. Thus, we can obtain in $O(kn^{3-(1/s)})$ time a tree decomposition of G that has treewidth $\leqslant (k+1)(2n)^{(s-1)/s}$. □

From Theorems 7 and 19, we have the following corollary.

Corollary 20. *Let $G=(V,E)$ be a graph and $w: V \to Q^+$ a function assigning a positive rational weight to each vertex in V. Then, if width(G) $\leqslant k$ and lc(G) $\leqslant s+2$, then there exist an $\frac{1}{2}$-separator of the function w in G with size at most $(k+1)(2n)^{(s-1)/s}$. Furthermore, such a separator is also a $\frac{2}{3}$-separator and can be found in* $O(kn^{3-(1/s)})$ *time.*

6. Conclusions

In this section, we give small upper bounds for the chordality of some well known classes of graphs. We also give an application of our separator theorem to the problem of approximating the maximum independent set problem.

Definition. A graph G is a *weakly chordal graph* iff neither G nor G^c contain a chordless cycle of length at least 5.

The class of weakly chordal graphs was introduced by Hayward in [23]. Clearly, all the weakly chordal graphs are 4-chordal (the same holds also for their complements). We mention that the class of weakly chordal graphs is quite a large one, as it contains the classes of co-chordal graphs, chordal bipartite graphs, permutation graphs, trapezoid graphs, tolerance graphs, 2-threshold graphs and others (see also [15]). It is also known that for chordal bipartite graphs, treewidth is polynomially computable in time $O(e^3)$ (see [28]) and pathwidth is NP-complete [27].

Definition. A graph $G=(V,E)$ is a *comparability* graph if there exist a partial order $<_*$ on V such that $\forall v,u \in V$, $\{v,u\} \in E$ iff $v <_* u$ or $u <_* v$ in P. A graph $G=(V,E)$ is a *cocomparability* graph if it is the complement of a comparability graph.

Gallai proved in [21] that if G is a cocomparability graph then G is 4-chordal. We mention that the class of cocomparability graphs properly contains the class of the cobipartite graphs where the problem of computing treewidth and pathwidth remains an NP-complete problem.

When additionally a degree restriction is put on the graphs, we have the following result, which can be derived directly from Theorem 14.

Corollary 21. *For any constant Δ, there exist*:
1. *A linear time algorithm that computes the treewidth of cocomparability graphs or weakly chordal graphs with maximum degree Δ.*
2. *A polynomial time algorithm that computes the pathwidth of graphs in cocomparability graphs or weakly chordal graphs with maximum degree Δ.*
3. *A $O(\log^2 n)$ time parallel algorithm that computes the treewidth of cocomparability graphs or weakly chordal graphs with maximum degree Δ, and that uses $O(n/\log^2 n)$ processors on an EREW PRAM.*

Each of the above algorithms outputs the corresponding tree or path decomposition of minimum treewidth or pathwidth.

Also, the next result follows from Corollary 20.

Corollary 22. *Let $G=(V,E)$ be a graph and let $w: V \to Q^+$ be a function assigning a positive rational weight to each vertex in V. If G is a 4-chordal graph and width$(G) \leqslant k$, then treewidth$(G) \leqslant (k+1)\sqrt{2n}$ and there is a $O(n^{2.5})$ time algorithm computing an $\frac{1}{2}$-separator (or a $\frac{2}{3}$-separator) of w in G of size at most $(k+1)\sqrt{2n}$.*

As an additional example of classes of graphs with a constant upper bound on the chordality, we mention the graphs that are complements of r-partite graphs (graphs with chromatic number at most r): these do not have a chordless cycle of length more than

$2r$. Also, the classes of diametral path graphs and dominating pair graphs introduced by Deogun and Kratsch in [19] (see also [18]) are 6-chordal. We also mention that the class of diametral path graphs properly contains the class of the asteroidal triple-free (AT-free) graphs which are 5-chordal (see [16]).

It is easy to prove that any graph that does not contain a specific graph as a minor has constant bounded width (see also [14]). Therefore, the separator result of Corollary 7 straightforwardly extents the results of Lipton and Tarjan in [32, 33] and Alon, Seymour, and Thomas in [1] in the setting of 4-chordal graphs with small width. Moreover, Theorem 20 can give applications for any class of graphs where chordality is bounded by a constant $s+2$ and width is small enough (width$(G) = O(n^{(1/s)-\varepsilon})$, $\varepsilon > 1$).

We present below the application of our separator theorem to the problem of approximating the independent set problem on $(s+2)$-chordal graphs with constant width.

We examine the non trivial case where $s+2>3$. Let G be a given $(s+2)$-chordal graph where width$\leqslant k$ where k is a fixed constant.

By repeatedly finding a $\frac{2}{3}$-separator as in Corollary 20 we can obtain the following immediate generalization of Theorem 3 in [33].

Proposition. *Let $s \geqslant 1$, k be constants. Let $\mathscr{G}$ be the set of $(s+2)$-chordal graphs G with width$(G) \leqslant k$, given with a function $w : V \to Q^+$ assigning positive rational weights to the vertices of G such that $\sum_{v\in V} w(v) = 1$. Then there is an $O(n^{3-(1/s)} \log n)$ algorithm, that when given an ε with $0 < \varepsilon \leqslant 1$, and a graph $G \in \mathscr{G}$, finds a set C of at most $O(n^{(s-1)/s}\varepsilon^{-1/s})$ vertices, whose removal leaves G with no connected component of total weight exceeding ε.*

We omit the proof of the above proposition as it is the same with the one given by Lipton and Tarjan in [33] for the case of the planar graphs (the only difference is that now, the size of the separators is $O(n^{(s-1)/s})$ and the time to split each component is $O(n^{3-(1/s)})$).

Applying now the previous proposition with $\varepsilon = \log n/n$ and giving each vertex weight $1/n$, we can find a set of vertices C of size $O(n/\log^{1/s} n)$, whose removal leaves no connected component with more than $\log n$ vertices. If now we apply exhaustive search to each connected component, we can find a collection of independent sets whose union is denoted as I. Let I^* be a maximum independent set in G. The restriction of I^* to one of the connected components formed when C is removed from G can be no larger than the restriction of I to the same component. Thus, $|I^*| - |I| = O(n/\log^{1/s} n)$. As the width of a graph minus one is greater that its chromatic number, it is easy to prove that, $|I^*| \geqslant n/(k+1)$. Thus $(|I^*| - |I|)/|I^*| = O(1/\log^{1/s} n)$ and the relative error in the size of I tends to zero with increasing n. From the proposition above, C can be found in $O(n^{3-(1/s)} \log n)$ time. Exaustive search in each of the, say r, connected components of $G[V-C]$ costs $O(n_i 2^{n_i})$ time (where n_i the size of the component). Thus, the total time required to find the independent sets of all the components is:

$$\mathrm{O}\left(\max\left\{\sum_{i=1,\dots,r} n_i 2^{n_i} : \sum_{i=1,\dots,r} n_i = \mathrm{O}(n),\ 0 \leqslant n_i \leqslant \log n\right\}\right)$$

$$= \mathrm{O}\left(\frac{n}{\log n}(\log n)2^{\log n}\right) = \mathrm{O}(n^2).$$

Thus, we can conclude to the following:

Theorem 23. *Given an $(s+2)$-chordal graph G that has constant width, there is an $\mathrm{O}(n^{3-(1/s)}\log n)$ algorithm that finds an independent set I in G with relative error $(|I^*| - |I|)/|I| = \mathrm{O}(1/\sqrt[s]{\log n})$, where I^* is a maximum independent set.*

7. Discussion

From Theorem 7 it follows that, in a graph whose treewidth is small comparatively to the number of its vertices (e.g., treewidth$(G) = \mathrm{O}(\sqrt{n})$), there exist also a (nearly) equal size separating set of vertices. Using this fact, it would be useful to determine classes of graphs where treewidth is small enough to provide a separator theorem.

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