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# Characterizing graphs of small carving-width☆

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ABSTRACT

We characterize all graphs that have carving-width at most $k$ for $k = 1, 2, 3$. In particular, we show that a graph has carving-width at most 3 if and only if it has maximum degree at most 3 and treewidth at most 2. This enables us to identify the immersion obstruction set for graphs of carving-width at most 3.



## 1. Introduction

All graphs considered in this paper are finite and undirected, have no self-loops but may have multiple edges. A graph that has no multiple edges is called *simple*. For undefined graph terminology we refer the reader to the textbook of Diestel [7]. A *carving* of a graph $G$ is a tree $T$ whose internal vertices all have degree 3 and whose leaves correspond to the vertices of $G$. For every edge $e$ of $T$, deleting $e$ from $T$ yields exactly two trees, whose leaves define a bipartition of the vertices of $G$; we say that the edge cut in $G$ corresponding to this bipartition is *induced by* $e$. The *width* of a carving $T$ is the maximum size of an edge cut in $G$ that is induced by an edge of $T$. The *carving-width* of $G$ is the minimum width of a carving of $G$.

Carving-width was introduced by Seymour and Thomas [17], who proved that checking whether the carving-width of a graph is at most $k$ is an NP-complete problem. In the same paper, they proved that there is a polynomial-time algorithm for computing the carving-width of planar graphs. Later, the problem of constructing carvings of minimum width was studied by Khuller [12], who presented a polynomial-time algorithm for constructing a carving $T$ whose width is within a $O(\log n)$ factor from the optimal. In [20] an algorithm was given that decides, in $f(k)\cdot n$ steps, whether an $n$-vertex graph $G$ has carving-width at most $k$ and, if so, also outputs a corresponding carving of $G$. We stress that the values of $f(k)$ in the complexity of the algorithm in [20] are huge, which makes the algorithm highly impractical even for trivial values of $k$.

A graph $G$ contains a graph $H$ as an immersion if $H$ can be obtained from some subgraph of $G$ after lifting a number of edges (see Section 2 for the complete definition). Recently, the immersion relation attracted a lot of attention both from the combinatorial [1,6,9,21] and the algorithmic [10,11] points of view. It can easily be observed (cf. [20]) that carving-width

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is a parameter closed under taking immersions, i.e., the carving-width of a graph is not smaller than the carving-width of any of its immersions. Combining this fact with the seminal result of Robertson and Seymour in [14] stating that graphs are well-quasi-ordered with respect to the immersion relation, it follows that the set $\mathcal{G}_k$ of graphs with carving-width at most $k$ can be completely characterized by forbidding a finite set of graphs as immersions. This set is called an *immersion obstruction set* for the class $\mathcal{G}_k$.

Identifying obstruction sets is a classic problem in structural graph theory, and its difficulty may vary, depending on the considered graph class. While obstructions have been extensively studied for parameters that are closed under minors (see [2,5,8,13,15,16,18,19] for a sample of such results), no obstruction characterization is known for any immersion-closed graph class. In this paper, we make a first step in this direction.

The outcome of our results is the identification of the immersion obstruction set for $\mathcal{G}_k$ when $k \leq 3$; the obstruction set for the non-trivial case $k = 3$ is depicted in Fig. 3. Our proof for this case is based on a combinatorial result stating that $\mathcal{G}_3$ consists of exactly the graphs with maximum degree at most 3 and treewidth at most 2. A direct implication of our results is a linear-time algorithm for the recognition of the class $\mathcal{G}_k$ when $k = 1, 2, 3$. This can be seen as a "tailor-made" alternative to the general algorithm of [20] for elementary values of $k$.

## 2. Preliminaries

Let $G = (V, E)$ be a graph, and let $S \subset V$ be a subset of vertices of $G$. Then the set of edges between $S$ and $V \setminus S$, denoted by $(S, V \setminus S)$, is an *edge cut* of $G$. Let the vertices of $G$ be in 1-to-1 correspondence to the leaves of a tree $T$ whose internal vertices all have degree 3. The correspondence between the leaves of $T$ and the vertices of $G$ uniquely defines the following edge weighting $w$ on the edges of $T$. Let $e \in E_T$, and let $C_1$ and $C_2$ be the two connected components of $T - e$. Let $S_i$ be the set of leaves of $T$ that are in $C_i$ for $i = 1, 2$; note that $S_2 = V \setminus S_1$. Then the weight $w(e)$ of the edge $e$ in $T$ is the number of edges in the edge cut $(S_1, S_2)$ of $G$. The tree $T$ is called a *carving* of $G$, and $(T, w)$ is a *carving decomposition* of $G$. The *width* of a carving decomposition $(T, w)$ is the maximum weight $w(e)$ over all $e \in E_T$. The *carving-width* of $G$, denoted by $\text{cw}(G)$, is the minimum width over all carving decompositions of $G$. We define $\text{cw}(G) = 0$ if $|V| = 1$. We refer to Fig. 4 for an example of a graph and a carving decomposition.

A *tree decomposition* of a graph $G = (V, E)$ is a pair $(\mathcal{T}, \mathcal{X})$, where $\mathcal{X}$ is a collection of subsets of $V$, called *bags*, and $\mathcal{T}$ is a tree whose vertices, called *nodes*, are the sets of $\mathcal{X}$, such that the following three properties are satisfied:

(i) for each $u \in V$, there is a bag $X \in \mathcal{X}$ with $u \in X$;
(ii) for each $uv \in E$, there is a bag $X \in \mathcal{X}$ with $u, v \in X$;
(iii) for each $u \in V$, the nodes containing $u$ induce a connected subtree of $\mathcal{T}$.

The *width* of a tree decomposition $(\mathcal{T}, \mathcal{X})$ is the size of a largest bag in $\mathcal{X}$ minus 1. The *treewidth* of $G$, denoted by $\text{tw}(G)$, is the minimum width over all possible tree decompositions of $G$.

Let $G = (V, E)$ be a graph, and let $uv$ be an edge of $G$. The *contraction* of the edge $uv$ is the operation that deletes $u$ and $v$ from $G$ and replaces them by a new vertex $x$ that is made adjacent to the neighbors of $u$ and of $v$ in $G$, such that for every vertex $w \in V \setminus \{u, v\}$, the number of edges between $x$ and $w$ in the new graph is equal to the number of edges between $w$ and $\{u, v\}$ in $G$. A graph $G$ contains a graph $H$ as a *minor* if $H$ can be obtained from $G$ by a sequence of vertex deletions, edge deletions and edge contractions. The following two well-known properties of treewidth will be used in the proof of our main result.

**Lemma 1** (*Cf. [4]*)**.** *Let $G$ be a simple graph and $k$ an integer. If $\text{tw}(G) \leq k$, then $G$ contains a vertex of degree at most $k$.*

**Lemma 2** (*Cf. [4,7]*)**.** *Let $G$ be a graph. Then $\text{tw}(H) \leq \text{tw}(G)$ for every minor $H$ of $G$.*

The *subdivision* of an edge $uv$ is the operation that deletes the edge $uv$ from the graph and adds a new vertex $w$ as well as two new edges $uw$ and $vw$. The reverse operation is called *vertex dissolution*; this operation removes a vertex $v$ of degree 2 that has two distinct neighbors $u$ and $w$, and adds a new edge between $u$ and $w$, regardless of whether or not there already exist edges between $u$ and $w$. A graph $G$ contains a graph $H$ as a *topological minor* if $H$ can be obtained from $G$ by a sequence of vertex deletions, edge deletions, and vertex dissolutions. Equivalently, $G$ contains $H$ as a topological minor if $G$ contains a subgraph $H'$ that is a *subdivision* of $H$, i.e., $H'$ can be obtained from $H$ by a sequence of edge subdivisions. The following lemma is obtained by combining some well-known properties of treewidth, minors, and topological minors.

**Lemma 3** (*Cf. [7]*)**.** *A graph has treewidth at most 2 if and only if it does not contain $K_4$ as a topological minor.*

Let $u, v, w$ be three distinct vertices in a graph such that $uv$ and $vw$ are edges. The operation that removes the edges $uv$ and $vw$, and adds the edge $uw$ (even in the case $u$ and $w$ are already adjacent) is called a *lift*. A graph $G$ contains a graph $H$ as an *immersion* if $H$ can be obtained from $G$ by a sequence of vertex deletions, edge deletions, and lifts. Note that dissolving a vertex $v$ of degree 2 that has two distinct neighbors $u$ and $w$ is equivalent to first lifting the edges $uv$ and $vw$ and then deleting the vertex $v$. Hence, it readily follows from the definitions of topological minors and immersions that every topological minor of graph $G$ is also an immersion of $G$.

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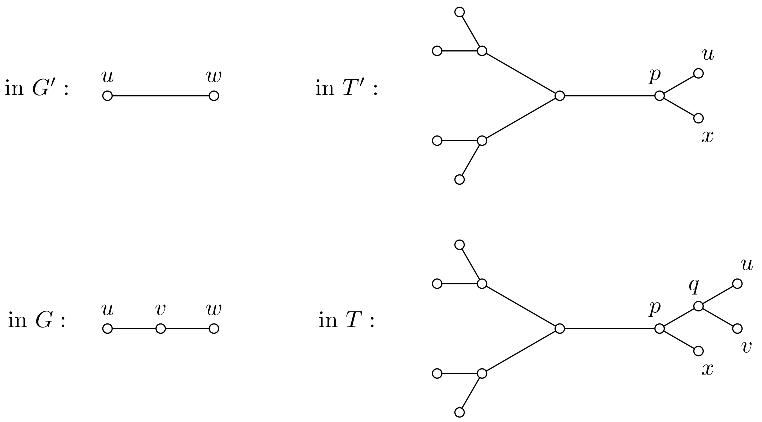


**Fig. 1.** A schematic illustration of how the tree $T'$ in the carving decomposition of $G'$ is transformed into a tree $T$ in the proof of Lemma 6 when the edge $uw$ in $G'$ is subdivided. The vertex $x$ is an arbitrary vertex of $G'$, possibly $w$.

## 3. The main result

We begin this section by stating some useful properties of carving-width. The following observation is known and easy to verify by considering the number of edges in the edge cut $(\{u\}, V \setminus \{u\})$ of a graph $G = (V, E)$.

**Observation 1.** *Let G be a graph. Then* $\mathrm{cw}(G) \geq \Delta(G)$.

We also need the following two straightforward lemmas. The first lemma follows from the observation that any subgraph of a graph is an immersion of that graph, combined with the observation that carving-width is a parameter that is closed under taking immersions (cf. [20]). We include the proof of the second lemma for completeness.

**Lemma 4.** *Let G be a graph. Then* $\mathrm{cw}(H) \leq \mathrm{cw}(G)$ *for every subgraph H of G.*

**Lemma 5.** *Let G be a graph with connected components* $G_1, \ldots, G_p$ *for some integer* $p \geq 1$. *Then* $\mathrm{cw}(G) = \max\{\mathrm{cw}(G_i) \mid 1 \leq i \leq p\}$.

**Proof.** Since the carving-width of a graph with only one vertex is defined to be 0, the lemma clearly holds if $G$ has no edges. Suppose $G$ has at least one edge. Lemma 4 implies that $\max\{\mathrm{cw}(G_i) \mid 1 \leq i \leq p\} \leq \mathrm{cw}(G)$. Now let $(T_i, w_i)$ be a carving decomposition of $G_i$ of width $\mathrm{cw}(G_i)$ for $i = 1, \ldots, p$. Since deleting isolated vertices does not change the carving-width of a graph, we may without loss of generality assume that $G$ has no isolated vertices. In particular, this means that each tree $T_i$ contains at least one edge. We construct a carving decomposition $(T, w)$ of $G$ from the $p$ carving decompositions $(T_i, w_i)$ as follows.

We pick an arbitrary edge $e_i = x_iy_i$ in each $T_i$. For each $i \in \{2, \ldots, p-1\}$, we subdivide the edge $e_i$ twice by replacing it with edges $x_iz_i$, $z_iz_i'$ and $z_i'y_i$, where $z_i$ and $z_i'$ are two new vertices. The edges $e_1$ and $e_p$ are subdivided only once: the edge $e_1$ is replaced with a new vertex $z_1$ and two new edges $x_1z_1$ and $z_1y_1$, and the edge $e_p$ is replaced with a new vertex $z_p'$ and two new edges $x_pz_p'$ and $z_p'y_p$. Finally, we add the edge $z_iz_{i+1}'$ for each $i \in \{1, \ldots, p-1\}$. This results in a tree $T$ whose internal vertices all have degree 3. Since there are no edges between any two connected components of $G$, the corresponding carving decomposition $(T, w)$ of $G$ has width $\max\{\mathrm{cw}(G_i) \mid 1 \leq i \leq p\}$. Hence, $\mathrm{cw}(G) \leq \max\{\mathrm{cw}(G_i) \mid 1 \leq i \leq p\}$. We conclude that $\mathrm{cw}(G) = \max\{\mathrm{cw}(G_i) \mid 1 \leq i \leq p\}$. □

The next lemma is the final lemma we need in order to prove our main result.

**Lemma 6.** *Let* $G'$ *be a graph with carving-width at least* 2, *and let* $uw$ *be an edge of* $G'$. *Let G be the graph obtained from* $G'$ *by subdividing the edge* $uw$. *Then* $\mathrm{cw}(G) = \mathrm{cw}(G')$.

**Proof.** Let $(T', w')$ be a carving decomposition of $G'$ of width $\mathrm{cw}(G') \geq 2$, and let $p$ be the unique neighbor of $u$ in $T'$. Let $v$ be the vertex that was used to subdivide the edge $uw$ in $G'$, i.e., the graph $G$ was obtained from $G'$ by replacing $uw$ with edges $uv$ and $vw$ for some new vertex $v$. Let $T$ be the tree obtained from $T'$ by first relabeling the leaf in $T'$ corresponding to vertex $u$ by $q$, and then adding two new vertices $u$ and $v$ as well as two new edges $qu$ and $qv$; see Fig. 1 for an illustration. Let us show that the resulting carving decomposition $(T, w)$ of $G$ has width at most $\mathrm{cw}(G')$.

Let $e$ be an edge in $T$. Suppose that $e = pq$. By definition, $w(e)$ is the number of edges between $\{u, v\}$ and $V \setminus \{u, v\}$ in $G$, which is equal to the number of edges incident with $u$ in $G'$. The latter number is the weight of the edge $pu$ in $T'$. Hence, $w(e) \leq \mathrm{cw}(G')$. Suppose that $e = qu$. By definition, $w(e)$ is the number of edges incident with $u$ in $G$, which is equal to the number of edges incident with $u$ in $G'$. Hence $w(e) \leq \mathrm{cw}(G')$. Suppose that $e = qv$. By definition, $w(e)$ is the number

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of edges incident with $v$ in $G$, which is 2. Hence $w(e) = 2 \leq \mathrm{cw}(G')$. Finally, suppose that $e \notin \{pq, qu, qv\}$. Let $C_1$ and $C_2$ denote the subtrees of $T$ obtained after removing $e$. Let $S_i$ be the set of leaves of $T$ in $C_i$ for $i = 1, 2$. Then $u$ and $v$ either both belong to $S_1$ or both belong to $S_2$. Without loss of generality, assume that both $u$ and $v$ belong to $S_1$. By definition, $w(e)$ is the number of edges between $S_1$ and $S_2$ in $G$, which is equal to the number of edges between $S_1 \setminus \{v\}$ and $S_2$ in $G'$. The latter number is the weight of the edge $e$ in $T'$. Hence, $w(e) \leq \mathrm{cw}(G')$. We conclude that $(T, w)$ has width at most $\mathrm{cw}(G')$, and hence $\mathrm{cw}(G) \leq \mathrm{cw}(G')$.

It remains to show that $\mathrm{cw}(G) \geq \mathrm{cw}(G')$. Let $(T^*, w^*)$ be a carving decomposition of $G$ of width $\mathrm{cw}(G)$. We remove the leaf corresponding to $v$ from $T^*$. Afterwards, the neighbor of $v$ in $T^*$ has degree 2, and we dissolve this vertex. This results in a tree $T''$. It is easy to see that the corresponding carving decomposition $(T'', w'')$ of $G'$ has width at most $\mathrm{cw}(G)$. Hence, $\mathrm{cw}(G) \geq \mathrm{cw}(G')$. This completes the proof of Lemma 6. □

We are now ready to show the main result of our paper.

**Theorem 1.** *Let G be a graph. Then the following three statements hold.*

(i) $\mathrm{cw}(G) \leq 1$ *if and only if* $\Delta(G) \leq 1$.
(ii) $\mathrm{cw}(G) \leq 2$ *if and only if* $\Delta(G) \leq 2$.
(iii) $\mathrm{cw}(G) \leq 3$ *if and only if* $\Delta(G) \leq 3$ *and* $\mathrm{tw}(G) \leq 2$.

**Proof.** Let $G = (V, E)$ be a graph. By Lemma 5 we may assume that $G$ is connected. We prove the three statements separately.

(i) If $\mathrm{cw}(G) \leq 1$, then $\Delta(G) \leq 1$ due to Observation 1. If $\Delta(G) \leq 1$, then $G$ is isomorphic to either $K_1$ or $K_2$. Clearly, $\mathrm{cw}(G) \leq 1$ in both cases.

(ii) If $\mathrm{cw}(G) \leq 2$, then $\Delta(G) \leq 2$ due to Observation 1. If $\Delta(G) = 1$, then $\mathrm{cw}(G) \leq 1$ follows from (i). If $\Delta(G) = 2$, then $G$ is either a graph consisting of two vertices with two edges between them, or a simple graph that is either a path or a cycle. In all three cases, it is clear that $\mathrm{cw}(G) \leq 2$.

(iii) First suppose that $\mathrm{cw}(G) \leq 3$. Then $\Delta(G) \leq 3$ due to Observation 1. We need to show that $\mathrm{tw}(G) \leq 2$. For contradiction, suppose that $\mathrm{tw}(G) \geq 3$. Then, by Lemma 3, $G$ contains $K_4$ as a topological minor, i.e., $G$ contains a subgraph $H$ such that $H$ is a subdivision of $K_4$. Since $\mathrm{cw}(K_4) = 4$, we have that $\mathrm{cw}(H) = \mathrm{cw}(K_4) = 4$ as a result of Lemma 6. Since $H$ is a subgraph of $G$, Lemma 4 implies that $\mathrm{cw}(G) \geq \mathrm{cw}(H) = 4$, contradicting the assumption that $\mathrm{cw}(G) \leq 3$.

For the reverse direction, we need to prove that every graph $G = (V, E)$ with $\Delta(G) \leq 3$ and $\mathrm{tw}(G) \leq 2$ has carving-width at most 3. We use induction on $|V|$. If $|V| \leq 2$, then $G$ is either isomorphic to $K_1$ or $K_2$, or $G$ consists of two vertices with exactly two edges between them. It is clear that $\mathrm{cw}(G) \leq 3$ in each of these cases. From now on, we assume that $|V| \geq 3$.

First, suppose that $G$ contains two vertices $u$ and $v$ with at least two edges between them. Since $|V| \geq 3$, we may without loss of generality assume that $v$ has a neighbor $t \neq u$. Then, because $\Delta(G) \leq 3$ and there are at least two edges between $u$ and $v$ in $G$, we find that $t$ and $u$ are the only two neighbors of $v$ in $G$ and that the number of edges between $u$ and $v$ is exactly 2. Let $G^*$ denote the graph obtained from $G$ by deleting one edge between $u$ and $v$, and let $G'$ denote the graph obtained from $G^*$ by dissolving $v$. Note that $G'$ is a connected graph on $|V| - 1$ vertices, and $\Delta(G') \leq 3$. Moreover, since $G'$ is a topological minor and hence a minor of $G$, Lemma 2 ensures that $\mathrm{tw}(G') \leq \mathrm{tw}(G) \leq 2$. Consequently, we can apply the induction hypothesis to deduce that $\mathrm{cw}(G') \leq 3$.

If $\mathrm{cw}(G') \leq 1$, then $\Delta(G') \leq 1$ by Observation 1. Since $G'$ is a connected graph on $|V| - 1 \geq 2$ vertices, $G'$ must be a path on two vertices. Then $G^*$ is a path on three vertices, implying that $\mathrm{cw}(G^*) = 2$. Since $G$ can be obtained from $G^*$ by adding a single edge, $\mathrm{cw}(G) \leq 3$ in this case. Suppose $2 \leq \mathrm{cw}(G') \leq 3$. Then, by Lemma 6, $\mathrm{cw}(G^*) = \mathrm{cw}(G') \leq 3$. Moreover, from the proof of Lemma 6 it is clear that there exists a carving decomposition $(T^*, w^*)$ of $G^*$ of width $\mathrm{cw}(G^*)$ such that $u$ and $v$ have a common neighbor $q$ in $T^*$. We consider the carving decomposition $(T, w)$ of $G$ with $T = T^*$. Let $e$ be an edge in $T$. First suppose that $e = uq$ or $e = vq$. Then $w(e) \leq 3$, as both $u$ and $v$ have degree at most 3 in $G$. Now suppose that $e \notin \{uq, vq\}$. Then $w(e) = w^*(e) \leq \mathrm{cw}(G^*) \leq 3$. We conclude that the carving decomposition $(T, w)$ of $G$ has width at most 3, which implies that $\mathrm{cw}(G) \leq 3$.

From now on, we assume that $G$ contains no multiple edges, i.e., we assume that $G$ is simple. Since $\mathrm{tw}(G) \leq 2$, $G$ contains a vertex of degree at most 2 due to Lemma 1.

Suppose $G$ contains a vertex $u$ of degree 1. Let $v$ be the neighbor of $u$ in $G$, and let $G'$ be the graph obtained from $G$ by deleting $u$. It is clear that $\Delta(G') \leq 3$ and $\mathrm{tw}(G') \leq 2$, so $\mathrm{cw}(G') \leq 3$ by the induction hypothesis. Let $(T', w')$ be a carving decomposition of $G'$ of width $\mathrm{cw}(G')$. Let $T$ be the tree obtained from $T'$ by first changing the label of the leaf of $T'$ corresponding to vertex $v$ into $p$, and then adding two new vertices $u$ and $v$ and two new edges $pu$ and $pv$; see Fig. 2. Since $v$ is the only neighbor of $u$ in $G$, it is easy to see that the width of the resulting carving decomposition $(T, w)$ of $G$ is at most $\mathrm{cw}(G')$, which implies that $\mathrm{cw}(G) \leq \mathrm{cw}(G') \leq 2$.

Finally, suppose that $G$ contains a vertex $u$ of degree 2. Since we assume $G$ to be simple, $u$ has two distinct neighbors $v$ and $t$. Let $G' = (V', E')$ denote the connected graph obtained from $G$ by dissolving $u$. Note that $G'$ has maximum degree at most 3, and that $\mathrm{tw}(G') \leq 2$ due to Lemma 2 and the fact that $G'$ is a minor of $G$. Hence, by the induction hypothesis, $\mathrm{cw}(G') \leq 3$. If $\mathrm{cw}(G') \leq 1$, then $\Delta(G') \leq 1$ by Observation 1. This, together with the observation that $G'$ is a connected graph on $|V| - 1 \geq 2$ vertices, implies that $G'$ is a path on two vertices. Consequently, $G$ is a path on three vertices, and hence $\mathrm{cw}(G) = 2 \leq 3$. If $2 \leq \mathrm{cw}(G') \leq 3$, then we can apply Lemma 6 to conclude that $\mathrm{cw}(G) = \mathrm{cw}(G') \leq 3$. This completes the proof of Theorem 1. □

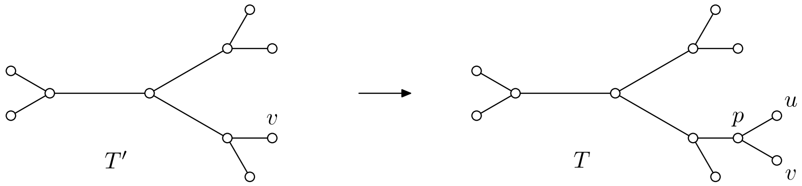


**Fig. 2.** A schematic illustration of how the tree $T$ is constructed from the tree $T'$ in the proof of Theorem 1.

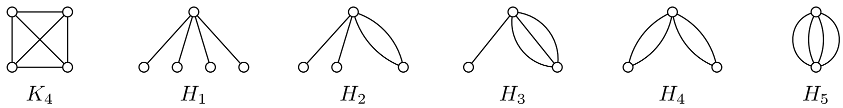


**Fig. 3.** The immersion obstruction set for graphs of carving-width at most 3.

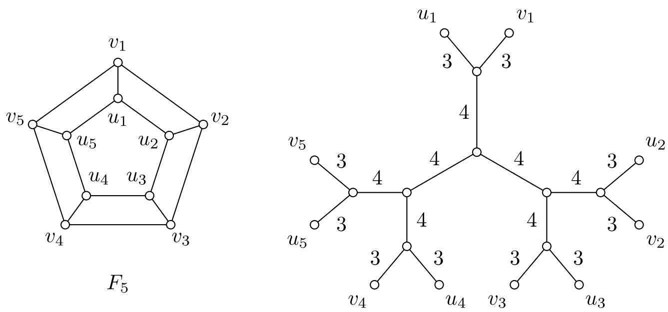


**Fig. 4.** The pentagonal prism $F_5$ and a carving decomposition $(T, w)$ of $F_5$ that has width 4.

Since graphs of treewidth at most 2 can easily be recognized in linear time, Theorem 1 implies a linear-time recognition algorithm for graphs of carving-width at most 3.

Thilikos, Serna and Bodlaender [20] proved that for any $k$, there exists a linear-time algorithm for constructing the immersion obstruction set for graphs of carving-width at most $k$. For $k \in \{1, 2\}$, finding such a set is trivial. We now present an explicit description of the immersion obstruction set for graphs of carving-width at most 3.

**Corollary 1.** *A graph has carving-width at most* 3 *if and only if it does not contain any of the six graphs in Fig. 3 as an immersion.*

**Proof.** Let $G$ be a graph. We first show that if $G$ contains one of the graphs in Fig. 3 as an immersion, then $G$ has carving-width at least 4. In order to see this, it suffices to observe that the graphs $K_4, H_1, \ldots, H_5$ all have carving-width 4. Hence, $G$ has carving-width at least 4, because carving-width is a parameter that is closed under taking immersions (cf. [20]).

Now suppose that $G$ has carving-width at least 4. Then, due to Theorem 1, $\Delta(G) \geq 4$ or $\text{tw}(G) \geq 3$. If $\Delta(G) \geq 4$, then $G$ has a vertex $v$ of degree at least 4. By considering $v$ and four of its incident edges, it is clear that $G$ contains one of the graphs $H_1, \ldots, H_5$ as a subgraph, and consequently as an immersion. If $\text{tw}(G) \geq 3$, then Lemma 3 implies that $G$ contains $K_4$ as a topological minor, and consequently as an immersion. □

From the proof of Corollary 1, we can observe that an alternative version of Corollary 1 states that a graph has carving-width at most 3 if and only if it does not contain any of the six graphs in Fig. 3 as a topological minor.

## 4. Conclusions

Extending Theorem 1 to higher values of carving-width remains an open problem, and finding the immersion obstruction set for graphs of carving-width at most 4 already seems to be a challenging task. We proved that for any graph $G$, $\text{cw}(G) \leq 3$ if and only if $\Delta(G) \leq 3$ and $\text{tw}(G) \leq 2$. We finish our paper by showing that the equivalence "$\text{cw}(G) \leq 4$ if and only if $\Delta(G) \leq 4$ and $\text{tw}(G) \leq 3$" does not hold in either direction.

To show that the forward implication is false, we consider the pentagonal prism $F_5$, which is displayed in Fig. 4 together with a carving decomposition $(T, w)$ of width 4. Hence, $\text{cw}(F_5) \leq 4$. However, $F_5$ is a minimal obstruction for graphs of treewidth at most 3 [4], implying that $\text{tw}(F_5) = 4$.

To show that the backward implication is false, we consider the graph $K_5^-$, which is the graph obtained from $K_5$ by removing an edge. Note that $\Delta(K_5^-) = 4$ and $\text{tw}(K_5^-) = 3$. It is not hard to verify that $\text{cw}(K_5) = 6$. Since removing an edge decreases the carving-width by at most 1, we conclude that $\text{cw}(K_5^-) \geq 5$.

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