GRAph Searching, Theory and Applications (GRASTA 2012)

Banff International Research Station
on Mathematical Innovation and discovery
Graph Searching (12w5055), October 8-12, 2012

Organizers

Fedor Fomin (University of Bergen)
Richard Nowakowski (Dalhousie University)
Pawel Pralat (Ryerson University)
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Overview of the Field

Graph searching can be seen as a game between fugitives and a set of searchers. The fugitives reside in a system of rooms and corridors represented by some graph $G$ (vertices of $G$ represent rooms while edges represent corridors between rooms). The purpose of searchers is to catch the fugitives who are trying to avoid capture. The first mathematical models on Graph Searching had the constraint that the searchers had imperfect information as to the location of the fugitives, but the fugitives were omniscient. Torrence Parsons and Nikolai Petrov in the 1970s introduced the first variants, along with the corresponding algorithmic and complexity results, appeared during the 80’s. The version with both sides having perfect information (fancifully called Cops and Robbers) was introduced in the 1980s by Nowakowski & Winkler, and Quilliot. Graph searching revealed the need to express in a formal mathematical way intuitive concepts such as avoidance, surrounding, sense of direction, hiding, persecution, and threatening. One of the most powerful combinatorial tools used in the study of such structures emerged from the Graph Minors theory, developed by Robertson and Seymour towards proving the long-standing Wagner’s Conjecture. The collection of results and methodologies derived from the Graph Minors Theorem are acknowledged as among the most influential results in modern combinatorics. They include deep graph-theoretic results and techniques with direct consequences to problems at the kernel of Graph Searching problems. For example, Robertson and Seymour showed that the number of Marshals needed in the Robber and Marshals game on a graph (a variant of Cops and Robbers) equalled the tree width of the graph. During the last years, the interest on this field increased dramatically as new challenging questions appeared that are not directly connected to the classic graph theoretical approach.

Various constraints can be placed on the participants. For example: both sides have perfect information but travel with different speeds. The fugitives can be also regarded as a contaminant (chemical or biological) initially present everywhere in the graph, if the contamination can only be contained then this is Firefighter or Seepage problem; if the contamination can be cleared then this equivalent to the Graph Searching Problem but depending upon the speed of recontamination, this might also be the Cleaning with Brushes problem (similar to the chip firing game) originating from the need to have robots clean networks with conditions that do not allow access to humans (e.g. cleaning the cooling pipes in a nuclear power plant, or cleaning biofilm from small pipes). There may also be secondary constraints applied to
the searchers, minimum cost or number of time steps allowed.

Overview of the week

Presentation Highlights

All the talks were of high quality, informative and were thought-provoking.

A. Bonato set the stage with an overview of the latest results and trends in Cops-&-Robbers. J. Kratochvil and T. Muller talked about the effect on the cop-number of playing in restricted classes of graphs. Of special interest were the talks that introduced a new model of searching: L. Erickson (location games), N. Nisse (the surveillance game, which is a direct translation of a web-server problem into a game), and D. West (spies and revolutionaries).

Outcomes

Each afternoon and evening, small groups convened and researched problems. Deserving special mention is D. Wests talks both formal and informal. In his formal talks, West introduced an exciting new metaphor, roughly that of spies ‘eavesdropping on revolutionaries. In informal discussions, West discussed the ‘acquisition problem. This model is in the chip firing, cleaning and pebbling class of models but with the constraint that, locally, the rich get richer.

The main outcome, one that would not have been possible without the BIRS Workshop, is the collaborations that started during the meeting and that are still active. Some of them include:

- Bonato, Gordinowicz, Kinnersley, Pralat, Capture time for cubes;
- Gordinowicz, Pralat, generalized Cleaning processes;
- Dudek, Gordinowicz, Pralat, Cops and robbers playing on edges;
- Diaz, Stacho, Widmayer, Searching for a round trip in a simple grid;
- Kinnersley, West, Boolean Cops and Robber;
- Finbow, Messinger, Seamone, Firefighter variations;
- Finbow, Messinger, Seamone, Fractional chip firing;
- Dereniowski, Dyer, Yang, Zero-visibility cop number problem;
- Dereniowski, Gavenciak, Kratochvil, Cops and infinitely-fast Robber played on (proper) interval graphs;
- Mitsche, Pralat, Acquisition problems (on graphs);
• Fisher, Nowakowski, Ottaway, Acquisition problems (on graphs as a combinatorial game).

These collaborations have led to papers that are in the ArXiv or that have been submitted:

• The Total Acquisition Number of Random Graphs (Bal, Bennett, Dudek, Pralat) arXiv:1402.2854 [math.CO].


• D. Dereniowski, O. Y. Diner, D. Dyer, Three-fast-searchable graphs.


• D. Bryant, N. Francetic, P. Gordinowicz, D. Pike, and P. Pralat, Brushing without capacity restrictions.


Another effect of bringing the researchers together directly resulted in the formation of an annual Canadian (although others are welcome) Graph Searching Meeting.

Open Problems

Cops and Robbers

1. Pawel Pralat, Ryerson University

*Meyniel’s conjecture*

The biggest open conjecture in the area of cops and robbers is the one of Meyniel, which asserts that for some absolute constant $C$, the cop number of every connected graph $G$ is at most $C \sqrt{n}$, where $n = |V(G)|$. Today we only know that the cop number is at most $n^{-1-o(1)} \sqrt{\log_2 n}$ (which is still $n^{1-o(1)}$) for any connected graph on $n$ vertices.
2. Gena Hahn, University of Montreal

Let $G$ be a finite graph, $c(G)$ its cop-number and $g(G)$ its genus. Schroeder proved that $c(G) \leq \lceil \frac{3}{2}g(G) \rceil + 3$ and conjectured that $c(G) \leq g(G) + 3$.

Find a toroidal graph that needs 4 cops to catch a robber. Note that Andreae thinks that $c(G) \leq 3$ for toroidal $G$, so another way to approach the question is to prove that he is right.

Prove (or disprove) Schroeder’s conjecture.

Let $T$ be a tournament obtained from a Steiner triple system by orienting the edges of each of the triples in a triangle decomposition of the appropriate complete graph in a cycle. Nowakowski asked if $c(T) \leq 2$ and Thériault found - by computer search - that this is not the case. Is there a constant $c$ such that $c(T) \leq c$ for each tournament obtained in the way described?

3. Richard Nowakowski, Dalhousie University
   Complementary Cops and Robber (Hill-Nowakowski)

Give a graph $G$ the cops move along the edges of $G$ and the robber along the non-edges or edges of $\overline{G}$. Given this move set, let $CCR(G)$ be the number of cops required to capture the robber on $G$. In general, $\gamma(G) - 1 \leq CCR(G) \leq \gamma(G)$. (See Hill PhD thesis (2008) and also Neufeld-Nowakowski, A vertex-to-vertex pursuit game played with disjoint sets of edges, Finite and infinite combinatorics in sets and logic, Kluwer Acad. Publ., Dordrecht, 1993)

Characterize $G$ such that $CCR(G) = 1$. On the last move, the cop and robber are on adjacent vertices that also dominate the graph.

4. Richard Nowakowski, Dalhousie University
   Boolean Cops and Robber (Hill-Nowakowski)

Given a boolean lattice $B_n$, i.e., the partial order of all subsets of an $n$-element set. The robber starts at the top and moves downwards, the cops start on the bottom and move upwards. No-one is allowed pass and all cops must move. (Hill PhD Thesis (2008) see also Nowakowski. Search and sweep numbers of finite directed acyclic graphs. Discrete Appl. Math., 41(1):111, 1993)

How many cops are required to capture the robber?

Known: for $n = 1, 2, 3, 4, 5, 6, 7, 9$ the number of cops required is 2, 2, 4, 3, 9, 6, 9 respectively.

5. Richard Nowakowski, Dalhousie University
   Stephen Finbow, St. Francis Xavier University
   Mafia

Given a graph $G$, all the vertices are originally ‘dark’. The cops choose vertices and the robbers choose vertices. If a cop passes the vertex he is on becomes light. If a
robber is on a light vertex and he passes then the vertex turns dark. The cops only have information about the robber’s whereabouts from light vertices. Hence, a cop and robber can be on the same dark vertex and the robber is not caught. (For example, the robber is on the light vertex \(x\) and the robber is on an adjacent vertex \(y\) and the robber passes, \(x\) goes dark and even if the cop moves to \(y\) the robber is now ‘hidden’ on \(x\) even though the cop knows where the robber is.)

a) Characterize those graphs in which one cop can capture 1 robber.

b) Characterize those graphs in which one cop can capture any number of robbers.

6. **Richard Nowakowski**, Dalhousie University  
**Shannon Fitzpatrick**, University of Prince Edward Island  
*Cops and Robber with signal delay*

a) Play Cops and Robber but the immediately before the cops move they are informed of the robber’s position on the previous turn. They do know immediately if they are on the same vertex as the robber.

i) Characterize the copwin graphs

b) The robber sends out a signal that propagates out to the \(k\)-th neighbourhood each time the robber moves. The cop that intercepts the signal knows the distance of the robber but not necessarily the position. For example on a path, if the robber at distance 4 from the cop moves toward the cop and the cop remains static, he will get two signals at once.

i) Characterize the copwin graphs.

ii) Is the number of cops smaller than a smallest resolving set?

7. **Nicolas Nisse**, INRIA Sophia Antipolis  
*Cops and Fast Robber*

Consider the cops and robber game with speed. Rules are the same as usual but at each step, the robber can move along at most \(s \geq 1\) edges and each cop can move along at most \(s' \geq 1\) edges. If \(s = s' = 1\), this is the classical game of Quilliot/Nowakowski and Winkler. Let \(c_{s,s'}(G)\) be the smallest number of cops with speed \(s'\) needed to capture a robber with speed \(s\).

a) Let \(G_n\) be the \(n\)-square grid \((n \geq 2)\), i.e., with \(n^2\) nodes. What is the value of \(c_{2,1}(G_n)\)? (It is known that, for any \(s > s'\), \(\Omega(\sqrt{\log n}) = c_{s,s'}(G_n) = O(n))

b) \(c_{s,1}(G)\) can be computed in polynomial-time in interval graphs. What about other graph classes?

8. **Ben Seamone**, Universite de Montreal  
*Cops and Robbers on Geometric Spanners*

We construct graphs and digraphs on a set \(S\) of \(n\) points (vertices) in the plane. Adjacency is defined for \(p \in S\) by dividing the plane into \(k\) regular cones having apex \(p\), and an arc is added from \(p\) to \(q\) if \(q\) is the “nearest point” in \(C\) to \(p\). The directed **Yao graph**, \(\overrightarrow{Y}_k\), defines the “nearest point” to be the one with minimal distance in the
In the directed Theta-graph $\Theta^\rightarrow_k$, the “nearest point” to $p$ is the one whose projection onto the bisecting ray of $C$ is minimal in the $L_2$ metric.

The underlying undirected graphs of $Y_k$ and $\Theta_k$ are denoted $Y_k$ (Yao graph) and $\Theta_k$ (Theta-graph), respectively. These undirected graphs are of particular interest since the shortest paths in $Y_k$ and $\Theta_k$ between two points have length no more than a constant times the Euclidean distance between them for large enough $k$ (i.e. $Y_k$ and $\Theta_k$ are geometric spanners).

Note that every vertex of $\overrightarrow{Y}_k$ has out-degree at most $k$ but may have unbounded in-degree. The directed Yao-Yao graph, $\overrightarrow{YY}_k$, is the subdigraph of $\overrightarrow{Y}_k$ having bounded in-degree that is constructed as follows: for each $p \in S$ and each cone $C$ with apex $p$, all but the shortest incoming arcs are removed. The underlying undirected graph of $\overrightarrow{YY}_k$ is denoted $YY_k$. It is not known whether or not $YY_k$ is a geometric spanner.

Problem 1. For a given $k \in \mathbb{Z}^+$, determine the cop number of $G$ if $G \in \{\Theta_k, Y_k, YY_k, \overrightarrow{Y}_k, \overrightarrow{YY}_k\}$.

Each of these graphs generalize naturally to higher dimension $d > 2$ and to arbitrary metric spaces.

Problem 2. Solve Problem 1 for higher dimensions and/or other metrics.

Graph Searching

1. Nicolas Nisse, INRIA Sophia Antipolis
   Another variant of graph searching

Consider the following variant of graph searching. An edge is cleared either if an agent slides along it or if both its ends are occupied (as in mixed-search). A clear edge is re-contaminated if it is incident to a contaminated edge and their common node is not occupied (classical recontamination). Let $G$ be a graph and $k \geq 1$. A strategy for $k$ agents starts by placing the $k$ agents on $k$ distinct nodes of $G$. Then, sequentially, an agent can slide from node $u$ to node $v$ only if $v$ is not occupied. In other words, a strategy for $k$ agents is defined by a set of $k$ initial nodes and by a sequence of sliding (one agent slides at each step) that ensures that no two agents can simultaneously occupy a same node.

Let $xs(G)$ be the smallest $k$ such that there is a strategy that clears all edges of $G$ using $k$ agents. It is known that, for any graph $G$, $s(G) - 1 \leq xs(G) \leq (\Delta - 1)s(G)$ where $\Delta$ is the maximum degree of $G$ and $s(G)$ is the mixed-search number of $G$. There is a Parson-like characterization of trees $T$ with $xs(T) = k$ and thus $xs(T)$ can be computed in polynomial-time in trees.

a) What is the complexity of computing $xs$?

b) If it is NP-hard can you give a polynomial-time approximation?

c) What about other graph classes?
Consider two version of searching on a graph:

- Inert invisible fugitive game.
- Agile Visible fugitive game.

Both problems are known to be monotone and the minimum number of searchers of a winning strategy is equal to the tree width of the graph. However, when the same games are defined in directed graphs the two parameters are different and both non-monotone. That ways three questions appear:

- What is the difference between the parameter corresponding to the Inert invisible fugitive game on directed graphs and its monotone counterpart?
- What is the difference between the parameter corresponding to the Agile Visible fugitive game on directed graphs and its monotone counterpart?
- What is the difference between between the parameter corresponding to the monotone Inert invisible fugitive game on directed graphs and its monotone Agile Visible counterpart?

#### Revolutionaries

1. **Douglas B. West**, University of Illinois at Urbana-Champaign

   **Revolutionaries and Spies**

   The game of Revolutionaries and Spies is played by \( r \) revolutionaries and \( s \) spies on a graph \( G \). Initially, revolutionaries and then spies occupy vertices. In each subsequent round, each revolutionary may move to a neighboring vertex or not move, and then each spy has the same option. The revolutionaries win if \( m \) of them meet at some vertex having no spy at the end of a round; the spies win if they can avoid this forever.

   Let \( \sigma(G, m, r) \) denote the minimum number of spies needed to win. To avoid degenerate cases, assume \( |V(G)| \geq r - m + 1 \geq \lfloor r/m \rfloor \geq 1 \). The easy bounds are then \( \lfloor r/m \rfloor \leq \sigma(G, m, r) \leq r - m + 1 \). A graph \( G \) is *spy-good* if \( \sigma(G, m, r) = \lfloor r/m \rfloor \) for all \( r \) and \( m \) satisfying \( |V(G)| \geq r - m + 1 \geq \lfloor r/m \rfloor \geq 1 \).

   **Question 1:** Is every interval graph spy-good? (Is it known that \( G \) is spy-good when \( G \) has a rooted spanning tree \( T \) such that every edge of \( G \) not in \( T \) joins vertices having the same parent in \( T \).)

   **Question 2:** What are good upper bounds on \( \sigma(Q_d, 2, r) \) for \( r > d \)? (It is known that \( \sigma(Q_d, 2, r) = r - 1 \) when \( d \geq r \).)

   **Question 3:** What is the smallest \( c \) such that \( \sigma(Q_d, m, r) > r - cm \) for \( d \geq r \)? (It is known that \( c < 39 \), meaning that \( r \) revolutionaries beat \( r - 39m \) spies when \( m \geq 3 \); also \( \sigma(Q_d, m, r) > r - \frac{3}{4}m^2 \).)
Question 4: For sparse random graphs, (such as \( p = c \ln n/n \)), how does \( \sigma(G, m, r) \) behave? (The behavior is well understood when \( p > n^{-1/3} \), where \( \sigma(G, m, r) \) is asymptotic to the trivial lower bound when \( r = \Omega(m \log n) \) and equals the trivial upper bound when \( r - m \) is bounded by a small multiple of \( \log n \).)

Question 5: For the complete bipartite graph \( G_2 \) with arbitrarily large partite sets, what is \( \lim_{r \to \infty} \frac{\sigma(G_2, m, r)}{r/m} \)? (It is known that the answer is between 1.5 and 1.58.)

Question 6: What is the complexity of determining who wins \( RS(G, m, r, s) \)? (The complexity is not known even for testing whether the revolutionaries can win \( RS(G, 2, r, s) \) in two moves when the initial position is specified.)

2. Pawel Pralat, Ryerson University

Revolutionaries and spies on random graphs

The behaviour of the spy number is analyzed for dense graphs (that is, graphs with average degree at least \( n^{1/2+\varepsilon} \) for some \( \varepsilon > 0 \)). For sparser graphs, only some bounds are provided and the picture is far from clear.

Other Models

1. Pawel Pralat, Ryerson University

The firefighter problem

Consider the following \( k \)-many firefighter problem on a finite graph \( G = (V, E) \). Suppose that a fire breaks out at a given vertex \( v \in V \). In each subsequent time unit, a firefighter protects \( k \) vertices which are not yet on fire, and then the fire spreads to all unprotected neighbors of the vertices on fire. The objective of the firefighter is to save as many vertices as possible.

The surviving rate \( \rho_k(G) \) of \( G \) is defined as the expected percentage of vertices that can be saved when a fire breaks out at a random vertex of \( G \). Let

\[
\tau_k = \begin{cases} 
\frac{30}{11} & \text{if } k = 1 \\
 k + 2 - \frac{1}{k+2} & \text{if } k \geq 2.
\end{cases}
\]

It is known that there exists a constant \( c > 0 \) such that for any \( \varepsilon > 0 \) and \( k \geq 1 \), each graph \( G \) on \( n \) vertices with at most \( (\tau_k - \varepsilon)n \) edges is not flammable; that is, \( \rho_k(G) > c \cdot \varepsilon > 0 \). Moreover, a construction of a family of flammable random graphs is proposed to show that the constants \( \tau_k \) cannot be improved.

It would be nice to find the threshold for other families of graphs, including planar graphs.

Problem 1: Determine the largest real number \( M \) such that every planar graph \( G \) with \( n \geq 2 \) vertices and \( \frac{2m}{n} \leq M - \varepsilon \) edges has \( \rho_1(G) \geq c \cdot \varepsilon \) for some \( c > 0 \). It is known that \( \frac{30}{11} \leq M \leq 4 \).
One can generalize this question to any number of firefighters. We know that all planar graphs are not \( k \)-flammable for \( k \geq 4 \). However, it is conjectured that, in fact, planar graphs are not 2-flammable but the techniques are too local to show it. Therefore, it seems that the question does not make sense for \( k \geq 2 \) (unless the conjecture is false).

**Problem 2:** Determine the least integer \( g^* \) such that there is a constant \( 0 < c < 1 \) such that every planar graph \( G \) with girth at least \( g^* \) has \( \rho(G) \geq c \). It is known that \( 5 \leq g^* \leq 7 \).

2. **Pawel Pralat**, Ryerson University  
*Chipping away at the edges: how long does it take?*

We introduce the single-node traffic flow process, which is related to both the chip-firing game and the edge searching process. Initially, real-valued weights (instead of chips) are placed on some vertices of a graph \( G \), and all the edges have zero weight. When a vertex is “fired”, the whole content accumulated in this vertex is sent uniformly to all its neighbours, and each edge increases its weight by the amount that is sent through this edge. We would like to discover the shortest firing sequence such that the total amount of traffic that has passed through each edge is at least some fixed value.

Suppose that initially each vertex has weight of \( \omega \). Let \( f(G) \) be the number of rounds of the shortest firing sequence such that the total amount of traffic that has passed through each edge is at least one. It is known that

\[
\frac{f(K_n)}{|E(K_n)|} \cdot \omega = \frac{1}{2} + o(1),
\]

\[
\frac{f(K_{n,n})}{|E(K_{n,n})|} \cdot \omega \leq 1 + o(1),
\]

\[
\frac{f(K_{1,n})}{|E(K_{1,n})|} \cdot \omega \leq \frac{1}{4} + o(1),
\]

for \( \omega \) small enough. In particular, complete bipartite graphs and stars are not fully investigated.

Let \( G(n) \) be a family of connected graphs on \( n \) vertices. It it natural to ask whether the following limits exist, and if so to find their values.

\[
M = \lim_{n \to \infty} \max_{G \in G(n)} \frac{f(G)}{|E(G)|} \cdot \omega,
\]

\[
m = \lim_{n \to \infty} \min_{G \in G(n)} \frac{f(G)}{|E(G)|} \cdot \omega.
\]

In particular, is it true that \( 0 < m < M = O(1) \)?

3. **Lawrence Erickson**, University of Illinois at Urbana-Champaign  
*Cops and robbers with distance queries*

A cop and robber game is played on a graph with the following rules:

- A robber is hiding at a vertex.
At the beginning of a round, the robber moves distance 0 or 1.
• The cop scans a vertex and receives the distance to the robber.
• The cop wins if it determines the robber’s location. Otherwise a new round begins.
• The robber wins if it can hide indefinitely.

Let $G^{1/m}$ be the graph formed by replacing each edge of $G$ with a path of length $m$. Let $n = |V(G)|$. Let $\mu(G)$ be the metric dimension of $G$.

It is known that:
• The cop wins in $G^{1/m}$ if $m > \min(\max(\mu(G) + 2\mu(G), \Delta(G)), n - 1)$.
• The cop wins in $G^{1/m}$ if $G$ is a grid and $m \geq 2$.
• The cop wins in $K^{1/m}_{a,b}$ if $m \geq a \geq b$ and $m > b$.
• The robber wins in $G$ if $G$ has girth 5 or less.

Open questions:
• Does the robber win in $K^{1/m}_{n}$ if $m \leq n - 1$?
• If the cop wins in $G$, does the cop win in every subdivision of $G$?
• Does the robber win in $G$ if $G$ has girth 6?

This game is studied in Carraher et al. (2012, Theoretical Computer Science). A very similar game, with a slightly stronger cop, is studied in Seager (2012, Discrete Mathematics).

4. Lawrence Erickson, University of Illinois at Urbana-Champaign
   Counting moving bodies with sparse sensor beams

Consider a directed graph $G$ with no sinks. A set of $m$ bodies is distributed among the vertices of $G$. The locations of these $m$ bodies are initially unknown. When a body moves between vertices, the edge traversed is returned as a sensor reading. Given $G$, an initial distribution $d$ of $m$ bodies, and a movement model for the bodies, what is the expected number of sensor readings required to determine the number of bodies in each vertex? Let this number be denoted by the random variable $X(G,d)$.

It is known that the accumulated sensor readings provide enough data to determine a count of the bodies in each vertex if and only if each vertex has been empty at least once.

If the movement model causes each body to have an equal chance of being the next one to move, regardless of earlier movements, then

$$mH_m \leq E[X(G,d)] \leq \frac{m^3 e^m}{\sqrt{2\pi e^{1/m}}}$$

with $H_m = \sum_{i=1}^{m} \frac{1}{m}$. The lower bound is sharp, as $m$ disjoint directed 2-cycles with one body in each produces an expectation of $mH_m$. If, however, the initial distribution
puts all bodies in a single 2-cycle, then the expectation is exponential in $m$, indicating the existence of a “phase transition” between polynomial and exponential for different distributions of bodies in the same graph.

Open problems include determining the properties (expectation, variance, etc.) of $X(G, d)$ for different movement models, specific graphs, and specific starting distributions. Also, for the movement model in which each body has an equal probability of being the next to move, what conditions on $d$ cause $E[X(G, d)]$ to become exponential (in $m$) as opposed to polynomial?

This problem was studied in Erickson and LaValle (2012, WAFR).

Conference photo