Geometrically continuous octahedron

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Abstract. Geometric continuity is a conceptually pleasant notion for constructing surfaces of arbitrary topology. On the other hand, parametric continuity allows straight convenient modeling techniques with B-splines. To combine the two concepts one would like to have some kind of geometrically continuous functions which could be blended into geometrically continuous surfaces without cumbersome manipulations with patches in $\mathbb{R}^3$. A way to define these functions is to glue a set of polygons in an abstract way by using some minimal data that defines "smoothness". This paper demonstrates this approach on one extensive example. We start with 8 triangles in $\mathbb{R}^2$ and identify their edges in the same way in which the faces of an octahedron meet each other. After geometrically continuous functions are defined, we demonstrate that by blending them one can model smooth surfaces formed by 8 triangles glued in the octahedral fashion. We compare the abstract differentiability structure with a corresponding differential manifold. At the end we give a general definition of a *geometrically continuous surface complex* which appears to be a good data structure for modeling geometrically continuous surfaces.

1. Introduction

The concept of *geometric continuity* applies to general situations when several parametric curves or surfaces are pieced together in a sufficiently smooth way. See [**Gre89, Hah89**]. For example, let Ω_1, Ω_2 be closed polygons in $\mathbb{R}^2$, and let $\Phi_1 : \Omega_1 \to \mathbb{R}^3$, $\Phi_2 : \Omega_2 \to \mathbb{R}^3$ be regular C^1 patches. Let $p \subset \Omega_1$, $q \subset \Omega_2$ be edges of the polygons. Then (loosely speaking) Φ_1 and Φ_2 *join with geometric continuity* GC^1 along the edges p, q if: (1) there is a homeomorphism $\mu : p \to q$ such that $\Phi_1 = \Phi_2 \circ \mu$ on p; (2) for any $X \in p$ the tangent plane of the first patch at $\Phi_1(X)$ coincides with the tangent plane of the second patch at $\Phi_2 \circ \mu(X)$; (3) the two patches do not meet at "zero angle" along the common boundary $\Phi_1(p)$. A lot of research is done in deriving explicit geometric continuity conditions for the most common surface patches; see [**Far82, Deg90, DeR90**], etc.

General definitions of geometric continuity for surfaces are based on *connecting diffeomorphisms* (or *reparametrizations*) between open neighborhoods of identified

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edges. This mimics manifold-theoretic definitions of differential surfaces in differential topology. General schemes for modeling geometrically continuous surfaces of arbitrary topology are presented in [**Hah89, DeR85, GH95**]. Since reparametrizations usually do not preserve the types of functions most widely used in geometric modeling (polynomial or rational functions, etc.) and deform the polygons, geometrically continuous gluing is done directly in $\mathbb{R}^3$. This is a considerably cumbersome procedure even for the first order GC^1 geometric continuity.

The alternative of *parametrically continuous* gluing allows one to use B-splines and flexible blending techniques. Two-dimensional B-splines, including tensor product or periodic B-splines, are locally supported piecewise polynomial (or rational, etc.) functions defined on a subdivided region in the plane. Surfaces modelled with B-splines are parametrically continuous since any two adjacent patches get glued in a parametrically continuous manner. For example, B-splines on closed surfaces are modelled by translating the polygonal pieces to bring them beside each other, which gives parametrically continuous patching again. The drawback of this approach is that parametric continuity preserves some metric structure of $\mathbb{R}^2$. Therefore only genus 1 surfaces can be satisfactorily modelled, whereas to model closed surfaces with other topology (say, sphere-like surfaces) one has to use singular patches.

The aim of this paper is to illustrate an approach that combines generality of geometrically continuous gluing and convenient techniques that are known within the framework of parametric continuity. The key notion is that of a *geometrically continuous surface complex*, which is a data structure that essentially defines a differential manifold (a differential surface). It glues a collection of polygons without a reference to concrete patches in $\mathbb{R}^3$. The importance for geometric modeling is that *geometrically continuous functions* can be defined before actual modeling. In other words, we suggest to start with a set of polygons with some additional continuity and "smoothness" data; this is our abstract "GC^1 surface". Then we compute piecewise polynomial (or rational, etc.) functions that are expected to be smooth on the abstract surface. Our main intention is to demonstrate that these functions can be used in geometric modeling as conveniently as traditional B-splines. An attempt to introduce this approach is present in [**Vid99**]. Reminiscent ideas in the context of curves are contained in [**GB89, GM89, Sei91**].

The paper considers one big example that illustrates the new approach. The example is a "smooth" octahedron $\mathcal{H}$. In the next section we define its combinatorial and differentiable structure and introduce GC^1 functions on it. In Section 3 we demonstrate possibilities of the new approach by computing some piecewise cubic GC^1 functions and giving several modeling examples. In Section 4 we define a differentiable surface $\mathcal{S}$ which naturally corresponds to our octahedron $\mathcal{H}$. In particular, C^1 functions on $\mathcal{S}$ are exactly the GC^1 functions on $\mathcal{H}$. In Section 5 we give a general definition of a GC^1 geometrically continuous surface complex.

2. The octahedron and functions on it

Here we specify the data structure that is used throughout the paper. Let $\mathbf{N}$ denote the set $\{1,2,3,4,5,6,7,8\}$.

DEFINITION 2.1. Our *geometrically continuous octahedron* $\mathcal{H}$ is defined by the following data (Ω,ρ,Ξ):

(i) Ω is a set of 8 triangles $P_i Q_i R_i \subset \mathbb{R}^2$, $i \in \mathbf{N}$. To avoid notational confusion, we assume that these triangles do not mutually intersect.

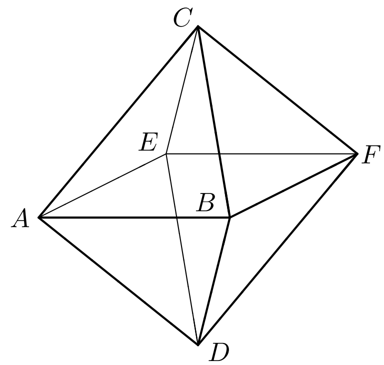

FIGURE 1. An octahedron

(ii) ρ is a set 12 linear maps between edges of those triangles:

$$\begin{array}{lll} \varrho_{12} : P_1\,Q_1 \to P_2\,Q_2, & \varrho_{13} : P_1\,R_1 \to P_3\,R_3, & \varrho_{15} : Q_1\,R_1 \to Q_5\,R_5, \\ \varrho_{34} : P_3\,Q_3 \to P_4\,Q_4, & \varrho_{24} : P_2\,R_2 \to P_4\,R_4, & \varrho_{48} : Q_4\,R_4 \to Q_8\,R_8, \\ \varrho_{56} : P_5\,Q_5 \to P_6\,Q_6, & \varrho_{68} : P_6\,R_6 \to P_8\,R_8, & \varrho_{26} : Q_2\,R_2 \to Q_6\,R_6, \\ \varrho_{78} : P_7\,Q_7 \to P_8\,Q_8, & \varrho_{57} : P_5\,R_5 \to P_7\,R_7, & \varrho_{37} : Q_3\,R_3 \to Q_7\,R_7. \end{array} \tag{2.1}$$

For $(i,j) \in \{(1,2),(3,4),(5,6),(7,8)\}$ we require that ϱ_{ij} is the linear homeomorphism such that $\varrho_{ij}(P_i) = P_j$ and $\varrho_{ij}(Q_i) = Q_j$, and similarly for other maps.

(iii) For $i \in \mathbf{N}$, Ξ assigns to each point X on the edge P_iQ_i the vector $\xi_{P_iQ_i}(X) = \overrightarrow{XR_i}$. Similarly, Ξ assigns the vectors $\xi_{P_iR_i}(X) = \overrightarrow{XQ_i}$ and $\xi_{Q_iR_i}(X) = \overrightarrow{XP_i}$ to all points on the edges P_iR_i and Q_iR_i respectively. Note that in total two vectors are assigned to the vertices P_i, Q_i, R_i.

To interpret the data structure $\mathcal{H}$ we define a topological space $\mathcal{S}$ as follows. We view the maps in (2.1) as identifications of edges of the 8 triangles. Then $\mathcal{S}$ is defined as the disjoint union of the triangles modulo the specified edge identifications. Our construction is designed with a picture of an octahedron $\mathcal{O}$ in Figure 1 in mind. The topological space $\mathcal{S}$ is homeomorphic to (the surface of) the octahedron by the following maps:

$$\begin{array}{ll} \psi_1 : P_1\,Q_1\,R_1 \to A\,B\,C, & \psi_2 : P_2\,Q_2\,R_2 \to A\,B\,D, \\ \psi_3 : P_3\,Q_3\,R_3 \to A\,E\,C, & \psi_4 : P_4\,Q_4\,R_4 \to A\,E\,D, \\ \psi_5 : P_5\,Q_5\,R_5 \to F\,B\,C, & \psi_6 : P_6\,Q_6\,R_6 \to F\,B\,D, \\ \psi_7 : P_7\,Q_7\,R_7 \to F\,E\,C, & \psi_8 : P_8\,Q_8\,R_8 \to F\,E\,D. \end{array} \tag{2.2}$$

Each map ψ_i is the linear homeomorphism such that

$$\psi_i(P_i) \in \{A, F\}, \qquad \psi_i(Q_i) \in \{B, E\}, \qquad \psi_i(R_i) \in \{C, D\}. \tag{2.3}$$

These homeomorphisms map the 12 pairs of identified edges onto the 12 edges of the octahedron. The triangle vertices are identified in groups of four into the 6 vertices of $\mathcal{O}$. For convenience, we refer to those 6 points on $\mathcal{S}$ as the *vertices* of $\mathcal{S}$ (or $\mathcal{H}$) and denote them by the same letters.

As we shall see, Ξ essentially endows the topological surface $\mathcal{S}$ with a structure of a C^1 *differential surface* in the sense of differential topology. At this stage we just define continuous and GC^1 geometrically continuous functions on $\mathcal{H}$. Our definition of a continuous function on $\mathcal{H}$ is equivalent to the notion of a continuous function on the topological surface $\mathcal{S}$. The GC^1 functions on $\mathcal{H}$ will correspond to the C^1 functions on $\mathcal{S}$ endowed with the promised differential surface structure.

A *continuous* function on $\mathcal{H}$ is a tuple $(f_i)_{i\in\mathbf{N}}$, where each f_i is a continuous function on the triangle $P_iQ_iR_i$, such that for any edge identification ϱ_{ij} in (2.2) we have $\varrho_{ij}(f_i) = f_j$ when restricted onto the glued edge of $P_jQ_jR_j$. We use barycentric coordinates to express functions on $\mathbb{R}^2$ and on $\mathcal{H}$. For $i \in \mathbf{N}$ consider the triangle $P_iQ_iR_i$. Any point $X \in \mathbb{R}^2$ can be written uniquely as an affine linear combination

$$X = u_i(X)\,P_i + v_i(X)\,Q_i + w_i(X)\,R_i \quad \text{with} \quad u_i(X)+v_i(X)+w_i(X) = 1. \tag{2.4}$$

The triple $(u_i(X), v_i(X), w_i(X))$ defines the *barycentric coordinates* of X with respect to the triangle $P_iQ_iR_i$. See [**Far90**]. Here are six continuous functions on $\mathcal{H}$ expressed in barycentric coordinates:

$$\begin{aligned} g_A &= (u_1,\ u_2,\ u_3,\ u_4,\ 0,\ 0,\ 0,\ 0)\,, & g_B &= (v_1,\ v_2,\ 0,\ 0,\ v_5,\ v_6,\ 0,\ 0)\,,\\ g_C &= (w_1,\ 0,\ w_3,\ 0,\ w_5,\ 0,\ w_6,\ 0)\,, & g_D &= (0,\ w_2,\ 0,\ w_4,\ 0,\ w_6,\ 0,\ w_8)\,,\\ g_E &= (0,\ 0,\ v_3,\ v_4,\ 0,\ 0,\ v_7,\ v_8)\,, & g_F &= (0,\ 0,\ 0,\ 0,\ u_5,\ u_6,\ u_7,\ u_8)\,. \end{aligned} \tag{2.5}$$

They can be used as *blending functions* to represent maps from the triangles $P_iQ_iR_i$ (or the whole $\mathcal{H}$) to $\mathbb{R}^3$. That means that the map is expressed as a linear expression of the blending functions, where the coefficients are *control points* in $\mathbb{R}^3$. For example, the homeomorphism $\psi_1 : P_1Q_1R_1 \to A\,B\,C$ can be represented as $\psi_1 = A\,u_1 + B\,v_1 + C\,w_1$. The overall homeomorphism $\mathcal{S} \to \mathcal{O}$ defined by (2.2) can be expressed as $A\,g_A + B\,g_B + C\,g_C + D\,g_D + E\,g_E + F\,g_F$.

To define geometrically continuous functions on $\mathcal{H}$, recall that if f is a C^1 function on $\mathbb{R}^2$ and $\vec{\mathbf{a}}$ is a vector in $\mathbb{R}^2$, then the *directional derivative* $\mathbf{D}_{\vec{\mathbf{a}}}$ of f at $X \in \mathbb{R}^2$ is defined as follows:

$$\mathbf{D}_{\vec{\mathbf{a}}}\, f(X) = \lim_{\zeta\to 0} \frac{f(X+\zeta\,\vec{\mathbf{a}}) - f(X)}{\zeta}, \tag{2.6}$$

Definition 2.2. A *geometrically continuous GC^1 function* on $\mathcal{H}$ is a continuous function $(f_i)_{i\in\mathbf{N}}$ on $\mathcal{H}$ that satisfies the following conditions:

(a) Each f_i is a nice differentiable function on the triangle $P_iQ_iR_i$. Technically we require that f_i is a C^1 function on the interior of $P_iQ_iR_i$, and that it can be extended to a C^1 function on some open neighborhood of $P_iQ_iR_i$.

(b) For each pair of identified edges $p \subset P_iQ_iR_i$, $q \subset P_jQ_jR_j$ we require

$$\mathbf{D}_{\xi_p(X)} f_i(X) = -\mathbf{D}_{\xi_q(Y)} f_j(Y) \qquad \text{for all } X \in p \text{ and } Y = \varrho_{ij}(X).$$

Here are two examples of geometrically continuous functions on $\mathcal{H}$:

$$G_A = \left(\frac{u_1^2}{u_1^2+v_1^2+w_1^2},\ \frac{u_2^2}{u_2^2+v_2^2+w_2^2},\ \frac{u_3^2}{u_3^2+v_3^2+w_3^2},\ \frac{u_4^2}{u_4^2+v_4^2+w_4^2},\ 0,\ 0,\ 0,\ 0 \right), \tag{2.7}$$

$$\begin{aligned} G_{uv} = \Bigg(&\frac{u_1v_1}{u_1^2+v_1^2+w_1^2},\ \frac{u_2v_2}{u_2^2+v_2^2+w_2^2},\ -\frac{u_3v_3}{u_3^2+v_3^2+w_3^2},\ -\frac{u_4v_4}{u_4^2+v_4^2+w_4^2},\\ &-\frac{u_5v_5}{u_5^2+v_5^2+w_5^2},\ -\frac{u_6v_6}{u_6^2+v_6^2+w_6^2},\ \frac{u_7v_7}{u_7^2+v_7^2+w_7^2},\ \frac{u_8v_8}{u_8^2+v_8^2+w_8^2} \Bigg). \end{aligned} \tag{2.8}$$

Directional derivatives can be expressed in terms of partial derivatives with respect to u_i, v_i, w_i that respect the relation $u_i + v_i + w_i = 1$. For example,

$$\mathbf{D}_{\overrightarrow{P_iQ_i}} = \frac{\partial}{\partial v_i} - \frac{\partial}{\partial u_i}, \quad \mathbf{D}_{\overrightarrow{P_iR_i}} = \frac{\partial}{\partial w_i} - \frac{\partial}{\partial u_i}, \quad \mathbf{D}_{\overrightarrow{R_iQ_i}} = \frac{\partial}{\partial v_i} - \frac{\partial}{\partial w_i}. \tag{2.9}$$

Differentiability condition *(b)* of Definition 2.2 can be rewritten more explicitly as follows. For $(i,j) \in \{(1,2),(3,4),(5,6),(7,8)\}$ we must have for all $\zeta \in [0,1]$:

$$\mathbf{D}_{\overrightarrow{P_iR_i}}\, f_i(1-\zeta,\,\zeta,\,0) + \mathbf{D}_{\overrightarrow{P_jR_j}}\, f_j(1-\zeta,\,\zeta,\,0) = 2\zeta\,\mathbf{D}_{\overrightarrow{P_jQ_j}}\, f_j(1-\zeta,\,\zeta,\,0). \tag{2.10}$$

Similarly, for $(i,j) \in \{(1,3),(2,4),(5,7),(6,8)\}$ and all $\zeta \in [0,1]$

$$\mathbf{D}_{\overrightarrow{P_iQ_i}}\, f_i(1-\zeta,\,0,\,\zeta) + \mathbf{D}_{\overrightarrow{P_jQ_j}}\, f_j(1-\zeta,\,0,\,\zeta) = 2\zeta\,\mathbf{D}_{\overrightarrow{P_jR_j}}\, f_j(1-\zeta,\,0,\,\zeta), \tag{2.11}$$

and for $(i,j) \in \{(1,5),(2,6),(3,7),(4,8)\}$ and all $\zeta \in [0,1]$

$$\mathbf{D}_{\overrightarrow{P_iQ_i}}\, f_i(0,\,1-\zeta,\,\zeta) + \mathbf{D}_{\overrightarrow{P_jQ_j}}\, f_j(0,\,1-\zeta,\,\zeta) = 2\zeta\,\mathbf{D}_{\overrightarrow{R_jQ_j}}\, f_j(0,\,1-\zeta,\,\zeta). \tag{2.12}$$

The following theorem shows direct relevance of GC^1 functions to geometric modeling. It follows directly from Theorem 4.2 here below, after we introduce a corresponding C^1 differential surface structure on $\mathcal{S}$. In Section 3 we introduce more GC^1 functions and demonstrate a few modeling examples.

THEOREM 2.3. *Let $\Phi = (F_1, F_2, F_3)$ be a map from $\mathcal{S}$ to $\mathbb{R}^3$ given by GC^1 functions F_1, F_2, F_3 on $\mathcal{H}$. Suppose that for each $i \in \mathbf{N}$ the restriction of Φ onto the the triangle $P_iQ_iR_i$ is a C^1 regular patch. Then the image of Φ is a GC^1 patch complex as defined in* [**Hah89**].

PROOF. (Sketch.) We have to show that the 8 patches join with CG^1 geometric continuity along the identified edges and around the six vertices. Consider a pair of triangles whose edges p, q are identified by a map in (2.1). Explicit connecting diffeomorphisms between open neighborhoods of p and q are present in our description of a C^1 surface structure on $\mathcal{S}$ in Section 4; see formulas (4.3)-(4.5) below. Here we note that if $X_1 \in p$, $X_2 \in q$ are two identified points then $\mathbf{D}_{\xi_p(X_1)}\Phi(X_1) = -\mathbf{D}_{\xi_q(X_2)}\Phi(X_2)$, so the two patches have the same tangent plane at $Y = \Phi(X_1) = \Phi(X_2)$ which is spanned by $\mathbf{D}_{\xi_p(X_1)}\Phi(X_1)$ and $\mathbf{D}_{\vec{p}}\,\Phi(X_1)$; here $\vec{p}$ is a vector along p. The minus sign before the derivative at X_2 ensures that the two patches meet smoothly at Y rather than at "zero angle".

To show that patches join with CG^1 continuity around vertices, we consider the concrete case of four identified vertices P_1, P_2, P_3, P_4. The tangent planes of all four patches at the common vertex coincide since each of them is spanned by $\mathbf{D}_{\overrightarrow{P_1Q_1}}\Phi(P_1) = \mathbf{D}_{\overrightarrow{P_2Q_2}}\Phi(P_2) = -\mathbf{D}_{\overrightarrow{P_3Q_3}}\Phi(P_3) = -\mathbf{D}_{\overrightarrow{P_4Q_4}}\Phi(P_4)$ and $\mathbf{D}_{\overrightarrow{P_1R_1}}\Phi(P_1) = -\mathbf{D}_{\overrightarrow{P_2R_2}}\Phi(P_2) = \mathbf{D}_{\overrightarrow{P_3R_3}}\Phi(P_3) = -\mathbf{D}_{\overrightarrow{P_4R_4}}\Phi(P_4)$. The tangent sectors of those four patches do not overlap, they are separated by two intersecting lines in the tangent plane. Therefore they surround the common vertex with GC^1 continuity. □

3. Geometrically continuous functions at work

In this section we consider mainly geometrically continuous functions $(f_i)_{i\in\mathbf{N}}$ with the property that each component f_i is a polynomial. We refer to them as *GC^1 splines* (or *geometrically continuous splines*). They form a linear space. The splines defined by polynomials of degree at most n form a linear subspace; we denote this subspace by $S^1_n(\mathcal{H})$. We give equations that define the splines and give several modeling examples using splines from $S^1_3(\mathcal{H})$.

We write components of a spline $(f_i)_{i\in\mathbf{N}} \in S^1_n(\mathcal{H})$ in the *Bernstein-Bezier form*:

$$f_i(u_i, v_i, w_i) = \sum_{\substack{j+k+\ell=n\\ j\ge 0, k\ge 0, \ell\ge 0}} c^{(i)}_{j,k,\ell}\,\frac{n!}{j!\,k!\,\ell!}\,u_i^j\,v_i^k\,w_i^\ell, \qquad \text{all} \quad c^{(i)}_{j,k,\ell} \in \mathbb{R}. \tag{3.1}$$

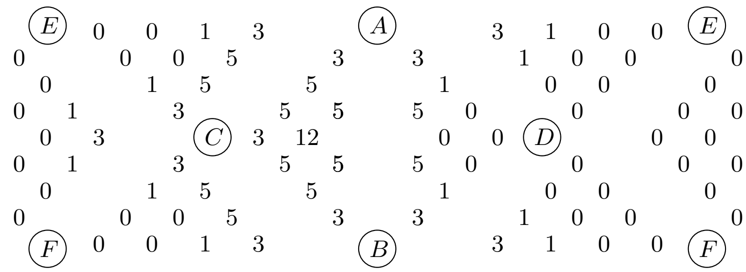

FIGURE 2. Bernstein-Bezier coefficients of $h_{(1)}$

Differentiability conditions (2.10)–(2.12) translate into the following equations for the Bernstein-Bezier coefficients:

- For $(i,j) \in \{(1,2),(3,4),(5,6),(7,8)\}$, $k \geq 1, \ell \geq 1$ with $k+\ell=n$ we have
$$c^{(i)}_{k,\ell,0} = c^{(j)}_{k,\ell,0} = \frac{k}{2n}\left(c^{(i)}_{k-1,\ell,1} + c^{(j)}_{k-1,\ell,1}\right) + \frac{\ell}{2n}\left(c^{(i)}_{k,\ell-1,1} + c^{(j)}_{k,\ell-1,1}\right),$$
$$c^{(i)}_{n,0,0} = c^{(j)}_{n,0,0} = \frac{c^{(i)}_{n-1,0,1} + c^{(j)}_{n-1,0,1}}{2}, \quad c^{(i)}_{0,n,0} = c^{(j)}_{0,n,0} = \frac{c^{(i)}_{0,n-1,1} + c^{(j)}_{0,n-1,1}}{2}.$$
- For $(i,j) \in \{(1,3),(2,4),(5,7),(6,8)\}$, $k \geq 1, \ell \geq 1$ with $k+\ell=n$ we have
$$c^{(i)}_{k,0,\ell} = c^{(j)}_{k,0,\ell} = \frac{k}{2n}\left(c^{(i)}_{k-1,1,\ell} + c^{(j)}_{k-1,1,\ell}\right) + \frac{\ell}{2n}\left(c^{(i)}_{k,1,\ell-1} + c^{(j)}_{k,1,\ell-1}\right),$$
$$c^{(i)}_{n,0,0} = c^{(j)}_{n,0,0} = \frac{c^{(i)}_{n-1,1,0} + c^{(j)}_{n-1,1,0}}{2}, \quad c^{(i)}_{0,0,n} = c^{(j)}_{0,0,n} = \frac{c^{(i)}_{0,1,n-1} + c^{(j)}_{0,1,n-1}}{2}.$$
- For $(i,j) \in \{(1,5),(2,6),(3,7),(4,8)\}$, $k \geq 1, \ell \geq 1$ with $k+\ell=n$ we have
$$c^{(i)}_{0,k,\ell} = c^{(j)}_{0,k,\ell} = \frac{k}{2n}\left(c^{(i)}_{1,k-1,\ell} + c^{(j)}_{1,k-1,\ell}\right) + \frac{\ell}{2n}\left(c^{(i)}_{1,k,\ell-1} + c^{(j)}_{1,k,\ell-1}\right),$$
$$c^{(i)}_{0,n,0} = c^{(j)}_{0,n,0} = \frac{c^{(i)}_{1,n-1,0} + c^{(j)}_{1,n-1,0}}{2}, \quad c^{(i)}_{0,0,n} = c^{(j)}_{0,0,n} = \frac{c^{(i)}_{1,0,n-1} + c^{(j)}_{1,0,n-1}}{2}.$$

These equations imply that for $n \geq 3$ the "edge" coefficients $c^{(i)}_{j,k,\ell}$ with $j\,k\,\ell = 0$ are uniquely determined by the "interior" coefficients $c^{(i)}_{j,k,\ell}$ with $j\,k\,\ell \neq 0$, and that the latter coefficients can be chosen freely. Therefore the dimension of $S^1_n(\mathcal{H})$ is equal to $4(n-1)(n-2)$ if $n \geq 3$ (and it is equal to 1 for $n = 0,1,2$). This result is present in Example 6.29 in [**Vid99**].

In particular, $\dim S^1_3(\mathcal{H}) = 8$. For $i \in \mathbf{N}$ let $h_{(i)}$ denote the function in $S^1_3(\mathcal{H})$ with $c^{(i)}_{1,1,1} = 12$ and all other "interior" coefficients equal to zero. The Bernstein-Bezier coefficients of their components can be easily computed from the equations above. The coefficients of $h_{(1)}$ are schematically depicted in Figure 2. Coefficients of each polynomial are represented by a triangular array in a natural way. The correspondence to the triangles $P_1Q_1R_1, P_2Q_2R_2, \ldots, P_8Q_8R_8$ can be seen from Figure 1 and homeomorphisms in (2.2). Monomials in u_i, v_i, w_i should be assigned according to (2.3) and (2.4). Similar expressions for $h_{(2)}, h_{(3)}, \ldots, h_{(8)}$ can be obtained by permuting the vertex labels in Figure 2 according to symmetries of $\mathcal{H}$.

The 8 functions $h_{(i)}$ form a basis for $S^1_3(\mathcal{H})$. They can be used as blending functions in geometric modeling of closed surfaces homeomorphic to a sphere. Note that $h_{(i)}$ (for fixed $i \in \mathbf{N}$) naturally corresponds to the ith triangle so that moving its

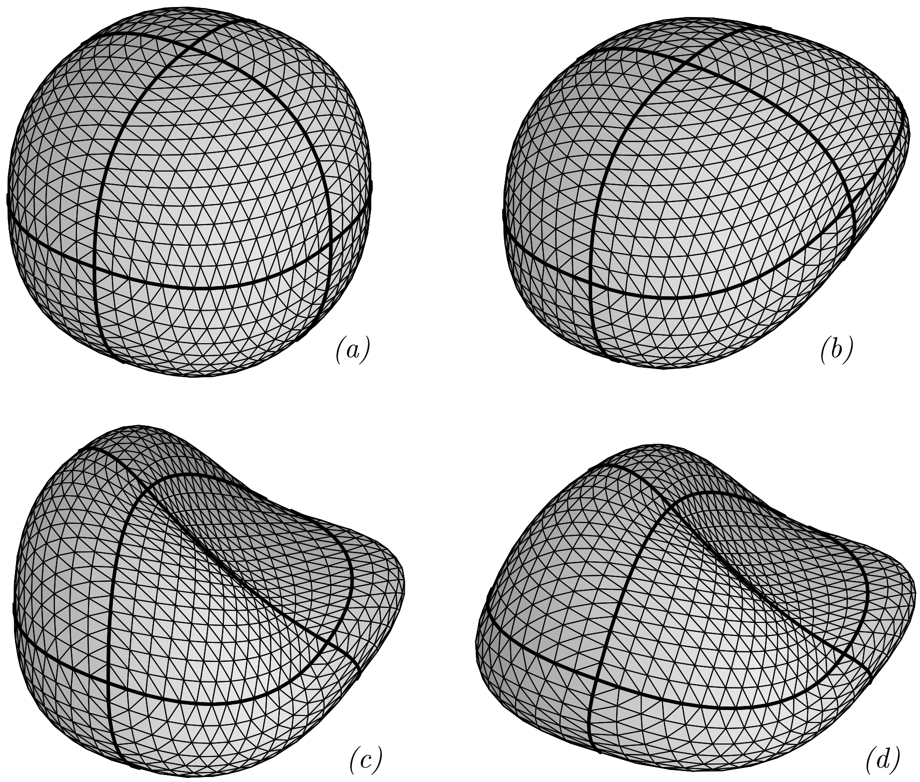

FIGURE 3. Modeling with $\mathcal{H}$

control point produces most change in the image of $P_iQ_iR_i$. Therefore modeling $\mathcal{H}$ by elements of $S_3^1(\mathcal{H})$ has more of the flavor of modeling a cubus (the Platonic body dual to the octahedron). By placing the control points of $h_{(i)}$'s at the vertices of the cubus $[-1,1]^3 \subset \mathbb{R}^3$ we get the most symmetric geometrically continuous surface that we can model using $S_3^1(\mathcal{H})$, see Figure 3*(a)*. The surface can be interpreted as a map $\mathcal{H} \to \mathbb{R}^3$ given by the following blending expression

$$\begin{array}{c}(1,1,1)\,h_{(1)} + (1,1,-1)\,h_{(2)} + (1,-1,1)\,h_{(3)} + (1,-1,-1)\,h_{(4)} + \\ (-1,1,1)\,h_{(5)} + (-1,1,-1)\,h_{(6)} + (-1,-1,1)\,h_{(7)} + (-1,-1,-1)\,h_{(8)}.\end{array} \tag{3.2}$$

This CG^1 surface is even curvature continuous, as it is shown in [**PK97**].

By moving the control points in (3.2) one can deform the surface in Figure 3*(a)*. Say, by moving the control point of $h_{(6)}$ to $(-2,1,0)$ we get picture *(b)* in Figure 3. (Scaling is different in the four pictures there. For orientation, assume that the three visible vertices in Figure 3*(a)* have coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.) Note that $h_{(6)}$ is identically zero on $P_3Q_3R_3$, so moving its control point does not affect the corresponding opposite patch at all. It looks like we work with B-splines! Consequently we may move the control point of $h_{(5)}$ to $(0,0,0)$ and get picture *(c)*, and then bring the control point of $h_{(7)}$ to $(1,-2,-1)$ and get picture *(d)*.

Theorem 2.3 ensures that these modelled surfaces are indeed geometrically continuous once the 8 patches do not have singularities. We constructed visually smooth surfaces without worrying about cumbersome geometric continuity restrictions that are usual in GC^1 patching directly in $\mathbb{R}^3$. Basically, we solve geometric continuity restrictions only once by computing the space of GC^1 functions. Besides, geometrical continuity is solved here as a one-dimensional problem rather than three-dimensional one. Recall that geometric continuity restrictions are linear equations between control points of the two patches that are glued, with unknown coefficients ("shape parameters"). In our approach we find GC^1 functions by solving basically the same linear equations, but the unknowns are just numbers rather than points, and the "shape parameters" are fixed by our choice of the differential structure Ξ. We can vary Ξ as well; this would change the space of GC^1 functions. To see what differential structures are possible we need to compare our data structure with similar constructions in differential topology. In the next section we construct a C^1 differential surface from the same combinatorial data and with equivalent differential structure. The equivalence manifests itself in the fact that the sets of C^1 functions and GC^1 functions coincide, see Theorem 4.2.

Apart from allowing convenient blending techniques in the framework of geometric continuity, our approach offers interesting possibilities that are difficult to realize with usual methods of geometric modeling. For example, write realization (3.2) of the most symmetric octahedron on Figure 3*(a)* as a map $(H_x, H_y, H_z) : \mathcal{H} \to \mathbb{R}^3$, where $H_x = h_{(1)}+h_{(2)}+h_{(3)}+h_{(4)}-h_{(5)}-h_{(6)}-h_{(7)}-h_{(8)}$, etc. Functions H_x, H_y, H_z look like projection functions to the "main axes" AF, BE, CD of the octahedron (consult Figure 1). For instance, H_x is positive on the hemisphere around A, negative on the opposite hemisphere, and it is zero on the "equator" $u_i = 0$. Let H_0 be a constant non-zero function on $\mathcal{H}$, and consider the functions

$$(3.3)\qquad \begin{aligned} H_{12} &= h_{(1)} - h_{(2)} - h_{(7)} + h_{(8)}, \quad & H_C &= h_{(1)} - h_{(3)} - h_{(5)} + h_{(7)},\\ H_{34} &= h_{(3)} - h_{(4)} - h_{(5)} + h_{(6)}, \quad & H_D &= h_{(2)} - h_{(4)} - h_{(6)} + h_{(8)}. \end{aligned}$$

The functions $H_0, H_x, H_y, H_z, H_{12}, H_{34}, H_C, H_D$ form a basis for $S^1_3(\mathcal{H})$. They appear to be pairwise orthogonal with respect to any positive definite scalar product on $S^1_3(\mathcal{H})$ that respects the octahedral symmetries of $\mathcal{H}$, with a possible exception for the pair (H_C, H_D). This can be an attractive feature for geometric modeling. For instance, consider the blending expression

$$(3.4)\qquad Z_0\, H_0 + Z_1\, H_x + Z_2\, H_y + Z_3\, H_z + Z_4\, H_{12} + Z_5\, H_{34} + Z_6\, H_C + Z_7\, H_D,$$

with $Z_i \in \mathbb{R}^3$ for $i = 0, 1, \ldots, 7$. It realizes a GC^1 surface $\mathcal{Z}$ in $\mathbb{R}^3$. Moving Z_0 changes the position of $\mathcal{Z}$ but does not affect its shape. The control points Z_1, Z_2, Z_3 determine direction of the three "main axes" with respect to Z_0. Moving other control points does not change position of the six vertices of $\mathcal{H}$ but deforms $\mathcal{Z}$ somehow. Say, moving Z_6 pushes two opposing patches around C in one direction and the other two patches around C in the opposite direction. Figure 4 depicts a few surfaces obtained by "deforming" the most symmetric octahedron in Figure 3*(a)*. Working with a blending expression like (3.4) can be considered as a multiresolution technique. This interpretation should appear more relevant when larger spaces of GC^1 functions are considered.

In principle, one can compute GC^1 functions $(f_i)_{i\in\mathbf{N}}$ on $\mathcal{H}$ given by rational functions f_i (or even more general functions). If one fixes the denominators of rational functions f_i and the degree of their numerators, then determining the set of such GC^1 functions is a linear algebra problem similar to computation of $S^1_n(\mathcal{H})$. For instance, consider the set $\widetilde{S}$ of GC^1 functions given by degree 2 rational functions with the denominators $u_i^2+v_i^2+w_i^2$. We have examples of these functions in (2.7)–(2.8). Computations show that $\widetilde{S}$ is a linear space of dimension 9. Six

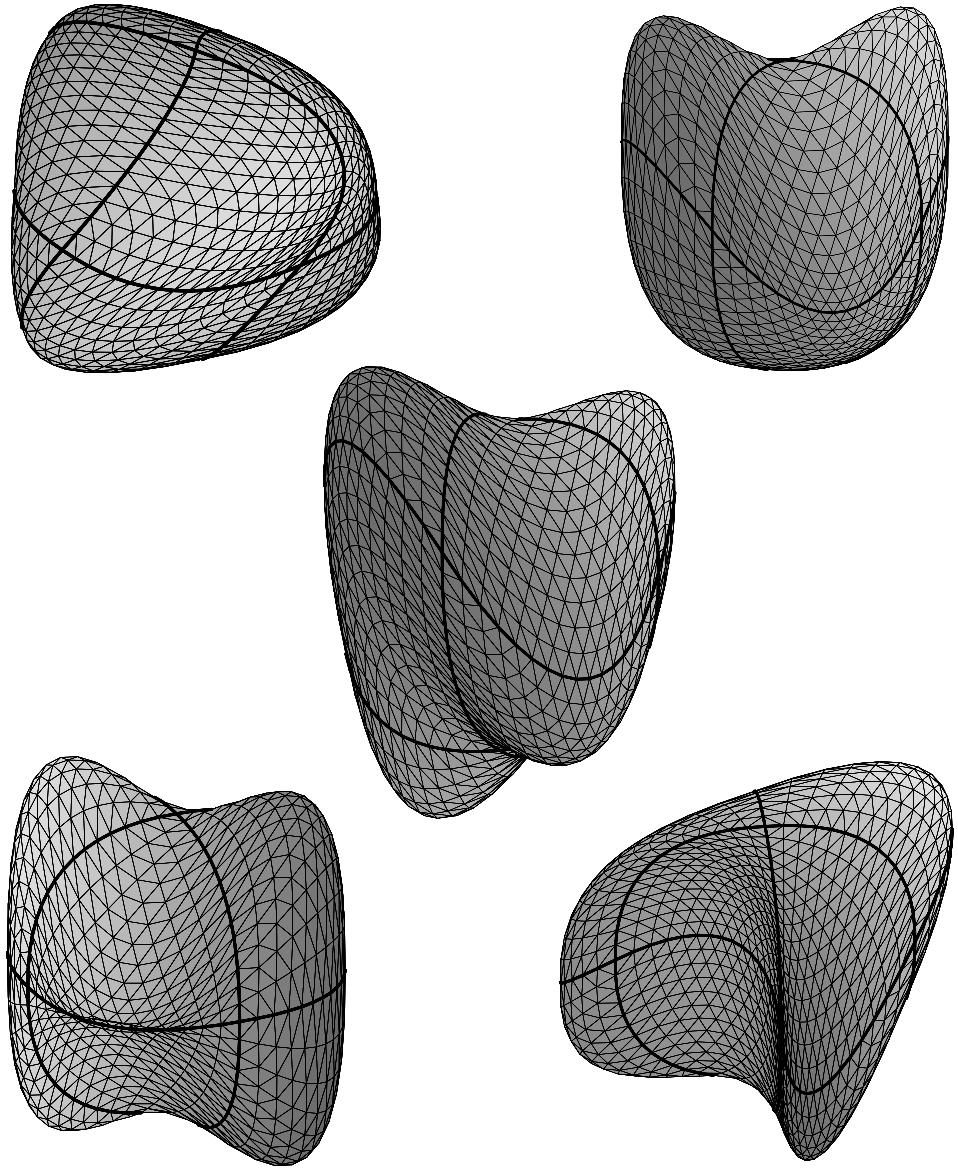

FIGURE 4. More modeling with $\mathcal{H}$

independent functions can be obtained by applying the symmetries of $\mathcal{H}$ to G_A, and three more independent functions can be similarly obtained from G_{uv}. However, it appears that GC^1 surfaces realized by functions from $\widetilde{S}$ always have singular patches. (Prove or confute this!) For computing general sets of "rational" GC^1 functions one can use Gröbner bases. This is quite cumbersome in general. On the other hand, GC^1 functions form an algebra: if f, g are two GC^1 functions on $\mathcal{H}$, then $f+g$, fg are GC^1 functions as well. If moreover g does not vanish anywhere, then f/g is a GC^1 function. For example, $G_A/(1+G_{uv})$ is a geometrically continuous function on $\mathcal{H}$; its components are rational functions of degree 2.

4. The differential surface

In this section we describe a C^1 differential surface that corresponds to our abstract "smooth" octahedron $\mathcal{H}$, and identify C^1 functions on this differential surface with GC^1 functions on $\mathcal{H}$. We use the definitions from [**War83**].

DEFINITION 4.1. Let J denote a finite set. A *differential surface* of class C^1 is a Hausdorff space $\mathcal{M}$ together with a collection $\{(U_p, \phi_p)\}_{p\in J}$ such that

- $\{U_p\}_{p\in J}$ is an open covering of $\mathcal{M}$.
- Each ϕ_p is a homeomorphism $\phi_p\colon V_p \to U_p$, where V_p is an open set in $\mathbb{R}^2$.
- For $p, q \in J$ such that $p \neq q$ and $U_p \cap U_q \neq \emptyset$, let $V_{p,q} := \phi_p^{-1}(U_p \cap U_q)$ and $V_{p,q} := \phi_q^{-1}(U_p \cap U_q)$. Then the map $\phi_q^{-1} \circ \phi_p : V_{p,q} \to V_{q,p}$ is required to be a C^1-diffeomorphism.

The collection $\{(U_p, \phi_p)\}_{p\in J}$ is a C^1 *atlas* of $\mathcal{M}$, and the maps $\phi_q^{-1} \circ \phi_p$ are called *transition maps*. Let W be an open subset of $\mathcal{M}$. A function $f : W \to \mathbb{R}$ is C^1 *continuous* if for any $p \in J$ the function $f \circ \phi_p$ is C^1 continuous on $W \cap V_p \subset \mathbb{R}^2$.

Let X be a point on $\mathcal{M}$. Let $C^1(X)$ denote the space of C^1 functions defined on some open neighborhood of X. A *point derivation* at X is an $\mathbb{R}$-linear map $\delta : C^1(X) \to \mathbb{R}$ that satisfies the Leibnitz rule $\delta(fg) = f\delta(g) + g\delta(f)$. The point derivations at X form a linear space which is the *tangent space* of $\mathcal{M}$ at X. We denote it by $T_{\mathcal{M},X}$. In the special case $\mathcal{M} = \mathbb{R}^2$ point derivations at $X \in \mathbb{R}^2$ are directional derivatives as defined in (2.6). The tangent space $T_{\mathbb{R}^2,X}$ is generated by any two of the three derivatives in (2.9).

Let $\mathcal{N}$ be other differentiable surface of class C^1. A map $\Phi : W \to \mathcal{N}$ is C^1 *continuous* if it is continuous and if for any function g that is C^1 on some open subset $\widetilde{W}$ of $\mathcal{N}$, the composition $g \circ \Phi$ is C^1 continuous on $W \cap \Phi^{-1}(\widetilde{W})$. Such a C^1 continuous map induces a linear transformation $d\Phi : T_{\mathcal{M},X} \to T_{\mathcal{N},\Phi(X)}$ by

$$d\Phi\,(\delta)\,(f) = \delta(f \circ \Phi) \tag{4.1}$$

for any $\delta \in T_{\mathbb{R}^2,X}$ and any C^1 function f in a neighborhood of $\Phi(X)$. This linear map is called the *Jabobi map* (or the *differential*) of Φ at X. If Φ is a C^1 diffeomorphism in a neighborhood of X, then $d\Phi$ is an isomorphism.

We start constructing our differential surface by taking the surface $\mathcal{S}$ of Section 2 as the underlying topological space. Let $J = J_1 \cup J_2 \cup J_3$, where J_1 is the set of the triangles $P_iQ_iR_i$ $(i \in \mathbf{N})$, J_2 is the set of the edges of these triangles, and J_3 is the set of vertices of the triangles. We choose the open sets $V_p \subset \mathbb{R}^2$ as follows:

- For $p \in J_1$, let V_p be the interior of the corresponding triangle.
- Suppose that $p \in J_2$. If $p = P_i\,Q_i$ for some $i \in \mathbf{N}$, let V_p be the open neighborhood of p defined by the inequality $w_i^2 < u_iv_i$. This is an interior of an ellipse (see Figure 5), since by setting $w_i = 1 - u_i - v_i$ we get the affine inequality $u_i^2 + u_i\,v_i + v_i^2 - 2\,u_i - 2\,v_i + 1 < 0$. If $p = P_i\,R_i$ for $i \in \mathbf{N}$, let V_p be the open neighborhood of p defined by $v_i^2 < u_iw_i$. If $p = Q_iR_i$ for some $i \in \mathbf{N}$, let V_p be the open neighborhood of p defined by $w_i^2 < u_iv_i$.
- Suppose that $p \in J_3$. If $p = P_i$ for some $i \in \mathbf{N}$, let V_p be the open neighborhood of p defined by the inequality $v_i^2 + w_i^2 < u_i^2/9$. One can check that this is an interior of an ellipse in the same way as above; see Figure 6. If $p = Q_i$ for some $i \in \mathbf{N}$, let V_p be the open neighborhood of p defined by the inequality $u_i^2 + w_i^2 < v_i^2/9$. If $p = R_i$ for some $i \in \mathbf{N}$, let V_p be the open neighborhood of p defined by the inequality $u_i^2 + v_i^2 < w_i^2/9$.

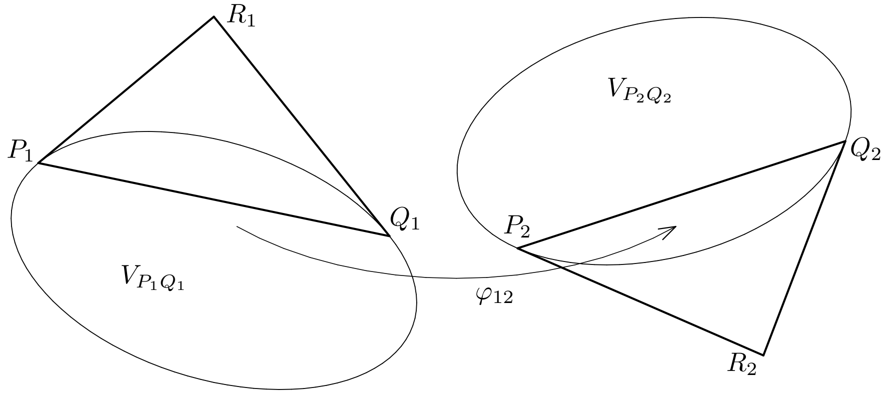

FIGURE 5. Gluing two triangle edges

Now we define some fractional-linear maps on $\mathbb{R}^2$. Suppose that $(i, j) \in \{(1,2),(3,4),(5,6),(7,8)\}$. Let X be a point in $\mathbb{R}^2$ with the barycentric coordinates (u_i, v_i, w_i) with respect to the triangle $P_i\,Q_i R_i$, and suppose that $w_i \neq 1/2$. We define $\varphi_{ij}(X)$ to be the point with the barycentric coordinates

$$(4.2)\qquad (u_j,\, v_j,\, w_j) = \left(\frac{u_i}{u_i{+}v_i{-}w_i},\, \frac{v_i}{u_i{+}v_i{-}w_i},\, -\frac{w_i}{u_i{+}v_i{-}w_i}\right)$$

with respect to the triangle $P_j\,Q_j\,R_j$. In a compact form, we write

$$(4.3)\qquad \varphi_{ij}\,(u_iP_i + v_iQ_i + w_iR_i) = \frac{u_i\,P_j + v_i\,Q_j - w_i\,R_j}{u_i + v_i - w_i}$$

By putting $w_i = 0$ we see that the restriction of φ_{ij} onto the edge $P_i\,Q_i$ is the homeomorphism ϱ_{ij} in (2.1). Further, φ_{ij} maps $V_{P_iQ_i}$ to $V_{P_jQ_j}$ since the inequality $w_i^2 < u_iv_i$ implies the inequality $w_j^2 < u_jv_j$ in the transformed coordinates (4.2). Note that φ_{ij} maps $V_{P_iQ_i}\cap V_{P_iQ_iR_i}$ to $V_{P_jQ_j}\setminus V_{P_jQ_jR_j}$, and it maps $V_{P_iQ_i}\setminus V_{P_iQ_iR_i}$ to $V_{P_jQ_j}\cap V_{P_jQ_jR_j}$; see Figure 5. Besides, φ_{ij} maps V_{P_i} to V_{P_j}, and it maps V_{Q_i} to V_{Q_j}; see Figure 6. We define the map φ_{ji} by interchanging i and j in (4.3). By inspecting transformations of the barycentric coordinates we see that φ_{ji} is an inverse of φ_{ij}. Similarly, for $(i, j) \in \{(1,3),(2,4),(5,7),(6,8)\}$ we define

$$(4.4)\qquad \varphi_{ij}\,(u_iP_i + v_iQ_i + w_iR_i) = \frac{u_i\,P_j - v_i\,Q_j + w_i\,R_j}{u_i - v_i + w_i}$$

and their inverses φ_{ji}. For $(i, j) \in \{(1,5),(2,6),(3,7),(4,8)\}$ we define

$$(4.5)\qquad \varphi_{ij}\,(u_iP_i + v_iQ_i + w_iR_i) = \frac{-u_i\,P_j + v_i\,Q_j + w_i\,R_j}{-u_i + v_i + w_i}$$

and their inverses φ_{ji}.

We define the open sets $U_p \subset \mathcal{S}$ and the homeomorphisms ϕ_p as follows:

- Suppose that $p \in J_1$. Let U_p be the interior of the corresponding triangle, and let $\phi_p : V_p \to U_p$ be the identity map.
- Suppose that $p \in J_2$. It is an edge of some triangle $P_iQ_iR_i$, $i \in \mathbf{N}$. Let $q \in J_2$ be the triangle edge to which p is identified by some homeomorphism in (2.1), and let $P_jQ_jR_j$ (with $j \in \mathbf{N}$) be the triangle of q. We define

$$U_p = (V_p \cap V_{P_iQ_iR_i}) \cup \left(V_q \cap V_{P_jQ_jR_j}\right).$$

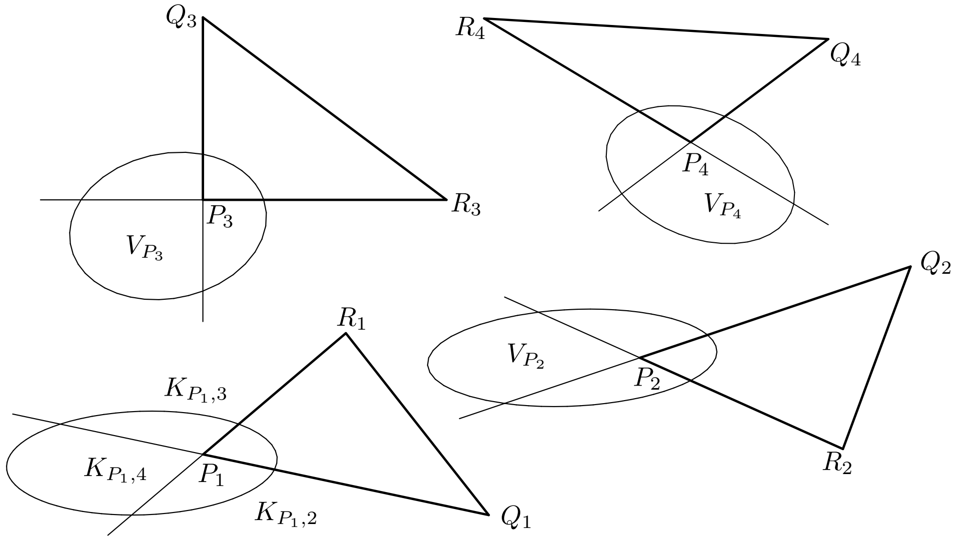

FIGURE 6. Gluing four triangle vertices

Here the union is taken on $\mathcal{S}$, so that p and q are identified. We define $\phi_p : V_p \to U_p$ by

$$\phi_p(X) = \begin{cases} X, & \text{if } X \in V_p \cap V_{P_iQ_iR_i}, \\ \varphi_{ij}(X), & \text{if } X \in V_p \setminus V_{P_iQ_iR_i}. \end{cases}$$

- Suppose that $p \in J_3$. It is a vertex of some triangle $P_iQ_iR_i$, $i \in \mathbf{N}$. Let $s, z \in J_2$ be the triangle edges incident to p. Let $P_jQ_jR_j$, $P_kQ_kR_k$ ($j, k \in \mathbf{N}$) be the triangles which have an edge identified by (2.1) with s and z respectively. Let $q, r \in J_3$ be the triangle vertices of $P_jQ_jR_j$, $P_kQ_kR_k$ respectively which are identified with p. There is one more triangle vertex identified with p; we denote it by t. Let $P_\ell Q_\ell R_\ell$ ($\ell \in \mathbf{N}$) be the triangle of t. We define U_p to be the set

$$\left(V_p \cap V_{P_iQ_iR_i}\right) \cup \left(V_q \cap V_{P_jQ_jR_j}\right) \cup \left(V_r \cap V_{P_kQ_kR_k}\right) \cup \left(V_t \cap V_{P_\ell Q_\ell R_\ell}\right).$$

Here the union is taken on $\mathcal{S}$. See Figure 6 for reference, with $i = 1$, $j = 2$, $k = 3$ and $\ell = 4$. Further, the two lines which contain s and z divide $\mathbb{R}^2$ into four sectors. Let $K_{p,i}$ denote the closed sector which contains $P_iQ_iR_i$. Let $K_{p,j}$, $K_{p,k}$ be the open sectors which are adjacent to $K_{p,i}$ and have non-empty intersection with V_s, V_z respectively. Let $K_{p,\ell}$ be the closed sector opposite to $K_{p,i}$. We define $\phi_p : V_p \to U_p$ by

$$\phi_p(X) = \begin{cases} X, & \text{if } X \in V_p \cap K_{p,i}, \\ \varphi_{ij}(X), & \text{if } X \in V_p \cap K_{p,j}, \\ \varphi_{j\ell} \circ \varphi_{ij}(X), & \text{if } X \in V_p \cap K_{p,\ell} \setminus \{p\}, \\ \varphi_{ik}(X), & \text{if } X \in V_p \cap K_{p,k}. \end{cases}$$

One can check that the image of this map is indeed U_p. Notice that $\varphi_{j\ell} \circ \varphi_{ij} = \varphi_{k\ell} \circ \varphi_{ik}$; we denote this map by $\varphi_{i\ell}$.

To see that we have a structure of a differentiable surface on $\mathcal{S}$, note that any transition map is either an identity map or a restriction of some φ_{ij} defined by us.

For example, if $p, q \in J_2$ are triangle edges identified by (2.1), and $i, j \in \mathbf{N}$ are the indices of their respective triangles, then $U_p = U_q$, and the transition map $\phi_q^{-1} \circ \phi_p$ is the restriction of φ_{ij} onto V_p. This completes our definition of the C^1 differential surface $\mathcal{S}$.

The following theorem says that the set of C^1 functions on $\mathcal{S}$ coincides with the set of GC^1 functions on $\mathcal{H}$. Theorem 6.2.5 in [**Vid99**] basically states that $\mathcal{S}$ is a unique C^1 differential surface (up to equivalence of C^1 atlases) with this property.

THEOREM 4.2. *Let $(f_i)_{i\in\mathbf{N}}$ be a continuous function on $\mathcal{S}$ (and a continuous function on $\mathcal{H}$). It is a C^1 function on $\mathcal{S}$ if and only if it is a GC^1 function on $\mathcal{H}$.*

PROOF. Assume that $(f_i)_{i\in\mathbf{N}}$ is a C^1 function on $\mathcal{S}$. To show condition (a) of Definition 2.2, take $i \in \mathbf{N}$ and consider the open set $W = V_{P_iQ_iR_i} \cup V_{P_iQ_i} \cup V_{P_iR_i} \cup V_{Q_iR_i} \cup V_{P_i} \cup V_{Q_i} \cup V_{R_i} \subset \mathbb{R}^2$. We extend f_i to a C^1 continuous function on W by using other components f_j and the corresponding maps ϕ_p. Now we show condition (b). For $(i,j) \in \{(1,2),(3,4),(5,6),(7,8)\}$ consider a point X on the edge P_iQ_i with barycentric coordinates $(u_i, v_i, w_i) = (1-\zeta, \zeta, 0)$, $\zeta \in [0,1]$. The points X and $\varphi_{ij}(X)$ represent the same point Y on $\mathcal{S}$. The Jacobi maps of $\phi_{P_iQ_i}$, $\phi_{P_jQ_j}$ identify three tangent spaces $T_{\mathcal{S},Y}$, $T_{\mathbb{R}^2,X}$ and $T_{\mathbb{R}^2,\varphi_{ij}(X)}$. Transformation between the latter two tangent spaces is given by $d\varphi_{ij}$. We take $\mathbf{D}_{\overrightarrow{P_jQ_j}}$, $\mathbf{D}_{\overrightarrow{P_jR_j}}$ as a basis for $T_{\mathbb{R}^2,\varphi_{ij}(X)}$. Note its straightforward *dual* action on the function pair (v_j, w_j); see (2.9). We take the similar basis for $T_{\mathbb{R}^2,X}$. Using (4.1) we compute the action of both $d\varphi_{ij}\big(\mathbf{D}_{\overrightarrow{P_iQ_i}}\big)$, $d\varphi_{ij}\big(\mathbf{D}_{\overrightarrow{P_iR_i}}\big)$ on the functions v_j, w_j and conclude that

$$d\varphi_{ij}\left(\mathbf{D}_{\overrightarrow{P_iQ_i}}\right) = \frac{1}{u_i+v_i-w_i}\,\mathbf{D}_{\overrightarrow{P_jQ_j}}, \tag{4.6}$$

$$d\varphi_{ij}\left(\mathbf{D}_{\overrightarrow{P_iR_i}}\right) = -\frac{1}{(u_i+v_i-w_i)^2}\,\mathbf{D}_{\overrightarrow{P_jR_j}} + \frac{2v_i}{(u_i+v_i-w_i)^2}\,\mathbf{D}_{\overrightarrow{P_jQ_j}}. \tag{4.7}$$

Here the coefficients should be evaluated at X, so $d\varphi_{ij}$ acts on $T_{\mathbb{R}^2,X}$ as follows:

$$\mathbf{D}_{\overrightarrow{P_iQ_i}} \mapsto \mathbf{D}_{\overrightarrow{P_jQ_j}}, \qquad \mathbf{D}_{\overrightarrow{P_iR_i}} \mapsto -\mathbf{D}_{\overrightarrow{P_jR_j}} + 2\zeta\,\mathbf{D}_{\overrightarrow{P_jQ_j}}. \tag{4.8}$$

The action on $\mathbf{D}_{\overrightarrow{P_iR_i}}$ gives (2.10). Similarly, equalities (2.11) and (2.12) hold for $(i,j) \in \{(1,3),(2,4),(5,7),(6,8)\}$ or $(i,j) \in \{(1,5),(2,6),(3,7),(4,8)\}$ respectively, and for all $\zeta \in [0,1]$.

Now suppose that $f = (f_i)_{i\in\mathbf{N}}$ is a GC^1 function on $\mathcal{H}$. If $Y \in \mathcal{S}$ is in the interior of some triangle $p = P_iQ_iR_i$, then $f \circ \phi_p = f_i$ is a C^1 function on the open neighborhood $U_{P_iQ_iR_i}$ of Y. Take now $Y \in \mathcal{S}$ represented by a point X_0 in the interior of an edge p, say $p = P_iQ_i$. Let $q = P_jQ_j$ be the edge identified with p. We have to prove that the function

$$\begin{cases} f_i(X), & \text{if } X \in V_p \cap V_{P_iQ_iR_i} \\ f_j \circ \varphi_{ij}(X), & \text{if } X \in V_p \setminus V_{P_iQ_iR_i} \end{cases} \tag{4.9}$$

is a C^1 function on an open neighborhood of X_0 inside V_p. By formula (4.1) we have to show $d\varphi_{ij}(\delta) f_j = \delta f_i$ at X_0 for any $\delta \in T_{\mathbb{R}^2,X_0}$. But $d\varphi_{ij}$ transforms the derivations as in (4.8) which suits us. Suppose now that $Y \in \mathcal{S}$ is represented by four vertices of triangles, say P_1, P_2, P_3, P_4. We have to prove that the function on V_{P_1}, given piecewise by f_1, $f_2 \circ \varphi_{12}$, $f_3 \circ \varphi_{13}$, $f_4 \circ \varphi_{14}$, is a C^1 function around P_1. This follows from the identifications $\mathbf{D}_{\overrightarrow{P_1Q_1}} = \mathbf{D}_{\overrightarrow{P_2Q_2}} = -\mathbf{D}_{\overrightarrow{P_3Q_3}} = -\mathbf{D}_{\overrightarrow{P_4Q_4}}$ and $\mathbf{D}_{\overrightarrow{P_1R_1}} = -\mathbf{D}_{\overrightarrow{P_2R_2}} = \mathbf{D}_{\overrightarrow{P_3R_3}} = -\mathbf{D}_{\overrightarrow{P_4R_4}}$ induced by $d\varphi_{12}$, $d\varphi_{13}$ and $d\varphi_{14}$. $\square$

5. Geometrically continuous surface complex

Here we define a CG^1 *geometrically continuous surface complex* and interpret the octahedron $\mathcal{H}$ as such an object. This definition has proper foundations in differential topology, and it gives a data structure that can be used effectively to work with general geometrically continuous surfaces and functions.

For a precise definition we use the notion of a tangent bundle. Let Ω_1 denote a polygon in $\mathbb{R}^2$, and let p denote an edge of Ω_1. The *tangent bundle* $T_{\mathbb{R}^2,p}$ of $\mathbb{R}^2$ along p is a continuous family of tangent spaces $T_{\mathbb{R}^2,X}$, $X \in p$. Technically, it is the restriction of the tangent bundle of $\mathbb{R}^2$ onto p. As a manifold, $T_{\mathbb{R}^2,p}$ is isomorphic to $p \times \mathbb{R}^2$. Let q be other edge on a polygon in $\mathbb{R}^2$. A map $\theta : T_{\mathbb{R}^2,p} \to T_{\mathbb{R}^2,q}$ is a *continuous isomorphism* of tangent bundles if it is continuous (as a map between manifolds) and for any $X \in p$ the fiber map $\theta|_X$ from $T_{\mathbb{R}^2,X}$ is a linear isomorphism.

It is not technically correct to speak of tangent bundles of p and q along themselves, since these are closed line segments. Instead we consider open neighborhoods $\widetilde{p} \supset p$ and $\widetilde{q} \supset p$ inside the lines containing p and q. The tangent bundle $T_{\tilde{p},p}$ of $\widetilde{p}$ along p is a subbundle of $T_{\mathbb{R}^2,p}$, so that for any $X \in p$ the fibre $T_{\tilde{p},X} \subset T_{\mathbb{R}^2,X}$ consists of those vectors that are tangent to $\widetilde{p}$. The same can be said about $T_{\tilde{q},q}$. We say that a map $\mu : p \to q$ is a C^1 *diffeomorphism* if there exists an extension $\widetilde{\mu} : \widetilde{p} \to \widetilde{q}$ of μ which is a C^1 diffeomorphism of $\widetilde{p}$ and $\widetilde{q}$. For $X \in p$ the *Jacobi map* $T_{\tilde{p},X} \to T_{\tilde{q},\mu(X)}$ is induced by $\widetilde{\mu}$ like in (4.1). The family of Jacobi maps gives a linear isomorphism $T_{\tilde{p},p} \to T_{\tilde{q},q}$ (in the identical sense as above) which does not depend on the extension $\widetilde{\mu}$ of μ. We denote this linear isomorphism by $d\mu$.

Here is the last piece of our notation. If X is an point on p, let $H_{\Omega_1,X}$ denote the set of those vectors $\vec{\mathbf{a}} \in T_{\mathbb{R}^2,X}$ for which $X + \zeta\vec{\mathbf{a}}$ lies in the interior of Ω_1 for all small enough $\zeta > 0$. If X is an endpoint of p, then $H_{\Omega_1,X}$ is a closed cone with its vertex at the origin of $T_{\mathbb{R}^2,X}$. If X is in the interior of p, then $H_{\Omega_1,X}$ is a closed half-plane (and the origin of $T_{\mathbb{R}^2,X}$ lies on its boundary).

Now we are ready to define CG^1 surface complexes. Here is a compact summary of Definitions 6.2, 6.5, 6.9 and 6.10 in [**Vid99**].

Definition 5.1. A GC^1 *geometrically continuous surface complex* $\mathcal{G}$ is given by the data $(\Omega, \sim, \rho, \Theta)$, where

1. Ω is a finite collection of polygons in $\mathbb{R}^2$. Some (or all) polygons may coincide but be considered as different elements of Ω. Edges and vertices of different polygons are considered as distinct.
2. $\sim$ is an equivalence relation between edges of the polygons, such that each edge is equivalent to at most one other polygon edge.
3. For each pair (p, q) of equivalent edges, ρ gives a C^1 diffeomorphism $\mu_{p,q}$ from p to q.
4. For each pair (p, q) of equivalent edges, Θ gives a continuous isomorphism $\theta_{p,q} : T_{\mathbb{R}^2,p} \to T_{\mathbb{R}^2,q}$ of the tangent bundles of $\mathbb{R}^2$ along p and q. Let Ω_1, Ω_2 be the polygons of p, q respectively. We require that:
   - (4a) $\theta_{p,q}$ maps the tangent bundle of p to the tangent bundle of q, and the restriction of $\theta_{p,q}$ to these tangent bundles coincides with $d\mu_{p,q}$.
   - (4b) If X is an interior point of p, let Y denote $\mu_{p,q}(X)$. Then the union of $\theta_{p,q}|_X(H_{\Omega_1,X})$ and $H_{\Omega_2,Y}$ must coincide with all of $T_{\mathbb{R}^2,Y}$.
   - (4c) If X is an endpoint of p, let Y denote $\mu_{p,q}(X)$. Then the intersection of $\theta_{p,q}|_X(H_{\Omega_1,X})$ and $H_{\Omega_2,Y}$ is a half-line through the origin of $T_{\mathbb{R}^2,Y}$.

Besides, we put the following restrictions.

(5) Suppose that $\Omega_1, \ldots, \Omega_n$ is a sequence of polygons taken from Ω and that for $i = 1, \ldots, n$ we have a vertex X_i of Ω_i and edges p_i, q_i of Ω_i meeting at X_i, such that the vertices $X_1, X_2, \ldots, X_{n-1}$ are distinct and for $i = 2, \ldots, n$ the edges p_{i-1}, q_i are equivalent.

(5a) If the vertices X_1, X_n are distinct, let $\widetilde{H}_n$ denote H_{Ω_n,X_n} and for $i = 1, \ldots, n-1$ let $\widetilde{H}_i$ denote the image of H_{Ω_i,X_i} under the composition $\theta_{p_{n-1},q_n}|_{X_{n-1}} \circ \ldots \circ \theta_{p_i,q_{i+1}}|_{X_i}$. Then the union of all $\widetilde{H}_i$ (for $i = 1, \ldots, n$) must be a proper subset of $T_{\mathbb{R}^2,X_n}$.

(5b) If $X_1 = X_n$ then the composition $\theta_{p_n,q_1}|_{X_n} \circ \theta_{p_{n-1},q_n}|_{X_{n-1}} \circ \ldots \circ \theta_{p_1,q_2}|_{X_1}$ must be the identity map on $T_{\mathbb{R}^2,X_1}$.

Part (4) of this Definition corresponds to [**Vid99**, Definition 6.2] and to CG^1 join of two patches along an edge (as defined in [**Hah89**]). Part (5) corresponds to [**Vid99**, Definition 6.5] and to CG^1 join of several patches at a vertex (as defined in [**Hah89**]). Compared with the definitions in [**Vid99**], we avoided here mentioning intersections of tangent cones in parts (4b), (5a), (5b) because of implications of parts (4a), (4c) and (5a) respectively. For instance, when part (5b) is needed then (5a) applies to subsequences of $\{(\Omega_i, X_i, p_i, q_i)\}_{i=1}^n$.

Now we transform the data structure $\mathcal{H} = (\Omega, \rho, \Xi)$ of Definition 2.1 to a geometrically continuous surface $(\Omega, \sim, \widehat{\rho}, \Theta)$.

- Ω is the same set of triangles $P_iQ_iR_i$, $i \in \mathbf{N}$.
- We say that two triangle edges are equivalent if there is a homeomorphism in (2.1) between them.
- $\widehat{\rho}$ is the set of linear homeomorphisms in (2.1) and their inverses.
- Let p, q be two triangle edges identified by a homeomorphism ϱ_{ij} in (2.1), and let $\vec{p} \in \{\overrightarrow{P_iQ_i}, \overrightarrow{P_iR_i}, \overrightarrow{Q_iR_i}\}$, $\vec{q} \in \{\overrightarrow{P_jQ_j}, \overrightarrow{P_jR_j}, \overrightarrow{Q_jR_j}\}$ be the vectors along p and q respectively. We require that the continuous isomorphisms $\theta_{p,q}, \theta_{q,p} \in \Theta$ should identify the tangent spaces $T_{\mathbb{R}^2,X}$ and $T_{\mathbb{R}^2,Y}$ for all $X \in p$, $Y = \varrho_{ij}(X) \in q$ by $\mathbf{D}_{\vec{p}} \leftrightarrow \mathbf{D}_{\vec{q}}$ and $\mathbf{D}_{\xi_p(X)} \leftrightarrow -\mathbf{D}_{\xi_q(Y)}$.

The difference between Ξ and Θ is that Ξ assigns *(transversal) continuous vector fields* to the triangle edges. The continuous isomorphisms in Θ are determined by condition (4a) and the specification that the pairs of vectors assigned by Ξ should map (up to the sign) to each other. On the other hand, Ξ is not determined uniquely by Θ, and conditions (5a)–(5b) are easier to state in terms of Θ.

Definition 5.2. Let $\mathcal{G} = (\Omega, \sim, \rho, \Theta)$ be a GC^1 surface complex. Suppose that f is an map from Ω which assigns to each polygon $\Omega_1 \in \Omega$ a function f_{Ω_1} on Ω_1. Then f is called a GC^1 *function* on $\mathcal{G}$ if the following conditions hold:

(1) For each $\Omega_1 \in \Omega$, the function f_{Ω_1} is a C^1 function on the interior of Ω_1, and it can be extended to a C^1 function on an open neighborhood of Ω^1.

(2) For each C^1 diffeomorphism $\mu \in \rho$ from an edge p of $\Omega_1 \in \Omega$ to an edge of $\Omega_2 \in \Omega$ we require that the restriction of f_{Ω_1} onto p coincides with $f_{\Omega_2} \circ \mu$.

(3) For each continuous isomorphism $\theta \in \Theta$ from the tangent bundle along an edge p of $\Omega_1 \in \Omega$ to the tangent bundle along an edge of $\Omega_2 \in \Omega$, and for each $X \in p$ we require that $\delta(f_{\Omega_1}) = \theta|_X(\delta)\,(f_{\Omega_2})$ for all $\delta \in T_{\mathbb{R}^2,X}$.

This definition applied to $\mathcal{H}$ coincides with Definition 2.2.

We have defined the notions of geometrically continuous surface complex and GC^1 functions on it. The example of this paper demonstrates that these notions can be used effectively in geometric modeling. They let us embrace the generality of geometrically continuity, and at the same time they allow handy blending methods that are available in the context of parametric continuity. In particular, geometrically continuous functions can be used as conveniently as usual B-splines. Computation of GC^1 functions is easier than geometrically continuous gluing of three-dimensional patches, and broad classes of these functions are computable. They may have various applications as just functions.

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