Transformations of Gauss hypergeometric functions

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Abstract

The paper classifies algebraic transformations of Gauss hypergeometric functions and pull-back transformations between hypergeometric differential equations. This classification recovers the classical transformations of degree 2, 3, 4, 6, and finds other transformations of some special classes of the Gauss hypergeometric function.

1 Introduction

An algebraic transformation of Gauss hypergeometric functions is an identity of the form

$$ {}_2\mathrm{F}_1\left(\left.\begin{array}{c}\widetilde{A},\,\widetilde{B}\\ \widetilde{C}\end{array}\right|\,x\right)=\theta(x)\;{}_2\mathrm{F}_1\left(\left.\begin{array}{c}A,\,B\\ C\end{array}\right|\,\varphi(x)\right). \tag{1}$$

Here $\varphi(x)$ is a rational function of x, and $\theta(x)$ is a radical function, i.e., product of some powers of rational functions. Examples of algebraic transformations are (see [Erd53, AAR99]):

$$\begin{aligned}
{}_2\mathrm{F}_1\left(\left.\begin{array}{c}a,\,b\\ \frac{a+b+1}{2}\end{array}\right|\,x\right) &= {}_2\mathrm{F}_1\left(\left.\begin{array}{c}\frac{a}{2},\,\frac{b}{2}\\ \frac{a+b+1}{2}\end{array}\right|\,4x\,(1-x)\right). &(2)\\
{}_2\mathrm{F}_1\left(\left.\begin{array}{c}a,\,\frac{2a+1}{2}\\ \frac{4a+5}{6}\end{array}\right|\,x\right) &= (1+3x)^{-a}\;{}_2\mathrm{F}_1\left(\left.\begin{array}{c}\frac{a}{3},\,\frac{a+1}{3}\\ \frac{4a+5}{6}\end{array}\right|\,\frac{27\,x\,(1-x)^2}{(1+3x)^3}\right), &(3)\\
{}_2\mathrm{F}_1\left(\left.\begin{array}{c}\frac{4a}{3},\,\frac{4a+1}{3}\\ \frac{4a+5}{6}\end{array}\right|\,x\right) &= (1+8x)^{-a}\;{}_2\mathrm{F}_1\left(\left.\begin{array}{c}\frac{a}{3},\,\frac{a+1}{3}\\ \frac{4a+5}{6}\end{array}\right|\,\frac{64\,x\,(1-x)^3}{(1+8x)^3}\right), &(4)\\
{}_2\mathrm{F}_1\left(\left.\begin{array}{c}a,\,\frac{a+1}{3}\\ \frac{2a+2}{3}\end{array}\right|\,x\right) &= \left(1+\omega^2x\right)^{-a}\;{}_2\mathrm{F}_1\left(\left.\begin{array}{c}\frac{a}{3},\,\frac{a+1}{3}\\ \frac{2a+2}{3}\end{array}\right|\,\frac{3(2\omega+1)\,x(x-1)}{(x+\omega)^3}\right). &(5)
\end{aligned}$$

In the last formula, ω is a primitive cubic root of unity. These identities are well-known classical transformations of hypergeometric functions. They hold in some neighborhood of $x=0$ in the complex plane, and can be continued analytically. For example, formula (2) holds for $\mathrm{Re}(x)<1/2$.

Algebraic transformations of Gauss hypergeometric functions are usually induced by pull-back transformations of their hypergeometric differential equations. The general relation between these two kinds of transformations is given in Lemma 2.1 here below. By that Lemma, if a pull-back transformation converts a hypergeometric equation to a hypergeometric equation as well, then there are identities of the form (1) between hypergeometric solutions of the two hypergeometric equations, unless the transformed equation has a trivial monodromy group. Conversely, an algebraic transformation (1) is induced by a pull-back

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transformation of the corresponding hypergeometric equations, unless the hypergeometric function on the left-hand side of (1) satisfies a simple first order differential equation.

In this paper we classify pull-back transformations between hypergeometric differential equations. At the same time we essentially classify algebraic transformations (1) of Gauss hypergeometric functions. Only transformations of elementary instances of hypergeometric functions are missed out.

A general pull-back transformation converts the hypergeometric equation to a Fucshian differential equation. There are simple rules to determine possible regular points and local exponent differences at the singular points for the transformed equation, see Lemma 2.2 here below. To classify transformations between hypergeometric equations, we have to look at the cases when the pull-backed Fuchsian equation has at most 3 singular points. This restriction allows us to investigate all possible pull-back transformations between hypergeometric equations.

Ultimately, the list of pull-back transformations of hypergeometric differential equations (and of their hypergeometric solutions) is the following.

- Classical algebraic transformations of hypergeometric functions due to Gauss, Euler, Kummer, Pfaff and Goursat. These include fractional-linear transformations, quadratic transformations, and Goursat's transformations of degree 3, 4 and 6.
- Transformations of hypergeometric equations with an abelian monodromy group. This is a degenerate case; the hypergeometric equations have 2 (rather than 3) actual singularities.
- Transformations of hypergeometric equations with a dihedral monodromy group.
- Transformations of algebraic Gauss hypergeometric functions.
- Transformations of hypergeometric functions that are expressible as (incomplete) elliptic integrals.
- Transformations of hypergeometric equations with the local exponent differences being $1/k_1$, $1/k_2$, $1/k_3$, where k_1, k_2, k_3 are positive integers such that $1/k_1 + 1/k_2 + 1/k_3 < 1$. We refer to the corresponding hypergeometric functions as *hyperbolic hypergeometric functions.*

The classification scheme is presented in Section 3. We follow the approach of Riemann and Papperitz [AAR99, Sections 2.3 and 3.9]. For basic theory of hypergeometric functions and Fucshian equations we also refer to [Beu02] or [vdW02, Chapters 1 and 2]. In Section 4 we outline more interesting types of algebraic transformations. All non-classical special cases are extensively considered in separate papers [Vid04a], [Vid04b], [Vid03], [Vid04c].

2 Preliminaries

The hypergeometric differential equation is [AAR99, Formula (2.3.5)]:

$$z\,(1-z)\,\frac{d^2y(z)}{dz^2}+\big(C-(A{+}B{+}1)\,z\big)\,\frac{dy(z)}{dz}-A\,B\,y(z)=0. \tag{6}$$

This is a Fuchsian equation with 3 regular singular points $z=0,1$ and ∞. The local exponent differences at these points are (up to a sign) $1-C$, $C-A-B$ and $A-B$ respectively. A basis of solutions for (6) is

$${}_2\mathrm{F}_1\left(\left.\begin{array}{c}A,\,B\\ C\end{array}\right|\,z\right),\qquad z^{1-C}\;{}_2\mathrm{F}_1\left(\left.\begin{array}{c}1+A-C,\,1+B-C\\ 2-C\end{array}\right|\,z\right). \tag{7}$$

A *pull-back transformation* of the hypergeometric equation has the form

$$z\longmapsto\varphi(x),\qquad y(z)\longmapsto Y(x)=\theta(x)\,y(\varphi(x)), \tag{8}$$

where $\varphi(x)$ and $\theta(x)$ have the same meaning as in formula (1). Geometrically, by such a transformation we pull-back the hypergeometric equation on the projective line $\mathbb{P}^1_z$ to a differential equation on the projective line $\mathbb{P}^1_x$, with respect to the finite covering $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ determined by the rational function $\varphi(x)$. We use the notations $\mathbb{P}^1_x$, $\mathbb{P}^1_z$ throughout the paper.

Pull-back transformations (8) between hypergeometric equations and algebraic transformations (1) of Gauss hypergeometric functions are related as follows.

Lemma 2.1 *1. Suppose that pull-back transformation* (8) *of hypergeometric equation* (6) *is a hypergeometric equation as well (with the new indeterminate x), and that the transformed equation has non-trivial monodromy. Then, possibly after fractional-linear transformations on $\mathbb{P}^1_x$ and $\mathbb{P}^1_z$, there is an identity of the form* (1) *between hypergeometric solutions of two hypergeometric equations.*

2. *Suppose that hypergeometric identity* (1) *holds in some region of the complex plane. Let $Y(x)$ denote the left-hand side of the identity. If $Y'(x)/Y(x)$ is not a rational function of x, then the transformation* (8) *converts the hypergeometric equation (6) into the hypergeometric equation for $Y(x)$.*

Proof. In the setting of the first statement, the transformed equation has either a logarithmic point or a singular point with non-integer local exponent difference. Such a point $P \in \mathbb{P}^1_x$ must lie above a point $Q \in \{0, 1, \infty\} \subset \mathbb{P}^1_z$. By suitable fractional-linear transformations on $\mathbb{P}^1_x$ and $\mathbb{P}^1_z$ one can keep the hypergeometric form of differential equations, and achieve that P is the point $x = 0$ and that Q is the point $z = 0$. Then identification of two hypergeometric solutions with the local exponent 0 and the value 1 at (respectively) $x = 0$ and $z = 0$ gives a two-term identity as in formula (1).

For the second statement, we have two second-order differential equations for the left-hand side of (1): the hypergeometric equation for $Y(x)$, and the pull-back transformation (8) of the hypergeometric equation (6). If these two equations are not $\mathbb{C}(x)$-proportional, then we can combine them to a first-order differential equation $Y'(x) = r(x)\,Y(x)$ with $r(x) \in \mathbb{C}(x)$, which would contradict the condition on $Y'(x)/Y(x)$. ∎

Since any Fuchsian equation with 3 singular points can be converted to a hypergeometric equation by a fractional-linear transformation [AAR99, Section 2.3], we essentially look for the pull-back transformations of hypergeometric equations into Fucshian equations with (at most) 3 singular points. Here is how the singular points and local exponents alter under pull-back transformation (8).

Lemma 2.2 *Let $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ be a finite covering. Let H_1 denote a Fuchsian equation on $\mathbb{P}^1_z$, and let H_2 denote the pull-back transformation of H_1 under* (8)*. Let $P \in \mathbb{P}^1_x$, $Q \in \mathbb{P}^1_z$ be points such that $\varphi(P) = Q$.*

1. *If the point Q is a regular point for H_1, then the point P is a regular point for H_2 only if the covering φ is unramified at P.*

2. *If the point Q is a singular point for H_1, then the point P is a regular point for H_2 only if the local exponent difference at Q is equal to $1/k$, where k is the branching index of φ at P.*

3. *Let d denote the degree of φ, and let Ξ denote a set of three points on $\mathbb{P}^1_z$. If all branching points of φ lie above Ξ, then there are exactly $d + 2$ distinct points on $\mathbb{P}^1_x$ above Ξ. Otherwise there are more than $d + 2$ distinct points above Ξ.*

Proof. Recall that the local exponent difference for regular points is necessarily 1. Let p, q denote the local exponents for H_1 at the point Q. Let k denote the branching index of φ at P, and let m denote the order of $\theta(x)$ at P. Then the local exponents for H_2 at P are equal to $kp + m$ and $kq + m$, so the local exponent difference gets multiplied by k. If the point Q is regular, the point P can be regular only if $k = 1$. If the point Q is singular, then the point P is regular only if $|p - q| = 1/k$. The first two statements follow.

The third part is a purely algebro-geometric statement. By Hurwitz formula [Har77, Corollary 2.4], the total branching degree is $2(d - 1)$. Therefore, there are at least $3d - 2(d - 1) = d + 2$ distinct points above Ξ; this is the exact number of points if φ branches above Ξ only. ∎

3 The classification scheme

We classify all two-term identities (1) of Gauss hypergeometric functions (and pull-back transformations between hypergeometric equations) in the following five principal steps:

1. Let H_1 denote hypergeometric equation (6), and let H_2 denote the pull-backed differential equation under (8). Let S denote the number of singular points of H_2, let Ξ denote the subset $\{0, 1, \infty\}$ of $\mathbb{P}^1_z$, and let d denote the degree of the covering $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ in (8). We consequently assume that exactly $N \in \{0, 1, 2, 3\}$ of the three local exponent differences for H_1 at Ξ are restricted.

2. In each assumed case, use Lemma 2.2 and determine all possible combinations of the degree d and local exponent differences for H_1. The restricted local exponent differences have the form $1/k$, where k is a positive integer. (If $k = 1$ then the corresponding point on $\mathbb{P}^1_z$ cannot be logarithmic, which means that local exponent differences at the other two points in Ξ should be equal [Vid04b].) Let $k_1, \ldots, k_N$ denote the denominators of the restricted differences. Then

$$S \geq d + 2 - \sum_{j=1}^{N} \left\lfloor \frac{d}{k_j} \right\rfloor. \tag{9}$$

   Since we wish $S \leq 3$, we get a restrictive inequality in integers. To skip specializations of the cases with smaller N, we may assume that $d \geq \max\{k_j\}_{j=1}^{N}$. A preliminary list of possibilities can be obtained by dropping the rounding down in (9) and using the weaker but more convenient inequality

$$\frac{1}{d} + \sum_{j=1}^{N} \frac{1}{k_j} \geq 1. \tag{10}$$

3. For each combination of d and local exponent differences for H_1, determine possible branching patterns for φ such that the transformed equation H_2 would have at most three singular points. In most cases we cannot allow branching points outside Ξ, and we have to take the maximal number $\lfloor d/k_j \rfloor$ of regular points above the point with the local exponent difference $1/k_j$.

4. For each possible branching pattern, determine all rational functions $\varphi(x)$ which determine a covering with that branching pattern. For $d \leq 6$ this can be done with a computer by a naive method of undetermined coefficients. In [Vid04c], a more appropriate algorithm is introduced which uses differentiation of $\varphi(x)$. This problem may have no solutions, or there may be several solutions (even up to fractional-linear transformations). To deal with infinite families of branching types, we can give a general, algorithmic or explicit characterization of the corresponding coverings.

5. Once we know a covering $\varphi(x)$, it is straightforward to compose it with relevant fractional-linear transformations and derive identities (1). According to the proof of the first part of Lemma 2.1, we consider all singular points on $\mathbb{P}^1_z$ above $\{0, 1, \infty\} \subset \mathbb{P}^1_z$ and move them to $x = 0$. If the transformed equation has less than 3 actual singular points, one can consider any point above $\{0, 1, \infty\} \subset \mathbb{P}^1_z$ in this manner. There are two identities (possibly the same up to transforming the free parameters) for each possibility to settle the point $x = 0$ above $z = 0$, since both solutions in (7) can be identified with the corresponding solutions of the transformed equation. The factor $\theta(x)$ in (8) should shift the local exponents at potentially regular points to the characteristic values 0 and 1. It is straightforward to determine this factor for each identity (1). Riemann's P-notation is very convenient for these purposes [AAR99, Section 3.9]. The parameters of hypergeometric functions are determined by local exponent differences.

Now we sketch appliance of the above procedure. Most interesting cases of algebraic transformations are illustrated in Section 4.

Local exponent differences		Degree	Branching pattern above	Covering
$(1/k_1,\ 1/k_2,\ p)$	above	d	the regular singular points	composition
$(1/2,\ 1/3,\ p)$	$(1/2,\ p,\ 2p)$	3	$2+1=3=2+1$	indecomposable
$(1/2,\ 1/3,\ p)$	$(1/3,\ p,\ 3p)$	4	$2+2=3+1=3+1$	indecomposable
$(1/2,\ 1/3,\ p)$	$(1/3,\ 2p,\ 2p)$	4	$2+2=3+1=2+2$	no covering
$(1/2,\ 1/3,\ p)$	$(p,\ p,\ 4p)$	6	$2+2+2=3+3=4+1+1$	2×3
$(1/2,\ 1/3,\ p)$	$(2p,\ 2p,\ 2p)$	6	$2+2+2=3+3=2+2+2$	2×3 or 3×2
$(1/2,\ 1/3,\ p)$	$(p,\ 2p,\ 3p)$	6	$2+2+2=3+3=3+2+1$	no covering
$(1/2,\ 1/4,\ p)$	$(p,\ p,\ 2p)$	4	$2+2=4=2+1+1$	2×2
$(1/3,\ 1/3,\ p)$	$(p,\ p,\ p)$	3	$3=3=1+1+1$	indecomposable

Table 1: Transformations of hypergeometric functions with 1 free parameter

When $N=0$, i.e., when no local exponent differences are restricted, then $d=1$ by formula (10). We get Pfaff's and Euler's fractional-linear transformations [AAR99, Theorem 2.2.5].

When $N=1$, we have the following cases:

- $k_1=2$, $d=2$. This gives the classical quadratic transformations [AAR99, Section 3.9].
- $k_1=1$, d any. To have regular points above the z-point with the local exponent difference $1/k_1$, that z-point cannot be logarithmic. This implies that the unrestricted local exponent differences must be equal [Vid04b]. The equation H_1 has only two actual singularities. The covering φ can branch only above the two points with unrestricted local exponent differences. If the triple of local exponent differences for H_1 is $(1,p,p)$, the transformed triple of local exponent differences is $(1,dp,dp)$.

When $N=2$, the most interesting possibilities are presented in Table 1. The first four columns form a snapshot after Step 3 in our scheme. (Two degenerate cases are discussed here immediately below.) The notation for branching pattern gives $d+2$ branching indices for the points above Ξ; branching indices at points in the same fiber are separated by the $+$ signs, different fibers are respectively separated by the $=$ signs. Step 4 of our scheme gives at most one covering (up to fractional-linear transformations) for each branching pattern. Possible compositions of small degree coverings are easy to list and identify. Ultimately, Table 1 yields precisely the classical transformations of degree 3, 4, 6 due to Goursat [Gou81]. Formulas (3)–(5) are examples of classical transformations for the three indecomposable coverings. The two degenerate cases are:

- $k_1=2$, $k_2=2$, d any. The monodromy group of H_1 is a dihedral group. The hypergeometric functions can be expressed very explicitly; their algebraic transformations are described in [Vid04b]. The triple $(1/2,1/2,p)$ of local exponent differences for H_1 is transformed either to $(1/2,1/2,dp)$ for any d, or to $(1,dp/2,dp/2)$ for even d.
- $k_1=1$, k_2 and d any. To have regular points above the z-point with the local exponent difference $1/k_1$, the triple of local exponent differences for H_1 must be $(1,1/k_2,1/k_2)$. The monodromy group for H_1 is a finite cyclic group. Transformations are explicitly described in [Vid04b].

When $N=3$, we have the following three very distinct cases:

- $1/k_1+1/k_2+1/k_3>1$. The monodromy groups of H_1 and H_2 are finite, the hypergeometric functions are algebraic. The degree d is unbounded. The most important transformations are those implied by Klein's theorem [Kle78]. In particular, any hypergeometric equation with the tetrahedral, octahedral or icosahedral monodromy group is a pull-back transformation of a *standard hypergeometric equation* with that monodromy group. The local exponent differences for standard equations are, respectively: $(1/2,1/3,1/3)$, $(1/2,1/3,1/4)$ or $(1/2,1/3,1/5)$.

- $1/k_1+1/k_2+1/k_3=1$. Non-trivial hypergeometric solutions of H_1 are elliptic integrals [Vid03]. The degree d is unbounded, different transformations with the same branching pattern are possible. The most interesting transformations pull-back the equation H_1 into itself, so that $H_2=H_1$. These transformations correspond to the endomorphisms of the corresponding elliptic curve.
- $1/k_1+1/k_2+1/k_3<1$. Here we have transformations of hyperbolic hypergeometric functions. The list of these transformations is finite [Vid04c], the maximal degree of their coverings is 24. Some of these transformations are anticipated in [Hod18], [Beu02].

4 Explicit transformations

Solutions of hypergeometric equations with an abelian or dihedral monodromy group are very explicit. Their transformations are extensively considered in [Vid04b], including the cases of finite cyclic and finite dihedral groups. Here we illustrate more interesting types of non-classical algebraic transformations.

Algebraic Gauss hypergeometric functions have been studied by many authors; see [Vid04a] for references. A convenient way to represent algebraic hypergeometric functions is to transform them to radical functions, for example:

$$ {}_2\mathrm{F}_1\left(\begin{matrix}1/4,-1/12\\2/3\end{matrix}\,\middle|\,\frac{x\,(x+4)^3}{4(2x-1)^3}\right)=\frac{1}{(1-2x)^{1/4}},\qquad {}_2\mathrm{F}_1\left(\begin{matrix}1/6,-1/6\\1/4\end{matrix}\,\middle|\,\frac{27\,x\,(x+1)^4}{2(x^2+4x+1)^3}\right)=\frac{(1+2x)^{1/4}}{\sqrt{1+4x+x^2}}, $$

$$ {}_2\mathrm{F}_1\left(\begin{matrix}7/20,\,-1/20\\4/5\end{matrix}\,\middle|\,\frac{64\,x\,(x^2-x-1)^5}{(x^2-1)\,(x^2+4x-1)^5}\right)=\frac{(1+x)^{7/20}}{(1-x)^{1/20}\,(1-4x-x^2)^{1/4}}. $$

These are so called *Darboux evaluations* of hypergeometric functions. Once a few such evaluations for each Schwartz type are known, any algebraic Gauss hypergeometric function can be evaluated in this way using contiguous relations [Vid04a]. These explicit evaluations can be used to pull-back a standard hypergeometric equation, with the local exponent differences $(1/2,1/3,1/3)$, $(1/2,1/3,1/4)$ or $(1/2,1/3,1/5)$, to any other hypergeometric equation with tetrahedral, octahedral or icosahedral monodromy group, as implied by Klein's theorem [Kle78]. Computations are quite straightforward [Vid04a]. For example, Klein's morphism for a hypergeometric equation with the local exponent differences $(4/3,4/3,2/3)$, with the tetrahedral monodromy group, is given by

$$ \varphi_{14}(x)=-\frac{108\,x^4\,(x-1)^4\,\left(27x^2-27x+7\right)^3}{(189x^4-378x^3+301x^2-112x+16)^3}. $$

Transformations between hypergeometric equations with different finite monodromy groups are usually compositions of know transformations. An interesting exception is the following transformation between standard tetrahedral and icosahedral equations:

$$ {}_2\mathrm{F}_1\left(\begin{matrix}1/4,-1/12\\2/3\end{matrix}\,\middle|\,x\right)=\left(1+\frac{7-33\sqrt{-15}}{128}\,x\right)^{1/12}{}_2\mathrm{F}_1\left(\begin{matrix}11/60,-1/60\\2/3\end{matrix}\,\middle|\,\varphi(x)\right), \tag{11} $$

where

$$ \varphi_5(x)=\frac{50(5+3\sqrt{-15})\,x\,(1024x-781-171\sqrt{-15})^3}{(128x+7+33\sqrt{-15})^5}. $$

Hypergeometric incomplete elliptic integrals are solutions of hypergeometric equations with the local exponent differences $(1/2,1/4,1/4)$, $(1/2,1/3,1/6)$ or $(1/3,1/3,1/3)$. For example, let H_4 denote hypergeometric equation (6) with $A=0$, $B=1/4$, $C=3/4$. It has a solution

$$ 2z^{1/4}\,{}_2\mathrm{F}_1\left(\begin{matrix}1/2,\,1/4\\5/4\end{matrix}\,\middle|\,z\right)=\frac{1}{2}\int_0^z t^{-3/4}\,(1-t)^{-1/2}\,dt=\int_{1/\sqrt{z}}^{\infty}\frac{dx}{\sqrt{x^3-x}}. \tag{12} $$

For the last integral we substituted $t \mapsto x^{-2}$. We recognize an integral of a holomorphic differential form on the genus 1 curve $y^2 = x^3 - x$. Let E_1 denote the corresponding elliptic curve in the standard Weierstrass form [Sil86]. If (ψ_x, ψ_y) is an endomorphism of E_1, then the substitution $x \mapsto \psi_x(x, \sqrt{x^3 - x})$ in (12) gives an integral of a holomorphic differential form again. Since the linear space of holomorphic differentials on E_1 is one-dimensional, the transformed differential form must be proportional to $dx/\sqrt{x^3 - x}$. The upper integration bound does not change. Then transformation of the lower integration bound gives the transformation $z \mapsto \psi_x(1/\sqrt{z})^{-2}$ of the hypergeometric function into itself, up to a radical factor. It turns out that $\psi_x(1/\sqrt{z})^{-2}$ is a rational function and that it gives a pull-back transformation of H_4 into itself [Vid03]. Conversely, any pull-back transformation of H_4 into itself is induced by an endomorphism of E_1. Examples of corresponding algebraic transformations are:

$$
\begin{aligned}
{}_2\mathrm{F}_1\left(\left.\begin{array}{c} 1/2,\ 1/4 \\ 5/4 \end{array}\right|\, z\right) &= \frac{\sqrt{1-z}}{1+z}\; {}_2\mathrm{F}_1\left(\left.\begin{array}{c} 1/2,\ 1/4 \\ 5/4 \end{array}\right|\, \frac{16\,z\,(z-1)^2}{(z+1)^4}\right). \\
{}_2\mathrm{F}_1\left(\left.\begin{array}{c} 1/2,\ 1/4 \\ 5/4 \end{array}\right|\, z\right) &= \frac{1-z/(1+2i)}{1-(1+2i)z}\; {}_2\mathrm{F}_1\left(\left.\begin{array}{c} 1/2,\ 1/4 \\ 5/4 \end{array}\right|\, \frac{z\,(z-1-2i)^4}{\big((1+2i)z-1\big)^4}\right).
\end{aligned}
$$

The ring of endomorphisms of E_1 is isomorphic to the ring $\mathbb{Z}[i]$ of Gaussian integers [Sil86]. The pull-back transformations of H_4 into itself form a group isomorphic to $\mathbb{Z}[i]^*/(\pm1, \pm i)$. The degree of such a transformation is equal to the norm of a corresponding Gaussian integer. Computation of endomorphisms of E_1 is equivalent to the group law computations on E_1 by the chord-and-tangent method.

Similarly, the pull-back transformations of hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/6)$ or $(1/3, 1/3, 1/3)$ into themselves correspond to endomorphisms of the elliptic curve $y^2 = x^3 - 1$. The group of these transformations is isomorphic to $\mathbb{Z}[\omega]^*/(\pm1, \pm\omega, \pm\omega^2)$, where ω is a primitive cubic root of unity as in (5). Additionally, hypergeometric equations with the exponent differences $(1/2, 1/3, 1/6)$ can also be transformed to equations with the exponent differences $(1/3, 1/3, 1/3)$ or $(2/3, 1/6, 1/6)$ by composing the mentioned transformations with quadratic ones.

Transformations of hyperbolic hypergeometric functions are extensively studied in [Vid04c]. We have the following finite list of transformations. Up to fractional-linear transformations, hypergeometric equations with the local exponent differences $(1/2,\ 1/3,\ 1/7)$ can be transformed: to equations with the exponent differences $(1/3,\ 1/3,\ 1/7)$ by the degree 8 transformation

$$
\varphi_8(x) = \frac{x\,(x-1)\,\big(27x^2 - (723+1392\omega)x - 496+696\omega\big)^3}{64\,\big((6\omega+3)x - 8 - 3\omega\big)^7};
$$

to equations with the exponent differences $(1/2,\ 1/7,\ 1/7)$ by the degree 9 transformation

$$
\varphi_9(x) = \frac{27\,x\,(x-1)\,(49x - 31 - 13\xi)^7}{49\,(7203x^3 + (9947\xi - 5831)x^2 - (9947\xi + 2009)x + 275 - 87\xi)^3} \quad \text{where } \xi^2 + \xi + 2 = 0;
$$

to equations with the exponent differences $(1/3,\ 1/7,\ 2/7)$ by the degree 10 transformation

$$
\varphi_{10}(x) = -\frac{x^2\,(x-1)\,(49x-81)^7}{4\,(16807x^3 - 9261x^2 - 13851x + 6561)^3};
$$

and to equations with the differences $(1/7,\ 1/7,\ 2/7)$ and $(1/7,\ 1/7,\ 1/7)$ by composite transformations of degree 18 and 24 respectively. A hypergeometric equation with the exponent differences $(1/2,\ 1/3,\ 1/8)$ can be transformed to an equation with the differences $(1/3,\ 1/3,\ 1/8)$ by the degree 10 transformation

$$
\widetilde{\varphi}_{10}(x) = \frac{4\,x\,(x-1)\,(8\beta x + 7 - 4\beta)^8}{(2048\beta x^3 - 3072\beta x^2 - 3264x^2 + 912\beta x + 3264x + 56\beta - 17)^3} \quad \text{where } \beta^2 + 2 = 0;
$$

and to an equation with the exponent differences $(1/4,\ 1/8,\ 1/8)$ by a composite degree 12 transformation. There is also a composite degree 12 transformation between hypergeometric equations with local exponent differences (1/2, 1/3, 1/9) and (1/9, 1/9, 1/9); and the indecomposable degree 6 transformation

$$\varphi_6(x) = \frac{4i\,x\,(x-1)\,(4x-2-11i)^4}{(8x-4+3i)^5}$$

between hypergeometric equations with the local exponent differences (1/2, 1/4, 1/5) and (1/4, 1/4, 1/5).

References

[AAR99] G.E. Andrews, R. Askey, and R. Roy. *Special Functions.* Cambridge Univ. Press, Cambridge, 1999.

[Beu02] F. Beukers. Gauss' hypergeometric function. Technical report, Utrecht University, http://www.math.uu.nl/people/beukers/MRIcourse93.ps, 2002.

[Erd53] A. Erdélyi, editor. *Higher Transcendental Functions*, volume I. McGraw-Hill Book Company, New-York, 1953.

[Gou81] E. Goursat. Sur l'équation différentielle linéaire qui adment pour intégrale la série hypergéométrique. *Ann. Sci. École Noprm. Sup.(2)*, 10:S3–S142, 1881.

[Har77] R. Hartshorne. *Algebraic Geometry.* Springer Verlag, New York, 1977.

[Hod18] J. Hodgkinson. An application of conformal representation to certain hypergeometric series. *Proc. London Math. Soc.(2)*, 17:17–24, 1918.

[Kle78] F. Klein. Uber lineare differentialgleichungen II. *Math. Annalen*, 12:167–179, 1878.

[Sil86] J. H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Grad. Texts in Math.* Springer Verlaag, New-York, 1986.

[vdW02] H.A. van der Waall. *Lamé Equations with Finite Monodromy.* PhD thesis, University of Utrecht, 2002.

[Vid03] R. Vidūnas. Transformations of hypergeometric elliptic integrals. Kyushu University, 2003.

[Vid04a] R. Vidūnas. Darboux evaluations of algebraic Gauss hypergometric functions. Kyushu University, 2004.

[Vid04b] R. Vidūnas. Degenerate and dihedral Gauss hypergeometric functions. Kyushu University, 2004.

[Vid04c] R. Vidūnas. Transformations os some Gauss hypergeometric functions. 2004. Submited for the proceedings of the 7th international symposium "Orthogonal Polynomials, Special Functions and Applications", Copenhagen, Denmark, August 18-22, 2003; also available at http://arXiv.org/abs/math.CA/0310436.