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Cyclic and ekpyrotic universes in modified Finsler osculating gravity on tangent Lorentz bundles

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Abstract
We consider models of an accelerating Universe elaborated for Finsler-like gravity theories constructed on tangent bundles to Lorentz manifolds. In the osculating approximation, certain locally anisotropic configurations are similar to those for $f(R)$ gravity. This allows us to generalize a proposal by Nojiri *et al* (2011 *AIP Conf. Proc.* **1458** 207–21) in order to reconstruct and compare two classes of Einstein–Finsler gravity (EFG) and $f(R)$ gravity theories using modern cosmological data and realistic physical scenarios. We conclude that EFG provides inflation, acceleration and little rip evolution scenarios with realistic alternatives to standard ΛCDM cosmology. The approach is based on a proof that there is a general decoupling property of gravitational field equations in EFG and modified theories which allows us to generate off-diagonal cosmological solutions.

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1. Introduction and preliminaries

Modern cosmology is based on observational data for two accelerating periods in the evolution of the Universe: the early-time inflation phase and the late-time acceleration with dark energy and dark matter effects. Such accelerating epochs are characterized by a number of differences with large and, respectively, small values of curvature, quite exotic states of matter at the 'beginning' and 'end', alternative classes of solutions of the gravitational and matter fields equations, with singularities and possible anisotropies etc. One of the main exploited ideas is that the Universe is under cyclic evolution with oscillating equations of matter [1–4]. It is considered that there may exist a unified theory which describes in different limits both the inflation and dark (energy/matter) periods.

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Modified gravity theories were elaborated as unifications and/or generalizations to Einstein gravity when cyclic cosmology with inflation and dark energy may be realized (see reviews [5–7]). The approaches with modified Lagrange density $R \to f(R, T, \ldots)$, where R is a scalar curvature and a functional $f(\ldots)$ is determined by traces of certain stress–energy tensors, additional torsion fields etc, are intensively developed in the modern literature. Nevertheless, there are other directions which at first site are alternative to $f(\ldots)$ gravity and present interest in modern classical and quantum gravity and cosmology. In this work, the so-called Finsler gravity and cosmology (for review of results and critical remarks, see [8–11]) are studied and possible connections to $f(R)$ theories are investigated. Perhaps, the first model with locally anisotropic inflation in a Finsler-like manner was proposed in [12]. Several Finsler cosmology and gravity models were developed in [13–22] following geometric and physical ideas related to quantum gravity and modified dispersion relations, broken local Lorentz symmetry, nonlinear symmetries etc.

Our purpose is to prove that a class of Einstein–Finsler gravity (EFG) theories can be considered as natural candidates for which cyclic cosmology with inflation and dark energy/matter epoches can be realized[4]. We shall formulate a reconstruction procedure where alternatively to $f(R)$ gravity [23] we can extract information on certain locally anisotropic (Finsler) like gravitational models. It should be noted that the general relativity (GR) theory and modifications can be re-written in the so-called Finsler-like variables which allow us to decouple and integrate the gravitational and matter field equations in very general forms. This is possible for nonholonomic/non-integrable $2+2$ splitting of a (pseudo)Riemannian manifold enabled with a formal fibred structure. We can mimic a (pseudo)Finsler geometry on a standard Einstein manifold if we adapt the constructions with respect to corresponding nonholonomic (non-integrable) distributions. Such constructions are rather formal but allow us to formulate a general geometric method of constructing exact solutions both in GR and $f(R, T, \ldots)$ modifications, when the generic off-diagonal metrics and various types of connections and frame variables depend on all coordinates via generating and integration functions and parameters.

Let us summarize some key ideas on modifications/generalizations of the Einstein gravity on (co)tangent bundles. In such theories, the fundamental geometric objects (metrics, frames and connections) depend on velocity/momentum-type variables which can be interpreted as fibre-like coordinates. The geometric constructions are derived from a nonlinear quadratic element $ds^2 = F^2(x, y)$, where $x = (x^i)$ and $y = (y^a)$ are coordinates on a tangent bundle TV to a Lorentz manifold $\mathbf{V}$. Such base manifolds are necessary if we want to obtain in a limit the standard GR theory[5]. The value F is called the fundamental/generating Finsler function. Usually, certain homogeneity conditions on y are imposed and the nondegenerated Hessian

$$\tilde{g}_{ab} := \frac{1}{2}\frac{\partial^2 F}{\partial y^a \partial y^b} \qquad (1)$$

is considered as the fibre metric. Gravity theories with anisotropies on $T\mathbf{V}$ cannot be determined only by F or $\tilde{g}_{ab}$. This is very different from GR and a class of $f(R, T, \ldots)$ theories (on pseudo-Riemannian spaces) when the geometric and physical models are completely defined by the metric structure data. To construct a Finsler geometry/gravity theory, we need a triple $(F : \mathbf{g}, \mathbf{N}, \mathbf{D})$ of fundamental geometric objects: the total metric structure, $\mathbf{g}$, the nonlinear connection (N-connection) structure, $\mathbf{N}$, and the distinguished connection structure, $\mathbf{D}$, which

[4] An EFG model is constructed on a manifold V, or its tangent bundle TV, similar to the GR theory when, roughly speaking, the Levi–Civita connection ∇ is substituted by a Finsler-like metric compatible connection $\mathbf{D}$, both defined by the same metric structure $\mathbf{g}$. The second linear connection $\mathbf{D}$ is with nontrivial torsion.

[5] The well-known pseudo-Riemannian geometry consists of a particular case with a quadratic element $ds^2 = g_{ij}(x)\,dx^i dx^j$, when $y^i \sim dx^i$.

is adapted to $\mathbf{N}$. There are necessary additional assumptions how such objects are defined by F, which types of compatible or noncompatible linear connections are involved, and how the corresponding curvature, torsion and nonmetricity fields must be computed. For realistic physical models, the experimental/observational effects can be analysed via the so-called osculating approximation (see examples in [13, 14])

$$\tilde{g}_{ij} = \tilde{g}_{ij}(x, y(x)) = \frac{1}{2}\frac{\partial^2 F}{\partial y^a \partial y^b}(x, y(x)). \tag{2}$$

For a review of Finsler geometry for physicists, see [8, 11] and references therein.

The paper is structured as follows. In section 2, we summarize the necessary results on Einstein–Finsler and $f(R)$-modified gravity and show how such theories can be modelled by nonholonomic distributions and/or off-diagonal metrics in GR. Section 3 is devoted to models of conformal cyclic universes in EFG. We consider FLRW metrics subjected to nonholonomic constraints and analyse the consequences of locally anisotropic models. A procedure of reconstructing EFG theories is provided. Several models with effective anisotropic fluids are studied. Scenarios of ekpyrotic and little rip cosmology governed by fundamental Finsler functions in osculating approximation are studied in section 4. The nonholonomic deformation method is applied for generating cosmological solutions in EFG. Conclusions are drawn in section 5.

2. Canonical EFG and $f(R)$ modifications

We consider a four-dimensional (4D) Lorentz manifold $\mathbf{V}$ in GR modelled as a pseudo-Riemannian space enabled with a metric $^{h}\mathrm{g} = \{g_{ij}(x^j)\}$ of local signature $(+,+,+,-)$. Physically motivated Finsler generalizations to metrics and other geometric objects depending anisotropically on velocity/momentum-type variables y^a can be constructed on a tangent bundle $T\mathbf{V}$. A (pseudo)Finsler geometry is characterized by its fundamental (equivalently, generating) functions when certain homogeneity conditions are imposed, $F(x^i, \beta y^j) = \beta F(x^i, y^j)$, for any $\beta > 0$, and det $|\tilde{g}_{ab}| \neq 0$; see Hessian (1) which defines the so-called vertical metric. In standard approaches, the matrix $\tilde{g}_{ab}$ is considered positively definite, but this condition has to be dropped in the (pseudo)Finsler geometry (hereafter, we shall omit the term 'pseudo').

2.1. The 'triple' of fundamental geometric objects

A canonical model of the Finsler–Cartan geometry is completely determined by F, and $\tilde{g}_{ij}$, up to necessary classes of frame/coordinate transform $e^{\alpha'} = e^{\alpha'}_{\ \alpha}(x, y)e^{\alpha}$, following such assumptions on the triple $(F : \mathbf{N}, \mathbf{g}, \mathbf{D})$ of fundamental geometric objects (see details in [8, 11]).

There is a canonical nonlinear connection (N-connection) structure,

$$\mathbf{N} : T\mathbf{TV} = h\mathbf{TV} \oplus v\mathbf{TV}, \tag{3}$$

which can be introduced as a nonholonomic (non-integrable/anholonomic) distribution with horizontal (h) and vertical (v) splitting[6]. We obtain an integrable/holonomic frame

[6] To define the canonical N-connection, we follow a geometric/variational principle for an effective regular Lagrangian $L = F^2$ and action $S(\tau) = \int_0^1 L(x(\tau), y(\tau))\,\mathrm{d}\tau$, for $y^k(\tau) = \mathrm{d}x^k(\tau)/\mathrm{d}\tau$. The Euler–Lagrange equations $\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = 0$ are equivalent to the 'nonlinear geodesic' (equivalently, semi-spray) equations $\frac{\mathrm{d}^2 x^k}{\mathrm{d}\tau^2} + 2\tilde{G}^k(x, y) = 0$. The canonical coefficients $\tilde{\mathbf{N}} = \{\tilde{N}^a_j\}$ are computed as $\tilde{N}^a_j := \frac{\partial \tilde{G}^a(x,y)}{\partial y^j}$, $\tilde{G}^k = \frac{1}{4}\tilde{g}^{kj}\left(y^i \frac{\partial^2 L}{\partial y^j \partial x^i} - \frac{\partial L}{\partial x^j}\right)$. In our works, we put 'tilde' on symbols (for instance, $\tilde{\mathbf{e}}_\alpha$) if the constructions are performed with respect to bases determined by $\tilde{N}^a_j$. 'Tilde' will be omitted for general N-adapted h–v-splitting when variables mix with each other.

configuration if $W^{\gamma}_{\alpha\beta} = 0$. Under frame transforms, the coefficient formulas transform into equivalent ones for arbitrary sets of coefficients $\mathbf{N} = \{\mathbf{N}^{a'}_{i'} = e^{a'}_{\ a} e_{i'}^{\ i} \tilde{N}^{a}_{i}\}$. We can adapt the geometric constructions via 'N-elongated' (co)frame structures,

$$e_{\nu} = (e_i, e_a), \quad e_i = \frac{\partial}{\partial x^i} - N^a_i(u)\frac{\partial}{\partial y^a} \quad \text{and} \quad e_a = \frac{\partial}{\partial y^a}; \tag{4}$$

$$e^{\mu} = (e^i, e^a), \ e^i = \mathrm{d}x^i \quad \text{and} \quad e^a = \mathrm{d}y^a + N^a_i(u)\,\mathrm{d}x^i. \tag{5}$$

In general, such frames are nonholonomic because

$$[e_{\alpha}, e_{\beta}] = e_{\alpha}e_{\beta} - e_{\beta}e_{\alpha} = W^{\gamma}_{\alpha\beta}\mathbf{w}_{\gamma}, \tag{6}$$

with $W^{b}_{ia} = \partial_a N^b_i$ and $W^{a}_{ji} = \Omega^{a}_{ij} = \frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b}$.

A canonical metric structure on TV can be introduced using data (g_{ij}, e_{α}), with $e_{\alpha'} = e^{\alpha}_{\ \alpha'} e_{\alpha}$ and $\mathbf{g}_{\alpha'\beta'} = e^{\alpha}_{\ \alpha'} e^{\beta}_{\ \beta'} \tilde{\mathbf{g}}_{\alpha\beta}$,

$$\mathbf{g} = h\mathbf{g} \oplus v\mathbf{g} = g_{ij}(x,y)\, e^i \otimes e^j + h_{ab}(x,y)\, e^a \otimes e^b. \tag{7}$$

The third fundamental geometric object in a Finsler geometry is the distinguished connections (d-connection) $\mathbf{D} = \{\mathbf{\Gamma}^{\alpha}_{\beta\gamma}\} = (hD, vD)$ which by definition is adapted to the N-connection structure, i.e. preserves the nonholonomic h–v-splitting (3). We can model physically viable 'almost' standard models (see discussions and critical remarks in [8, 9]) for d-connections which are compatible with the metric structure, $\mathbf{D}\mathbf{g} = 0$ and completely defined by data $(\mathbf{g}, \mathbf{N})$. Let us consider how such a canonical d-connection $\mathbf{D}$ can be constructed[7]. By definition, such a connection is with zero h- and v-torsions, $T^{i}_{\ jk} = 0$ and $T^{a}_{bc} = 0$. There are also nontrivial torsion coefficients. In the Finsler geometry, it is also possible to introduce the Levi-Civita connection $\nabla = \{\Gamma^{\alpha}_{\beta\gamma}\}$ in a standard form (as a unique one which is metric compatible and zero torsion), but it does not preserve the N-connection splitting (3) under parallelism. There is a canonical distortion relation

$$\mathbf{D} = \nabla + \mathbf{Z}, \tag{8}$$

where $\mathbf{D}$, ∇ and the distortion tensor $\mathbf{Z}$ are uniquely defined by the same metric structure $\mathbf{g}$.

Finally, we note that there is an important argument to work with $\mathbf{D}$ with respect to N-adapted frames (4) and (5) even in GR. This is possible for a conventional nonholonomic 2+2 spitting. The priority is that $\mathbf{D}$ allows us to decouple and integrate the gravitational field equations in very general forms [20–22] (see section 4). Such solutions define standard Lorentz/Einstein manifolds if we constrain at the end the generating/integration functions in such a form that $\mathbf{Z} = 0$ which means zero torsion. For such configurations, we obtain $\mathbf{D}_{|\mathcal{T}=0} = \nabla$ in N-adapted form, see (8).

[7] Using N-adapted differential forms and the d-connection, 1-form is $\mathbf{\Gamma}^{\alpha}_{\ \beta} = \mathbf{\Gamma}^{\alpha}_{\ \beta\gamma} e^{\gamma}$; we compute the torsion, $\mathcal{T}^{\alpha}$, and curvature 2-forms, $\mathcal{R}^{\alpha}_{\beta}$, $\mathcal{T}^{\alpha} := \mathbf{D}e^{\alpha} = \mathrm{d}e^{\alpha} + \mathbf{\Gamma}^{\alpha}_{\ \beta} \wedge e^{\beta}$, $\mathcal{R}^{\alpha}_{\beta} := \mathbf{D}\mathbf{\Gamma}^{\alpha}_{\ \beta} = \mathrm{d}\mathbf{\Gamma}^{\alpha}_{\ \beta} - \mathbf{\Gamma}^{\gamma}_{\ \beta} \wedge \mathbf{\Gamma}^{\alpha}_{\ \gamma} = \mathbf{R}^{\alpha}_{\ \beta\gamma\delta} e^{\gamma} \wedge e^{\delta}$. The torsion coefficients $\mathbf{T}^{\alpha}_{\ \beta\gamma} = \{T^{i}_{\ jk}, T^{i}_{\ ja}, T^{a}_{\ ji}, T^{a}_{\ bi}, T^{a}_{\ bc}\}$ are $T^{i}_{\ jk} = L^{i}_{\ jk} - L^{i}_{\ kj}$, $T^{i}_{\ ja} = -T^{i}_{\ aj} = C^{i}_{\ ja}$, $T^{a}_{\ ji} = \Omega^{a}_{\ ji}$, $T^{a}_{\ bi} = \frac{\partial N^a_i}{\partial y^b} - L^{a}_{\ bi}$, $T^{a}_{\ bc} = C^{a}_{\ bc} - C^{a}_{\ cb}$; for N-adapted coefficients $\mathbf{R}^{\alpha}_{\ \beta\gamma\delta}$, see [8, 11].In our former works, the symbol $\widehat{\mathbf{D}}$ was used for the canonical d-connection. We shall omit 'hats' if that will not result in ambiguities. The N-adapted coefficients of $\mathbf{D}$ can be computed in the form $\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta} = (L^{i}_{jk}, L^{a}_{bk}, C^{i}_{jc}, C^{a}_{bc})$,

$$L^{i}_{jk} = \tfrac{1}{2} g^{ir}(e_k g_{jr} + e_j g_{kr} - e_r g_{jk}),\ \widehat{C}^{a}_{bc} = \tfrac{1}{2} h^{ad}(e_c h_{bd} + e_c h_{cd} - e_d h_{bc}),$$
$$L^{a}_{bk} = e_b(N^a_k) + \tfrac{1}{2} h^{ac}\left(e_k h_{bc} - h_{dc}\, e_b N^d_k - h_{db}\, e_c N^d_k\right),\ C^{i}_{jc} = \tfrac{1}{2} g^{ik} e_c g_{jk}.$$

2.2. The EFG field equations and $f(R)$ gravity

EFG theory is constructed for a Finsler-like d-connection $\mathbf{D}$ following standard geometric and/or variation rules as in GR. In this work, we shall use only the canonical d-connection. Such models can be constructed, for instance, on a Lorentz manifold $\mathbf{V}$ (considering an 2+2 splitting); in this case, we introduce Finsler-like variables in GR, when $\mathbf{T} = \mathbf{0}$ is considered as nonholonomic constraint for $\mathbf{D} \to \nabla$, or on its tangent bundle TV. In the second case, we can relate the constructions to certain models of Finsler–Cartan gravity (via nonholonomic deformations, we can transform $\mathbf{D}$ into the Cartan connection for Finsler space)[8].

The Ricci tensor, $Ric = \{\mathbf{R}_{\beta\gamma} := \mathbf{R}^{\alpha}_{\ \beta\gamma\alpha}\}$, of $\mathbf{D} = (hD, vD)$, splits into certain h- and v-components $(R_{ij}, R_{ai}, R_{ia}, R_{ab})$. The scalar curvature is

$$ {}^{F}R := \mathbf{g}^{\beta\gamma}\,\mathbf{R}_{\beta\gamma} = g^{ij}R_{ij} + h^{ab}R_{ab} = \hat{R} + \check{R}. \tag{9}$$

The gravitational field equations can be postulated in a geometric form (and/or derived via N-adapted variational calculus),

$$ \mathbf{R}_{\beta\delta} - \tfrac{1}{2}\mathbf{g}_{\beta\delta}\ {}^{F}R = \mathbf{\Upsilon}_{\beta\delta}. \tag{10}$$

Such equations transform into the Einstein ones in GR if $\mathbf{\Upsilon}_{\beta\delta} \to T_{\beta\delta}$ and $\mathbf{D} \to \nabla$. So, we can consider necessary N-adapted variations of actions for scalar, electromagnetic, spinor etc fields in order to derive the certain matter source in (10). The solutions of the field equations in EFG are with nontrivial torsion induced by $(\mathbf{g}, \mathbf{N})$. The Levi–Civita (zero torsion) conditions with respect to N-adapted frames are

$$ L^{c}_{aj} = e_a(N^{c}_{j}), C^{i}_{jb} = 0,\ \ \Omega^{a}_{\ ji} = 0. \tag{11}$$

If such constraints are imposed at the end, after we have constructed certain classes of solutions of (10) on a 4D manifold and for physically motivated sources, then we generate (in general, off-diagonal) solutions in GR.

It is not clear what types of sources $\mathbf{\Upsilon}_{\beta\delta}$ should be considered in EFG models on TV. We note that, in general, such tensors are not symmetric because $\mathbf{R}_{\beta\delta}$ is not symmetric. It reflects the nonholonomic and locally anisotropic character of Finsler gravity theories. For certain toy models, we can approximate $\mathbf{\Upsilon}_{\beta\delta}$ to a cosmological constant with possible locally anisotropic polarizations depending on (x^i, y^a). The fundamental Finsler function $F(x, y)$ is encoded into geometric objects of (10) and solutions of such PDE. In general, we cannot 'see' explicit dependences on $F(x, y)$ because of the principles of generalized covariance and relativity: arbitrary frame/coordinate transforms on $\mathbf{V}$ and TV mix the variables. Fixing a system of local coordinates, we can measure experimentally only an osculating (pseudo) Riemannian metric (2) on h-subspace. Mathematically, we can construct exact solutions of (10) for 8D metrics (7), but to verify possible physical implications in a direct form is only possible for the h-components.

EFG theories on $\mathbf{V}$, or TV, are explicit examples of modified GR. In a different manner, such theories of gravity are constructed for modified Lagrange densities when $R \to f(R, T, \ldots)$; see reviews of results in[5–7]. For instance, the corresponding field equations with $f(R)$ can be written as effective Einstein field equations

$$ R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi\ {}^{ef}G\ {}^{ef}T_{\mu\nu}, \tag{12}$$

for ${}^{ef}T_{\mu\nu} = (\partial_R f)^{-1}\left[\tfrac{1}{2}[f - R\partial_R f]g_{\mu\nu} - (g_{\mu\nu}\nabla^{\alpha}\nabla_{\alpha} - \nabla_{\mu}\nabla_{\nu})\partial_R f\right]$ and ${}^{ef}G = (\partial_R f)^{-1}$.

Using observational/experimental data, we can measure some geometric configurations determined by functional dependences of type $g_{ij}[F, f, \ldots]$. In general, it may not be clear if

[8] We use such conventions on indices $\alpha = (i, a)$: for a $2+2$ splitting on $\mathbf{V}$, in GR, we consider $i, j, \ldots = 1, 2$ and $a, b, \ldots = 3, 4$. On TV, $i, j, \ldots = 1, 2, 3, 4$ and $a, b, \ldots = 5, 6, 7, 8$. Here, we note that formal integrations of gravitational field equations in general forms are possible for splitting of type $2+2+2+2+\cdots$ and/or $3+2+\cdots$.

such a metric is a solution of (10) or (12), i.e. we cannot say exactly what kinds of modifications of GR would result in dark energy/matter effects. If a fundamental Finsler function F is considered as a nonholonomic distribution on a pseudo-Riemannian space V, or TV, then modifications of type $R \to f(R)$ result in $(F;\ {}^{F}R) \to f(R, F) \simeq f(\tilde{R}) \simeq f({}^{F}R)$, see formulas (10) for the Finsler curvature scalar and its h- and v-splitting. In general, such modifications can be performed for any model of the Finsler spacetime geometry with a scalar curvature ${}^{F}R$. A physically realistic theory close to GR and MG can be constructed in the simplest way for the scalar curvature $\tilde{R}$ of the Cartan d-connection. For a prescribed N-connection structure, we can define via nonholonomic deformations (8) functional dependences of type $\tilde{R}(R)$ and, inversely, $R(\tilde{R})$. In general, EFG and $f(\ldots)$ are different modifications of GR described by different Lagrange densities and derived field equations. For certain conditions, we can transform a class of solutions of (10) into (12), and inversely, using frame/coordinate transforms. It is known that a reconstruction technique [23] allows us to recover data for $f(R)$ using observational data from modern cosmology. In this work, we formulate a similar approach for extracting data for EFG theories and speculate on conditions when we can distinguish a Finsler configuration from an f-modification.

3. Conformal cyclic Finsler-like universes

In this section, we show how to construct cycling universes in models with $f(R, F) \simeq f(\tilde{R})$ when the theory is determined by actions of type

$$S = \int d^4x\sqrt{|g_{ij}|}\left[\frac{f(\tilde{R})}{2\kappa^2} + {}^{m}L\right] \quad \text{and/or} \quad S = \int d^4x\sqrt{|g_{ij}|}\left[\frac{\tilde{R}(R)}{2\kappa^2} + {}^{m}L\right], \tag{13}$$

where κ is the gravitational constant, ${}^{m}L$ is the Lagrangian density and the osculating approximation (2) is encoded in $g_{ij}, \tilde{R}$ and (for simplicity, we shall consider models with functional dependence on metric for matter Lagrangians) ${}^{m}L$.

A scalar field χ can be introduced via conformal transforms $\mathbf{g}_{\alpha\beta} \to e^{-\chi(x)}\mathbf{g}_{\alpha\beta}$ with respect to N-elongated frames (4) and (5)[9]. Denoting

$${}_{\chi}\tilde{R} = e^{\chi}\left\{\tilde{R} + 3g^{ij}\left[(\mathbf{D}_i\chi)(\mathbf{D}_j\chi) - \tfrac{1}{2}(e_i\chi)(e_j\chi)\right]\right\},$$

we obtain a new action

$$S = \int d^4x\, e^{-2\chi}\sqrt{|g_{ij}|}\left[\frac{1}{2\kappa^2}f({}_{\chi}\tilde{R}, \chi) + {}^{m}L\right]. \tag{14}$$

Choosing some 'proper' functionals $\mathbf{P}(\chi)$ and $\mathbf{Q}(\chi)$ when the equation

$$\partial_{\chi}\mathbf{P}(\chi)\,\tilde{R} + \partial_{\chi}\mathbf{Q}(\chi) = 0 \tag{15}$$

can be solved as $\chi = \chi(\tilde{R})$, we can express $f(\tilde{R}) = \mathbf{P}[\chi(\tilde{R})]\,\tilde{R} + \mathbf{Q}[\chi(\tilde{R})]$.

For 'pure' Finsler modifications, the functionals (15) are of type $\tilde{R} = \tilde{R}(R)$, determined in a nonholonomic form via distortions (8), where R is the scalar curvature of $\mathbf{V}$. In such cases, actions of type (14) are parameterized in the form

$$S = \int d^4x\, e^{-2\chi}\sqrt{|g_{ij}|}\left[\frac{1}{2\kappa^2}\,\tilde{R}(R, \chi) + {}^{m}L\right], \tag{16}$$

when

$$\tilde{R}(R) = \tilde{\mathbf{P}}[\chi(R)]R + \tilde{\mathbf{Q}}[\chi(R)]. \tag{17}$$

[9] We do not write ϕ for scalar fields as, for instance, in [23] but use χ because in our studies ϕ is considered as a generating function for constructing off-diagonal solutions [20–22] (see also the following section).

We conclude that EFG theory with an osculating approximation is a variant of modified gravity with $f \rightarrow \tilde{R}$. Such theories can be studied by the similar methods if we work with respect to N-adapted frames. In former approaches to $f(R,T)$ modifications of gravity, such 'preferred' bases and geometric constructions were not considered.

3.1. Reconstructing EFG theories

There are three possibilities of introducing locally anisotropic scalar fields in $\tilde{R}$-modified theory: (1) via conformal transforms as in (14); (2) with certain compactification of extra dimensions of models on $T\mathbf{V}$; (3) postulating scalar field Lagrangians with operators $\mathbf{e}_i$ and the canonical d-connection $\mathbf{D}_i$. Models of types (1)–(3) mutually transform from one into another one under frame/coordinate transforms.

Let us introduce a system of local coordinates $(x^1, x^2, x^3 = t, x^4)$, with time-like coordinate t, on $\mathbf{V}$, and the osculating approximation of Finsler type $\tilde{g}_{ij} = \tilde{g}_{ij}(t, y^a(t))$ and $\tilde{N}_i^a = \tilde{N}_i^a(t, y^b(t))$ with 'non-tildes' with respect to arbitrary frames. On h-subspace, we can consider an FLRW ansatz of type (7) when

$$ds^2 = -(e^3)^2 + \tilde{a}^2(t + \widehat{t})[(e^1)^2 + (e^2)^2 + (e^4)^2] + \{\text{unknown } v\text{-components}\}, \qquad (18)$$

with

$$\tilde{a}(t) = \exp[H_0 t + b(t)], \qquad (19)$$

where the constant $H_0 > 0$ and $\widehat{t}$ is the period. We use a tilde on a in order to emphasize that this value contains certain Finsler information. Such a generic off-diagonal metric is considered to result in physical identical scenarios at t and $t + \widehat{t}$. For instance, we can take $b(t) = {}^0b\cos(2\pi t/\widehat{t})$ when the condition ${}^0b < H_0$ results in monotonic expansion. We emphasize that the h-part of the quadratic element on $T\mathbf{V}$ is the same as in the 4D flat cosmology. Nevertheless, such models are with osculating dependence on $t, y^a(t)$ and contain certain information on a possible Finsler spacetime structure with N-elongated partial derivatives of type (4). To show this we consider a locally anisotropic scalar–tensor model. The action is taken following the possibility (3) above,

$$S = \int d^4x\sqrt{|g_{ij}|}\left[\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}\omega(\chi)g^{ij}(e_i\chi)(e_j\chi) - \mathcal{V}(\chi) + {}^mL\right], \qquad (20)$$

for some functionals $\omega(\chi)$ and $\mathcal{V}(\chi)$ on χ. If both values are determined by a single function $\varsigma(\chi)$, then we express

$$\omega(\chi) = -(2/\kappa^2)\partial^2_{\chi\chi}\varsigma(\chi) \text{ and } \mathcal{V}(\chi) = \kappa^{-2}\{3[\partial_t\varsigma(\chi)]^2 + \partial^2_{\chi\chi}\varsigma(\chi)\}$$

for a cosmological solution with the h-part of FLRW type, when $\chi = t$ and the Hubble function $H = \partial_t\varsigma$.[10]

For instance, if we parameterize the cycling factor (19) $\tilde{a}(t) = \exp[\varsigma(\chi)] = \exp[\varsigma(t)]$, we recover a scalar–tensor model with spherical symmetry on $hT\mathbf{V}$ and reproduce the cyclic universe via formulas which are similar to that presented in section II of [23],

$$\omega(\chi) = (2\,{}^0b/\kappa^2)(2\pi\widehat{t}^{-1})^2\cos(2\pi\chi\widehat{t}^{-1}),$$
$$\mathcal{V}(\chi) = \kappa^{-2}\{3[H_0 - (2\pi\,{}^0b\widehat{t}^{-1})\sin(2\pi\chi\widehat{t}^{-1})]^2 - (2\pi\widehat{t}^{-1})^2\cos(2\pi\chi\widehat{t}^{-1})\}.$$

[10] We note that for cosmological models, the rule of N-elongation of partial derivatives (4) for $A(x(t), y(x(t)))$ is

$$\partial_t A \rightarrow e_t A = \partial_t A - N_t^a\frac{\partial A}{\partial y^a},$$

which points to possible contributions from the fibred structure. Considering a scaling factor $a(t)$ in a nontrivial nonholonomic (Finsler) background, we introduce $H(t) := [e_t a(t)]/a(t)$.

It is not possible to say too much on the underlying Finsler structure using such solutions. We can only conclude that a model with h-scalar $\tilde{R}$ may be of cyclic behaviour. In order to extract more information, we should perform a rigorous analysis on how F modifies the GR.

We apply the methods of reconstructing modified gravity theories [5, 6, 23, 24] reformulated for models of type (13) with $\tilde{R}(R)$ and functionals $\tilde{\mathbf{P}}[\chi]R + \tilde{\mathbf{Q}}[\chi]$ (17) and the scaling factor parameterized in the form $\tilde{a}(t) = {}^0a\, e^{\varpi(t)}$, ${}^0a = $ const. The solutions for $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Q}}$ can be found expanding such values in the Fourier series and defining certain recurrent formulas for coefficients expressed in terms of H_0 and ${}^0b$. For simplicity, we provide the equation for $\tilde{\mathbf{P}}$ when the matter can be neglected,

$$\partial_\chi[\varpi(\chi)/\sqrt{\tilde{\mathbf{P}}(\chi)}\,] = -\tfrac{1}{2}[\partial_{\chi\chi}\tilde{\mathbf{P}}(\chi)]/(\sqrt{\tilde{\mathbf{P}}(\chi)}\,)^3.$$

The equation for $\varpi(\chi)$ is

$$\partial_\chi\varpi = -\frac{1}{2}\sqrt{\tilde{\mathbf{P}}}\int d\chi(\partial_{\chi\chi}\tilde{\mathbf{P}})/(\sqrt{|\tilde{\mathbf{P}}|})^3. \tag{21}$$

We can consider a functional $\tilde{\mathbf{P}}$ in (15) for certain values of $\tilde{R}$ as a generating function for solutions of (21). Here, we provide a well-known example with divergences corresponding to $a(t_0) = 0$ which can be identified with a moment of singularity for a big bang or crunch effect. Fixing $\tilde{\mathbf{P}} = P_0\,[\cos(P_1\chi)]^4$, for some constants P_0 and P_1, we express the last equation as

$$\partial_\chi\varpi = \varpi_0\,[\cos(P_1\chi)]^2 + 2P_1\,[\sin(2P_1\chi) - \tan(2P_1\chi)],$$

for an integration constant ϖ_0. The term with $\tan(2P_1\chi)$ appears in divergences, as is well known for cyclic scenarios and ekpyrotic effects. This allows us to reproduce partially various models of modified gravity including EFG. If the Levi–Civita condition $\mathbf{D}_{|\mathbf{T}=0} = \nabla$ is imposed on $\tilde{R}$, then we obtain modifications which are equivalent to that for $f(R)$. In general, the cycling/ekpyrotic models with nontrivial F are derived for certain nonholonomically induced torsion configurations.

3.2. Models with effective anisotropic fluids

To understand what is the difference between a usual $f(R)$ theory and a Finsler-type one with $\tilde{R}(R)$, we have to consider physical equations with N-elongated partial derivatives and certain information of osculating approximation (2). For simplicity, we shall study a model of effective locally anisotropic perfect fluid elaborated as follows. This allows us to write directly generalizations of FLRW equations in the N-adapted form,

$$3H^2 = \kappa^2\tilde{\rho},\ 3H^2 + 2(e_tH) = -\kappa^2\tilde{p}, \tag{22}$$

where the energy–density and pressure of the locally anisotropic perfect fluid are defined in such a way that for $\tilde{R} = \tilde{R}(R)$,

$$\begin{aligned}\tilde{\rho} &= \kappa^{-2}\{(\partial_R\tilde{R})^{-1}\left[\tfrac{1}{2}\tilde{R} + 3He_t(\partial_R\tilde{R})\right] - 3(e_tH)\},\\ \tilde{p} &= -\kappa^{-2}\{(\partial_R\tilde{R})^{-1}\left[\tfrac{1}{2}\tilde{R} + 2He_t(\partial_R\tilde{R}) + e_t(e_t(\partial_R\tilde{R}))\right] + (e_tH)\}.\end{aligned} \tag{23}$$

We use such an N-adapted system of reference when (4) and (5) are different in a form when pressure $\tilde{p}$ of such a 'dark' fluid is the same in all space-like directions. (We remember that 'tilde' is used for certain values encoding contributions from a nontrivial F.) The equations of state (EoS) and their parameters in this Finsler model can be written/computed, respectively, as

$$\tilde{p} = -\tilde{\rho} + {}^1\tilde{p} \quad \text{and} \quad \tilde{w} = \tilde{p}/\tilde{\rho},$$

where

$$^1\tilde{p} := -\kappa^{-2}[4(e_t H) + e_t(e_t \ln|\partial_R \tilde{R}|) + (e_t \ln|\partial_R \tilde{R}|)^2] - H e_t \ln|\partial_R \tilde{R}|] \quad (24)$$

has to be defined from a combination of FLRW equations,

$$2e_t H = \kappa^2 \times \ ^1\tilde{p}(H, e_t H, e_t(e_t H), \ldots). \quad (25)$$

It should be noted that in local coordinates such a system transforms into a very cumbersome combination of functional and partial derivatives on t, with nontrivial N-coefficients, which is quite difficult to be solved in an explicit form for some prescribed values N_i^a. We have to introduce additional frame/coordinate transforms and assumptions on $H(t)$ in order to construct exact solutions for certain locally anisotropic 'cosmic' functions (24).

If the time variable is written as $t(\tilde{R}(R))$, then it is possible to construct solutions of (25) following the approach developed in [25, 23]. For a new variable $\tilde{r}(t, y(t)) = e_t \ln|\partial_R \tilde{R}(t)|$, we transform the last equation into

$$e_t \tilde{r} + \tilde{r}^2 - H\tilde{r} = 2e_t H, \ \text{for } e_t \tilde{r} = \partial_t \tilde{r} - N_i^a(t, y(t))\partial_a \tilde{r}.$$

To reconstruct the Hubble parameter for such locally anisotropic configurations, we can consider typical examples with power-law or oscillating solutions. For instance, we show how we could generate oscillating Finsler configurations. We assume a particular behaviour when $\tilde{r} = r + \ ^1r$, for $r(t)$ being the solution of

$$\partial_t r + r^2 - Hr = 2\partial_t H \quad (26)$$

and a small value ^1r is to be found from (neglecting terms $(\ ^1r)^2$ and $N \times (\ ^1r)$),

$$\partial_t(\ ^1r) + (\ ^1r)(2r - Hr) = n, \quad (27)$$

where $n := N_t^a \partial_a (r - 2H)$. For $n = 0$ and $^1r = 0$, a solution of (26) was found in [23], when for $r(t) = r_0 \cos \omega_0 t$ it was reproduced

$$f(R) = \int dR \ \exp[-r_0 \omega_0^{-1} t(R)].$$

In the case of Finsler modified gravity, the oscillating solutions of (26) and (27) can be expressed in the form $\tilde{r}(t) = r_0 \cos \omega_0 t + r_1 \cos \omega_1 t$. This results in reconstructions of type

$$\tilde{R}(R) = \int dR \exp\left\{-r_0 \omega_0^{-1} \sin[\omega_0 t(R)] - r_1 \omega_1^{-1} \sin[\omega_1 t(R)]\right\}. \quad (28)$$

The function $t(R)$ is an inversion of $R(t) = 12H^2 + 6\partial_t H$.[11] This corresponds to cyclic evolution reproduced both in $f(R)$ and/or EFG gravity theories. The complexity of such solutions does not allow us to obtain explicit forms of $\tilde{R}(R)$ contributions. Nevertheless, we can distinguish from 'pure' $f(R)$ theory and that with a mixed with F, when $f(R, F) \simeq \tilde{R}(R)$: in oscillations of type (28), there is the second term induced by off-diagonal terms summarized in n as a source of (27).

4. Tangent Lorentz bundle cosmology

The goal of this section is to show how using $\tilde{R}(R)$ and related $f(R, F)$ cosmological scenarios on Lorentz manifolds $\mathbf{V}$ we can derive canonical models of Finsler gravity and cosmology on $T\mathbf{V}$. Via Sasaki lifts of metrics (7), we shall construct theories on total bundle spaces.

[11] A similar formula was derived in [26] for a model of osculating Finsler cosmology with weak anisotropy.

4.1. Ekpyrotic and little rip Finsler cosmology

Let us analyse scenarios of ekpyrotic/cyclic Universe [27, 28] in our case derived from Finsler-like modifications of GR. In this subsection, the constructions will be performed on hTV components of tangent Lorentz bundles. Cyclic solutions solve the problems of standard cosmological model and provide a more complete theory. In ekpyrotic models, a scalar field is necessary for reproducing the cycling behaviour and the $f(R)$ gravity admits such phases in time evolution. Because the $\tilde{R}(R)$ locally anisotropic gravity can be modelled as the $f(R)$ theory, it is clear that both types of theories contain cyclic configurations.

It is possible to reconstruct a canonical Finsler-like model with a phantom phase and free of future singularity when it is generated a little rip cosmology similar to [29–31]. We can understand how such models may contain locally anisotropic modifications following arguments. Using the first formula in (22), $3H^2 = \kappa^2\tilde{\rho}$, the effective density function $\tilde{\rho}$ (23) and EoS parameter $\tilde{w} = -1 - 2(e_t H)/3H^2$, we find terms of types $e_t(\partial_R \tilde{R})$ and $e_t H$ with contributions from the N-connection coefficients. Big rip singularities (by definition) occur in a finite time t_s when $a(t)$ and the energy–density diverge. In $f(R)$ models, this can be analysed in local coordinate frames. Finsler modifications $\tilde{R}(R)$ result in N-elongated partial derivatives e_t and nonholonomic frames of reference.

It is difficult to formulate a general condition for $\tilde{\rho}$ and $\tilde{R}(R)$ to avoid divergences. We can consider an enough but not sufficient condition that $\tilde{R}(R) > 0$ but an additional investigation of contributions with N_i^a for a little rip. Using the reconstruction techniques from the previous section, we shall prove that there are Finsler models of general acceleration which do not contain future singularities and drive to a stronger growth in time than evolutions for big rip singularities. We can use the generating function $\tilde{\mathrm{P}}(\chi) = \exp[4\tilde{\beta}\,\mathrm{e}^{\tilde{\alpha}\chi}]$, $\tilde{\alpha} = \text{const}$, $\tilde{\beta} = \text{const}$, resulting in cycling factors of type (19). A similar function can be considered for different constants when we test the 'cycling–accelerating' system for trivial N-coefficients, $\mathrm{P}(\chi) = \exp[4\beta\,\mathrm{e}^{\alpha\chi}]$, $\alpha = \text{const}$, $\beta = \text{const}$. For small times $t \ll \tilde{\alpha}$, de Sitter solutions are reproduced and a Hubble parameter is approximated to a constant. We obtain states with $\tilde{w} < -1$, for a locally anisotropic phantom phase without big rip singularity.

Let us show that a dissolution of bound structures (little rip) induced by a Finsler structure can occur. Using expansions of exponential functions into power series and solutions (15) for certain values of $\tilde{R}$ and/or R as a generating function for solutions of (21), we compute three reconstruction models (see similar details in the beginning of section V of [23] and references therein). The solutions depend on what type of scalar curvature we use, $R(t) = 12H^2 + 6\partial_t H$, or $\tilde{R}(t) = 12H^2 + 6e_t H$, and can be written in the form

$$
\begin{aligned}
f(R) &= \alpha^2\left(c_1 + c_2\sqrt{4R/\alpha^2 + 75}\right)\mathrm{e}^{\sqrt{R/12\alpha^2 + 25/16}} \\
&\sim \kappa_1 R + \kappa_2 R^2/\alpha^2 + \kappa_3 R^3/\alpha^4 + \cdots, \\
\tilde{R}(R) &= \tilde{\alpha}^2\left(c_1 + c_2\sqrt{4R/\tilde{\alpha}^2 + 75}\right)\mathrm{e}^{\sqrt{R/12\tilde{\alpha}^2 + 25/16}} \\
&\sim \tilde{\kappa}_1 R + \tilde{\kappa}_2 R^2/\tilde{\alpha}^2 + \tilde{\kappa}_3 R^3/\tilde{\alpha}^4 + \cdots, \\
f(\tilde{R}) &= \check{\alpha}^2\left(c_1 + c_2\sqrt{4\tilde{R}/\check{\alpha}^2 + 75}\right)\mathrm{e}^{\sqrt{\tilde{R}/12\check{\alpha}^2 + 25/16}} \\
&\sim \check{\kappa}_1 \tilde{R} + \check{\kappa}_2 \tilde{R}^2/\check{\alpha}^2 + \check{\kappa}_3 \tilde{R}^3/\check{\alpha}^4 + \cdots,
\end{aligned}
\tag{29}
$$

where $c_1 = -24\exp(-39/12)$ and $c_2 = 2\sqrt{3}$ are taken in order to obtain the same approximations if $\tilde{R} \to R$ and $f(\ldots) \to$GR; constants of types $\kappa_i, \tilde{\kappa}_i, \ldots$ depend on c_1 and c_1 and respectively on α and $\tilde{\alpha}$ etc.

The above formulas with series decompositions distinguish three possible cycling universes with rip evolution. In all cases, the quadratic terms on curvature allow us to cure the singularities. But approximations are different: in the first case, we can obtain a recovering of

GR form the 'standard' $f(R)$ gravity theory; in the second case, we model Finsler modifications as the $f(\ldots)$ theory without much information on the Finsler generating function F; in the third case, the rip evolution starts from a Finsler modified spacetime. We suppose that such different cosmological scenarios can be verified experimentally. For instance, for a Sun–Earth system with densities of type $\rho = \rho_0 e^{2\alpha t}$, $\tilde{\rho} = \tilde{\rho}_0 e^{2\tilde{\alpha} t}, \ldots$ and $t_0 = 13.7$ Gyr, we can obtain three different approximations for the time of little rip (decoupling) ${}^{\mathrm{LR}}t$,

$$ {}^{\mathrm{LR}}t = 13.7\,\mathrm{Gyr} + 29.9/\alpha, \quad {}^{\mathrm{LR}}\tilde{t} = 13.7\,\mathrm{Gyr} + 29.9/\tilde{\alpha}, \quad {}^{\mathrm{LR}}\check{t} = 13.7\,\mathrm{Gyr} + 29.9/\check{\alpha}. $$

For oscillating solutions of type (28), we obtain possible resonant behaviour and shifting of decoupling etc. In general, models with $f(R)$, $\tilde{R}(R)$, $f(\tilde{R})$ possess different little rip properties.

4.2. Reproducing a canonical model of EFG

A metric compatible Finsler model on $T\mathbf{V}$ can be completely defined by an action

$$ S = \int \mathrm{d}^4x\,\delta^4y \sqrt{|g_{ij}h_{ab}|}\left[\frac{f({}^F R)}{2\kappa^2} + {}^m\tilde{L}\right]. $$

This would result in functional dependences of type ${}^F R = {}^F R(R)$ for a canonical scalar Finsler curvature (9). It is not clear how we could extract information on a generalized gravitational constant(s) and matter–field interactions via ${}^m\tilde{L}$ in extra 'velocity/momentum'-type dimensions. Nevertheless, we can encode such contributions into certain polarized cosmological constants derived for certain very general parameterizations of possible matter interactions with 'velocity/acceleration' variables and their duals. A 'recovering' of Finsler cosmology observational data on $\mathbf{V}$ to $T\mathbf{V}$ can be performed following such a procedure for two distinguished cases.

(1) Certain models of Finsler gravity can be associated with modified dispersion relations[12]

$$ \omega^2 = c^2[g_{\widehat{i}\widehat{j}}k^{\widehat{i}}k^{\widehat{j}}]^2\left(1 - \frac{1}{r}\rho_{\widehat{i}_1\widehat{i}_2\ldots\widehat{i}_{2r}}y^{\widehat{i}_1}\cdots y^{\widehat{i}_{2r}}/[g_{\widehat{i}\widehat{j}}y^{\widehat{i}}y^{\widehat{j}}]^{2r}\right), $$

where a corresponding frequency ω and wave vector k_i are computed locally when the local wave vectors $k_i \rightarrow p_i \sim y^a$ are related to variables p_i which are dual to 'fibre' coordinates y^a. These relations can be associated with a nonlinear quadratic element (see details in [32, 10, 11]),

$$ \mathrm{d}s^2 = F^2(x,y) \approx -(c\,\mathrm{d}t)^2 + g_{\widehat{i}\widehat{j}}(x^k)y^{\widehat{i}}y^{\widehat{j}}\left[1 + \frac{1}{r}\frac{q_{\widehat{i}_1\widehat{i}_2\ldots\widehat{i}_{2r}}(x^k)y^{\widehat{i}_1}\cdots y^{\widehat{i}_{2r}}}{(g_{\widehat{i}\widehat{j}}(x^k)y^{\widehat{i}}y^{\widehat{j}})^r}\right] + O(\rho^2). $$

For physical applications related to 'small' deformations of GR, we can consider that $g_{ij} = (-1, g_{\widehat{i}\widehat{j}}(x^k))$ in the limit $q \rightarrow 0$ correspond to a metric on a (pseudo)Riemannian manifold.

(2) Finsler variables with a generating function $F(x,y)$ can be introduced in GR and $f(R)$ modifications nonholonomic with 2 + 2 splitting. Such an F can be partially recovered in cosmological models using observational data on h-subspace. Understanding possible physical implications of theories on $T\mathbf{V}$ is important to construct exact solutions for locally anisotropic black holes, brane trapping/warping and anisotropic cosmological solutions [22, 19]. The generic off-diagonal metrics, in both cases 1 and 2, are for Sasaki lifts (7).

[12] For $y^i = \mathrm{d}x^i/\mathrm{d}\tau$, when $x^i(\tau)$ is for a real parameter τ; $\rho_{\widehat{i}_1\widehat{i}_2\ldots\widehat{i}_{2r}}(x)$ are parameterized by 3D space coordinates with 'hats' on indices running values $\widehat{i} = 1, 2, 3$.

On tangent bundles, all fields depend on coordinates $u^\alpha = (x^i, y^a)$ and the scalar curvature for the canonical d-connection $\mathbf{D}$ transforms into $\tilde{R} \to {}^F R = \tilde{R} + \check{R}$ (9). Having recovered the spacetime metric $hg = \{ g_{ij}(t) = g_{ij}(t, y(t))\}$ for a cosmological model, we can construct up to frame transform a metric (7),

$$\mathbf{g} = h\mathbf{g} \oplus v\mathbf{g} = g_{ij}(t)\, e^i \otimes e^j + h_{ab}(x, y)\, \mathbf{e}^a \otimes \mathbf{e}^b, \qquad (30)$$

for certain coefficients $h_{ab} = e^{a'}_{a}(x, y)e^{b'}_{b}(x, y)g_{a'b'}(t)$ and $\mathbf{e}^a$ being determined by a canonical N-connection $\tilde{N}^a_i(x, y)$ induced via a chosen $F(x, y)$, see footnote 5. For cosmological models, the coefficients of v-metric can be transformed via frame transform to $h_{ab}(t, y)$. In general, such models are inhomogeneous on fibre coordinates y^a. Extending the scalar fields $\chi(x^i) \to \chi(x^i, y^a)$, we can construct values of type (17), ${}^F\mathbf{P}[\chi({}^F R)] > 0$ and ${}^F\mathbf{Q}[\chi({}^F R)]$, when ${}^F R(R) = {}^F\mathbf{P}[\chi(R)]R + {}^F\mathbf{Q}[\chi(R)]$.

Actions of type (20) can be generalized in the form

$$S = \int \mathrm{d}^4x\delta^4y\sqrt{|g_{ij}||h_{ab}|}\left[\frac{1}{2\kappa^2}\, {}^F R(R, \chi) + {}^m L\right],$$

where $\delta y^a = \mathbf{e}^a$ (5). We do not consider a factor $\mathrm{e}^{-2\chi}$ in this action because we shall work with another type of conformal transform when the action is re-defined in Finsler generalized Einstein frames in order to remove strong coupling. On TV, a quintessence locally anisotropic action is postulated in the form

$$S = \int \mathrm{d}^4x\delta^4y\sqrt{|\widehat{g}_{ij}||\widehat{h}_{ab}|}\left[{}^F R - \frac{1}{2}\omega(\chi)(\widehat{\mathbf{D}}_\alpha\chi)(\widehat{\mathbf{D}}^\alpha\chi) - U(\chi)\right],$$

where $\widehat{\mathbf{g}}_{\alpha\beta} = {}^F\mathbf{P}(\chi)\mathbf{g}_{\alpha\beta}$, $\omega = 12\left(\partial_\chi\sqrt{{}^F\mathbf{P}}\right)^2 / {}^F\mathbf{P}$, $U = {}^F\mathbf{Q}/({}^F\mathbf{P})$ and $\widehat{\mathbf{D}}$ is the conformal transform of $\mathbf{D}$. For cosmological models, (in our case, Finsler analogues) the Jordan frames with

$$\widehat{a}(\widehat{t}) = \sqrt{{}^F\mathbf{P}(\chi(t))}a(t), \quad \text{for} \quad \mathrm{d}\widehat{t} = \sqrt{{}^F\mathbf{P}(\chi(t))}\,\mathrm{d}t,$$

and locally anisotropic configurations $\chi(t, y^a)$ are used where the fibre coordinates y^a can be parameterized $y^a(t)$. In EFG, the evolutions with $\widehat{a}(\widehat{t})$ and $\widehat{a}(t)$ are with respect to N-adapted frames (4) and (5).

We reproduce cycling universes with a rip evolution of type (29) prescribing[13]

$$\omega(\chi) = 4\widehat{\alpha}\widehat{\beta}^2\,\mathrm{e}^{2\widehat{\alpha}\chi} \quad \text{and} \quad U(\chi) = -6\widehat{\alpha}^2(3 + 4\widehat{\beta}\,\mathrm{e}^{\widehat{\alpha}\chi})(3 + 8\widehat{\beta}\,\mathrm{e}^{\widehat{\alpha}\chi})\exp(-4\widehat{\beta}\,\mathrm{e}^{\widehat{\alpha}\chi}).$$

Keeping only terms with t-evolution and $\mathrm{e}^{\widehat{\alpha}t} \sim 1 + \widehat{\alpha}t + O(t^2)$ for small t, the solutions for scaling factors are respectively constructed

$$\widehat{a}(t) = a_0 e[6(\widehat{\beta}\,\mathrm{e}^{\widehat{\alpha}\chi} + \widehat{\alpha}t)], \quad \text{for } a_0 = \text{const} \quad \text{and} \quad \widehat{t} = \int_{-\infty}^{2\widehat{\beta}\exp[\widehat{\alpha}\chi]} \mathrm{e}^z/z;$$

$$\widehat{a}(\widehat{t}) = \widehat{a}_0\widehat{t}^6\exp[6\widehat{\alpha}\widehat{\beta}\,\mathrm{e}^{-2\widehat{\beta}}\widehat{t}], \quad \text{for } \widehat{a}_0 = \text{const} \quad \text{and} \quad \widehat{H}(\widehat{t}) = \widehat{\alpha}\widehat{\beta}\,\mathrm{e}^{-2\widehat{\beta}} + 6/\widehat{t}. \qquad (31)$$

Exact solutions in EFG with such scaling factors will be constructed in the following section. Here we observe that the Universe with Einstein–Finsler frames describe both a type of (initial) big bang singularity and a super-accelerating evolution. This is a manifestation of scenarios with little rip determined by possible locally anisotropic character of gravitational interactions on TV. It seems that singularities can be removed in Jordan frames adapted to N-connections. We conclude that Finsler configurations extended on tangent Lorentz bundles may result in the dissolution of bound structures of certain classes of FLRW models originally defined in GR and then extended to EFG.

[13] We extend on TV similar formulas (59) in [23]; the coefficients are redefined with 'hats' and fixing in such a form on h-subspaces we obtain little rips as in $f(R)$; nevertheless, the parametric dependence is modified on velocity-like coordinates and parameters for EFG.

4.3. *Cosmological solutions in Finsler gravity*

On TV of a Lorentz manifold $\mathbf{V}$, the Einstein–Finsler equations (10) for the canonical d-connection $\mathbf{D}$ are for an 8D spacetime endowed with a nontrivial N-connection structure. Such systems of nonlinear partial derivative equations (PDEs) can be solved in very general forms using the anholonomic frame method [22, 32, 21]. Locally anisotropic and Finsler-like solutions in 4D models of gravity were constructed in [11, 19, 12]. In this subsection, we provide several examples of 8D exact solutions which possess cyclic/ekpyrotic and little rip properties.

4.3.1. Decoupling of EFG cosmological equations. We label local coordinates $x^i = (x^1 = r, x^2 = \theta), y^{a_1} = (y^3 = t, y^4 = \varphi)$; $y^a = (y^{a_2}, y^{a_3})$, where indices run respective values $a_1 = 3, 4$; $a_2 = 5, 6$; $a_3 = 7, 8$. We consider (x^i, y^{a_1}) as coordinates on a 4D Lorentz manifold $\mathbf{V}$, and the coordinates y^a as fibre coordinates in $T\mathbf{V}$. This reflects a conventional $2+2+2+2$ splitting of coordinates which will give us the possibility of integrating the gravitational field equations 'shell by shell' increasing dimensions by 2. The FLRW cosmological solution can be written in the form

$$h\mathbf{g} = a^2(t)\left(\frac{\mathrm{d}r \otimes \mathrm{d}r}{1-\kappa r^2} + r^2\,\mathrm{d}\theta \otimes \mathrm{d}\theta\right) - \mathrm{d}t \otimes \mathrm{d}t + a^2(t) r^2 \sin^2\theta\,\mathrm{d}\varphi \otimes \mathrm{d}\varphi, \qquad (32)$$

with $\sigma = 0, \pm 1$. This metric is an exact solution of the Einstein equations with a perfect fluid stress–energy tensor, $T^{\alpha}{}_{\beta} = \mathrm{diag}[-p, -p, \rho, -p]$, where ρ and p are the proper energy density and pressure in the fluid rest frame. For simplicity, we consider as a 'prime' an h-metric ansatz containing a conformal transform of (32), multiplying on a^{-2}, with $\sigma = 0$, and in Cartesian coordinates, $h\mathring{\mathbf{g}} = \mathring{g}_i\,\mathrm{d}x^i \otimes \mathrm{d}x^i + \mathring{h}_a\,\mathrm{d}y^a \otimes \mathrm{d}y^a$, where $\mathring{g}_i = 1, \mathring{h}_3(y^3) = -a^{-2}(t), \mathring{h}_4 = 1$. To construct 8D cosmological Finsler-like solutions, we shall use the ansatz

$$\begin{aligned}
\mathbf{g} &= \eta_i(x^j)\mathring{g}_i\,\mathrm{d}x^i \otimes \mathrm{d}x^i + \eta_{a_1}(x^j, t)\mathring{h}_{a_1} e^{a_1} \otimes e^{a_1} \\
&\quad + h_{a_2}(x^k, t, y^5) e^{a_2} \otimes e^{a_2} + h_{a_3}(x^k, t, y^{a_2}, y^7) e^{a_3} \otimes e^{a_3}, \\
e^{a_1} &= \mathrm{d}y^{a_1} + N_i^{a_1}(x^k, t)\,\mathrm{d}x^i, \\
e^{a_2} &= \mathrm{d}y^{a_2} + N_i^{a_2}(x^k, t, y^5)\,\mathrm{d}x^i + N_{a_1}^{a_2}(x^k, t, y^5)\,\mathrm{d}y^{a_1}, \\
e^{a_3} &= \mathrm{d}y^{a_3} + N_i^{a_3}(x^k, t, y^{a_2}, y^7)\,\mathrm{d}x^i + N_{a_1}^{a_3}(x^k, t, y^{a_2}, y^7)\,\mathrm{d}y^{a_1} + N_{a_2}^{a_3}(x^k, t, y^{a_2}, y^7)\,\mathrm{d}y^{a_2},
\end{aligned} \qquad (33)$$

where $h_{a_1} = \eta_{a_1}(t)\mathring{h}_{a_1}$ are defined by polarization functions $\eta_{a_1} = 1 + \varepsilon\chi_{a_1}(x^i, y^3)$, $\varepsilon < 1$. This class of generic off-diagonal metrics depends on time-like coordinate, t, and on fibre-like ones, y^{a_2} and y^7, and possesses a Killing symmetry on $\partial/\partial y^7$ because the coefficients do not contain the coordinate y^7. In Cartesian coordinates, the h-part of ansatz (33) is modified by $\eta_i = 1 + \varepsilon\chi_i(x^j)$, when $g_i = \eta_i\mathring{g}_i = \mathrm{e}^{\psi(x^j)}$.[14]

We do not have explicit observational/experimental pieces of evidence to determine what kinds of sources $\Upsilon_{\beta\delta}$ may have physical importance for models of matter–field interactions in the total space $T\mathbf{V}$. From a formal point of view, we can extend a geometric/variational formalism for deriving energy–momentum tensors on $\mathbf{V}$ (for instance, for scalar, spinor, gauge etc fields) to construct similar values using Sasaki lifts for metrics and adapting the constructions to N-elongated frames (4) and (5). For simplicity, we shall approximate such possible 'extra velocity' contributions by matter interactions with an effective cosmological constant Λ when $\Upsilon^{\alpha}_{\beta} = \Lambda\delta^{\alpha}_{\beta}$. Then, we shall see what kinds of 8D generic off-diagonal

[14] We note that it is possible to construct more general classes of solutions without Killing symmetries, nonhomogeneous cosmological metrics etc. Such an ansatz is a natural one when on the 4D base spacetimes the cosmological metrics depend only on t and can be modified by 3D space coordinates and/or several fibre-type coordinates.

cosmological solutions would possess cyclic, ekpyrotic and/or little rip properties in the h-part of the total metric. As a matter of principle, such an effective cosmological constant can be 'polarized' and depends on base and fibre coordinates. Nevertheless, it can be transformed into a constant value using re-definition of generating functions for various classes of solutions as we shall prove below. Certain mechanisms of Finsler brane trapping/warping of extra 'velocity' type coordinates are also possible when N-adapted geometric constructions are considered [22, 32].

The gravitational field equations (10) for the ansatz (33) decouple in this form.

In 4D cosmology, we obtain a system of nonlinear PDEs which with respect to N-adapted frames is written as

$$\partial_{11}^2\psi + \partial_{22}^2\psi = \Lambda, \tag{34}$$

$$\phi^*(\ln|h_4|)^* = \Lambda h_3, \tag{35}$$

$$\beta N_i^3 + \alpha_i = 0, \tag{36}$$

$$\left(N_i^4\right)^{**} + \gamma\left(N_i^4\right)^* = 0, \tag{37}$$

with $\phi^* = \partial_t\phi$, $\partial_1 = \partial_{x^1}$, $\partial_{11}^2 = \partial^2_{x^1x^1}$, where the coefficients

$$\gamma = (\ln|h_4|^{3/2} - \ln|h_3|)^*, \ \alpha_i = h_4^*\partial_i\phi \quad \text{and} \quad \beta = h_4^*\phi^*, \tag{38}$$

are determined by h_3 and h_4 via the generating function

$$\phi(x^i, t) = \ln|2(\ln\sqrt{|h_4|})^*| - \ln\sqrt{|h_3|}. \tag{39}$$

The equations on the first 2D shell (with coordinates y^5 and y^6) are

$$(\partial_5\, {}^1\phi)\partial_5(\ln|h_6|) = \Lambda h_5, \tag{40}$$

$${}^1\beta N_i^5 + {}^1\alpha_i = 0, \ {}^1\beta N_{a_1}^5 + {}^1\alpha_{a_1} = 0, \tag{41}$$

$$\partial_{55}^2\left(N_i^6\right) + {}^1\gamma\partial_5\left(N_i^6\right) = 0, \ \partial_{55}^2\left(N_{a_1}^6\right) + {}^1\gamma\partial_5\left(N_{a_1}^6\right) = 0, \tag{42}$$

for coefficients

$$\begin{aligned} &{}^1\gamma = \partial_5(\ln|h_6|^{3/2} - \ln|h_5|), \ {}^1\alpha_i = (\partial_5 h_6)(\partial_i\, {}^1\phi), \\ &{}^1\alpha_{a_1} = (\partial_5 h_6)(\partial_{a_1}{}^1\phi), \ {}^1\beta = (\partial_5 h_6)(\partial_5\, {}^1\phi), \end{aligned} \tag{43}$$

determined by h_5 and h_6 via

$${}^1\phi(x^k, t, y^5) = \ln|2\partial_5(\ln\sqrt{|h_6|})| - \ln\sqrt{|h_5|}. \tag{44}$$

In a similar form, using coefficients h_7 and h_8 (and third 'shell coordinates' y^7 and y^6) we obtain

$$(\partial_7\, {}^2\phi)\partial_7(\ln|h_8|) = \Lambda h_7, \tag{45}$$

$${}^2\beta N_i^7 + {}^2\alpha_i = 0, \ {}^2\beta N_{a_1}^7 + {}^2\alpha_{a_1} = 0, \ {}^2\beta N_{a_2}^7 + {}^2\alpha_{a_2} = 0, \tag{46}$$

$$\begin{aligned} &\partial_{77}^2\left(N_i^8\right) + {}^2\gamma\partial_7\left(N_i^8\right) = 0, \ \partial_{77}^2\left(N_{a_1}^8\right) + {}^2\gamma\partial_7\left(N_{a_1}^8\right) = 0, \\ &\partial_{77}^2\left(N_{a_2}^8\right) + {}^2\gamma\partial_7\left(N_{a_2}^8\right) = 0, \end{aligned} \tag{47}$$

for coefficients

$$\begin{aligned} &{}^2\gamma = \partial_7(\ln|h_8|^{3/2} - \ln|h_7|), \ {}^2\alpha_i = (\partial_7 h_8)(\partial_i\, {}^2\phi), \\ &{}^2\alpha_{a_1} = (\partial_7 h_8)(\partial_{a_1}{}^2\phi), \ {}^2\alpha_{a_2} = (\partial_7 h_8)(\partial_{a_2}{}^2\phi), \ {}^2\beta = (\partial_7 h_8)(\partial_7\, {}^2\phi), \end{aligned} \tag{48}$$

generated by

$${}^2\phi(x^k, t, y^{a_1}, y^7) = \ln|2\partial_7(\ln\sqrt{|h_8|})| - \ln\sqrt{|h_7|}. \tag{49}$$

We can see that the system of equations (35)–(37) is similar, respectively, to (40)–(42) and (45)–(47). Such equations can be integrated consequently by adding additional dependences on next shell coordinates.

4.3.2. Generating off-diagonal cosmological Finsler solutions. The h-metric is given by $e^{\psi(x^k)} dx^i \otimes dx^j$, where $\psi(x^k)$ is a solution of (34) considered as a 2D Laplace equation (34). It depends on the effective cosmological constant Λ.

We can integrate the system (35) and (39), for $\phi^* \neq 0$. Such a condition can be satisfied by choosing a corresponding system of frames/coordinates; it is possible to construct solutions choosing ϕ with $\phi^* = 0$, as particular cases (for simplicity, we omit such considerations in this paper). Defining $A := (\ln|h_4|)^*$ and $B = \sqrt{|h_3|}$, we re-write

$$\phi^* A = \Lambda B^2, \; B e^{\phi} = 2A. \tag{50}$$

If $B \neq 0$, then we obtain $B = (e^{\phi})^*/2\Lambda$ as a solution of a system of quadratic algebraic equations. This formula can be integrated on dt which results in

$$h_3 = {}^0h_3(1 + (e^{\phi})^*/2\Lambda\sqrt{|{}^0h_3|})^2. \tag{51}$$

Introducing this h_3 in (50) and integrating on t, we obtain the coefficients

$$h_4 = {}^0h_4 \exp\left[\frac{e^{2\phi}}{8\Lambda}\right], \tag{52}$$

for an integration function ${}^0h_4 = {}^0h_4(x^k)$. We can fix ${}^0h_a = \mathring{h}_a$ as in (33).

Having defined h_a we can compute the N-connection coefficients as solutions of (36) and (integrating two times on t) (37),

$$w_i = -\partial_i\phi/\phi^*,$$
$$n_k = {}^1n_k + {}^2n_k \int dt\, h_3/(\sqrt{|h_4|})^3, \tag{53}$$

for integration functions ${}^1n_k(x^i)$ and ${}^2n_k(x^i)$.

Introducing solutions (51), (52) and (53) for the h-metric of ansatz (33), we obtain a quadratic element for nonhomogeneous 4D cosmologies,

$$ds^2 = e^{\psi}(dx^i)^2 + \mathring{h}_3\left(1 + \frac{(e^{\phi})^*}{2\Lambda\sqrt{|\mathring{h}_3|}}\right)^2\left[dt - \frac{\partial_i\phi}{\phi^*}dx^i\right]^2$$
$$+ \mathring{h}_4 \exp\left[\frac{e^{2\phi}}{8\Lambda}\right]\left[dy^4 + \left({}^1n_k + {}^2n_k\int dt \frac{h_3}{(\sqrt{|h_4|})^3}\right)dx^i\right]^2. \tag{54}$$

The solutions depend on generating functions $\phi(x^i, t)$ and $\psi(x^k)$ and on integration functions ${}^1n_k(x^k)$ and ${}^2n_k(x^k)$. We approach the FLRW metric (32) if we choose values such that $\eta_i = 1 + \varepsilon\chi_i(x^i) \to 1$ and $\eta_{a_1} = 1 + \varepsilon\chi_{a_1}(x^i, y^3) \to 1$ and the N-connection coefficients vanish[15]. This class of modified Finsler-like spacetimes (effectively modelled for 2 + 2 decompositions) is characterized by a nontrivial torsion field determined only the coefficients of metric (and respective N-connection). So, extra-dimensional Finsler contributions can be via off-diagonal extension of solutions, nonlinear polarizations of physical constants and metric coefficients and induced torsion. Such metrics (being generic off-diagonal) cannot be diagonalized via coordinate transforms because the anholonomy coefficients in (6) are not zero for arbitrary generating/functions.

Metric of type (54) can be constrained additionally in order to construct exact solutions for the Levi-Civita connection ∇. We have to consider solutions with ${}^2n_k = 0$, $\partial_i({}^1n_k) = \partial_k({}^1n_i)$ and $w_i = -\partial_i\phi/\phi^*$ and h_4 subjected to

$$w_i^* = e_i \ln|h_4|, \; \partial_i w_j = \partial_j w_i, \; n_i^* = 0;$$

[15] Here we note that the FLRW solution is for perfect fluid stress–energy tensor and not for a cosmological constant. Nevertheless, using generating and integration functions for coupled nonlinear systems we can define limits to necessary type diagonal metrics via corresponding nonholonomic constraints and frame transforms.

see details how to solve such equations in [20, 22, 32, 21] (this is possible, for instance, for separation of variables). Even for solutions with ∇, weobtain only effective Einstein spaces with locally anisotropic polarizations and off-diagonal terms induced from Finsler gravity.

Let us consider extra shell fibre contributions. Equations (40)–(42) and (45)–(47) can be integrated following the above procedure but for corresponding coefficients, (43) and (48), and generating functions, (44) and (49). Such a class of 8D solutions is parameterized by the quadratic element

$$
\begin{aligned}
ds^2 = {} & e^{\psi}(dx^i)^2 + \mathring{h}_3\left(1+\frac{\partial_t(e^{\phi})}{2\Lambda\sqrt{|\mathring{h}_3}}\right)^2\left[dt - \frac{\partial_i\phi}{\phi^*}dx^i\right]^2 \\
& + \mathring{h}_4 \exp\left[\frac{e^{2\,\phi}}{8\Lambda}\right]\left[dy^4 + \left({}^1N_k^4 + {}^2N_k^4\int dt \frac{h_3}{(\sqrt{|h_4})^3}\right)dx^i\right]^2 \\
& + {}^0h_5\left(1+\frac{\partial_5 e^{\,{}^1\phi}}{2\Lambda\sqrt{\,{}^0h_5|}}\right)^2\left[dy^5 - \frac{\partial_i({}^1\phi)}{\partial_5({}^1\phi)}dx^i - \frac{\partial_{a_1}({}^1\phi)}{\partial_5({}^1\phi)}dy^{a_1}\right]^2 \\
& + {}^0h_6 \exp\left[\frac{e^{2\,{}^1\phi}}{8\Lambda}\right]\left[dy^6 + \left({}^1N_k^6 + {}^2N_k^6\int dy^5 \frac{h_5}{(\sqrt{|h_6})^3}\right)dx^i\right. \\
& \left.+ \left({}^1N_{a_1}^6 + {}^2N_{a_1}^6\int dy^5 \frac{h_5}{(\sqrt{|h_6})^3}\right)dy^{a_1}\right]^2 + {}^0h_7\left(1+\frac{\partial_7 e^{\,{}^2\phi}}{2\Lambda\sqrt{\,{}^0h_7|}}\right)^2 \\
& \left[dy^7 - \frac{\partial_i({}^2\phi)}{\partial_7({}^2\phi)}dx^i - \frac{\partial_{a_1}({}^2\phi)}{\partial_7({}^2\phi)}dy^{a_1} - \frac{\partial_{a_2}({}^2\phi)}{\partial_7({}^2\phi)}dy^{a_2}\right]^2 \\
& + {}^0h_8 \exp\left[\frac{e^{2\,{}^2\phi}}{8\Lambda}\right]\left[dy^8 + \left({}^1N_k^8 + {}^2N_k^8\int dy^7 \frac{h_7}{(\sqrt{h_8})^3}\right)dx^i\right. \\
& \left.+ \left({}^1N_{a_1}^8 + {}^2N_{a_1}^8\int dy^7 \frac{h_7}{(\sqrt{|h_8})^3}\right)dy^{a_1} + \left({}^1N_{a_2}^8 + {}^2N_{a_2}^8\int dy^7 \frac{h_7}{(\sqrt{h_8|})^3}\right)dy^{a_2}\right],
\end{aligned}
\tag{55}
$$

where the integration functions ${}^1N_k^4$, ${}^2N_k^4$ are used depending on $(x^i, y^3 = t)$; ${}^0h_{a_1}$, ${}^1N_k^6$, ${}^2N_k^6$, ${}^1N_{a_1}^6$, ${}^2N_{a_1}^6$ depending on (x^i, y^{a_1}, y^5); ${}^0h_{a_2}$, ${}^1N_k^8$, ${}^2N_k^8$, ${}^1N_{a_1}^8$, ${}^2N_{a_1}^8$, ${}^1N_{a_2}^8$, ${}^2N_{a_2}^8$ depending on $(x^i, y^{a_1}, y^{a_2}, y^7)$. The generating functions are

$$\phi(x^i, t), \partial_t\phi \neq 0;\ {}^1\phi(x^i, y^{a_1}, y^5), \partial_5\,{}^1\phi \neq 0;\ {}^2\phi(x^i, y^{a_1}, y^{a_2}, y^7), \partial_7\,{}^2\phi \neq 0.$$

We can fix such configurations which model certain scenarios in cosmology and/or gravitational and matter–field interactions.

4.3.3. Extracting realistic cosmological configurations in EFG. Metrics of type (55) define off-diagonal exact solutions for 8D locally anistoropic generalizations of FLRW cosmology. In general, it is not clear what kind of physical significance such nonhomogeneous solutions may have. Playing with values and parameters in generating and integration functions and source Λ, we can mimic different scenarios and compare them, for instance, with modifications derived for other types of gravity theories.

We note that we can generate off-diagonal solutions depending only on t and fibre coordinates (y^{a_2}, y^7) if via frame/coordinate transforms and corresponding fixing of generating/integration functions when the coefficients of (55) do not depend on coordinates (x^i). In such cases, a series of coefficients $N_i^a = 0$ and the integration functions are of types $\phi(t), \partial_t\phi \neq 0;\ {}^1\phi(t, y^5), \partial_5\,{}^1\phi \neq 0;\ {}^2\phi(t, y^5, y^6, y^7), \partial_7\,{}^2\phi \neq 0$. The nonholonomic/nonlinear gravitational dynamics on fibre variables may mimic scalar and other matter–field interactions in observable 4D spacetime. It is possible to apply various trapping/warping mechanisms such as in brane gravity [22, 32] or using osculating approximations in Finsler gravity.

Introducing a cycling factor $\tilde{a}(t) = \exp[H_0 t + b(t)]$ (19) instead of $a(t)$ in $\mathring{h}_3(y^3) = -a^{-2}(t)$, we can model such a cycling scenarios with respect to N-elongated frames (4) and (5) when the nontrivial coefficients N_i^a are functions on some variables from the set (t, y^5, y^6, y^7). For small values of N (assuming that fibre gravity results only in small corrections) we can put certain symmetries and boundary conditions on a subclass of solutions (55), which possess cycling behaviour and may not only limit the FLRW solution (32) to four dimensions, but also contain certain nontrivial anisotropic polarizations caused by possible Finsler-like interactions. In the case of anisotropic spacetimes, we have to recover not an $f(R)$ theory and certain 'exotic' states of matter but to choose a corresponding set of generating/integration functions.

To reproduce little rip evolution by off-diagonal Finsler metrics we can introduce a scaling factor $\widehat{a}(t) = a_0 e[6(\widehat{\beta} e^{\widehat{\alpha}\chi} + \widehat{\alpha} t)]$ (31) instead of $a(t)$ in $\mathring{h}_3(y^3) = -a^{-2}(t)$, or try to get such a term in h_3 and h_4, for (55). All geometric constructions will evolve with respect to certain nonholonomic frames of reference, with small corrections by N-coefficients.

Finally, we note that a solution of type (55) is not written in the well-known Finsler form (7) with the standard Hessian $\tilde{g}_{ab}$ (1) and N-connection structure $\tilde{N}_i^a$. Nevertheless, we can relate the coefficients of both representations using frame transforms of type $g_{a'b'} = e^a{}_{a'} e^b{}_{b'} \tilde{g}_{ab}$ if a fundamental Finsler function F is taken from certain experimental data or fixed following certain theoretical arguments. It is not convenient to find exact solutions working directly with data $(\tilde{g}_{ab}, \tilde{N}_i^a)$ because the gravitational field equations contain in such cases fourth-order derivatives on F. To derive a general geometric method of constructing exact solutions is possible for ansatz of type (55) with general dependence on certain classes of generating/integration functions. Then, constraining the integral varieties of solutions and after corresponding frame transforms and distorting of connections (8) we can derive exact solutions for necessary types of connections, symmetries, cosmological evolution behaviour which may explain cyclic and little rip properties etc.

5. Discussion and conclusions

The procedure for an explicit reconstruction of modified gravity using cosmological observation data [23–25] was generalized for a comparative study of $f(R)$ and Einstein–Finsler gravity (EFG) theories. We have shown how such models can be extended around general relativity (GR) action and various classes of cosmological solutions. The conclusion on the existence of cyclic evolution scenarios driven by Finsler fundamental functions was made using locally anisotropic variants of scalar–tensor gravity theories. Nonholonomic constraints and off-diagonal metrics may mimic matter–field interactions and evolution processes. Geometrical actions, off-diagonal nonlinear dynamics and the conditions usually considered for inflation models were shown that may lead to cyclic universes and ekpyrotic effects.

We proved that both $f(R)$ and EFG theories encode scenarios with little rip universe with dark energy represented by nonholonomically induced non-singular phantom cosmology. So, such classes of theories are consistent with observational/experimental data and present a realistic alternative to the ΛCDM model. Various types of non-singular super-accelerating and/or locally accelerated universes can be reconstructed in modified gravity theories. The solutions presented here can be extended to Finsler-like brane and/or Hořava–Lifshitz theories [22, 32]. Such theories modelled as GR plus corrections pass various local tests.

It is a very complicated technical problem to construct and modify generic off-diagonal solutions in GR. We generalized our anholonomic frame method [20, 21] of constructing exact solutions in a form which would allow us to decouple and solve the gravitational field equations for modified theories in off-diagonal forms. Following such an approach, it is possible to generate anisotropic off-diagonal cosmological scenarios with dependence on 'velocity–momentum'-type coordinates y^a, see metrics (55). Such models can be considered as low-energy limits of Finsler stochastic metrics of types (3.4), (3.10) and (4.18) in [16], when y^a can be associated with momentum transfers by quantum-gravitational fluctuations in the spacetime and D-brane/-particle foam. It is possible to provide a microscopic background following this approach. Our class of solutions allows us to model cosmological evolution scenarios with acceleration and dark energy/matter effects in a form adapted to the nonlinear connection structure (with is very important in Finsler gravity). For various types of Finsler gravity and cosmology theories with y^a treated as extra dimension coordinates, the physical meaning of such variables is model dependent. For instance, we cannot perform compactification of velocity-type coordinates because there is a constant speed of light and nontrivial N-connection structure. It is important to find solutions with trapping/warping effects. Having constructed certain physically important Finsler gravity/cosmology models, we can perform the osculating approximation $y(x)$ to metrics of type (2). This allows us to compare observational data for different modified theories of gravity and GR which in all case are considered in real/effective 4D spacetime[16].

Finally, we note that in this paper the reconstruction procedure is involved in a new form when the cosmological models are determined by certain classes of generating/integration functions. The cyclic/ekpyrotic/little rip universe scenarios are possible in EFG, but it remains to understand how natural and realistic are solutions with locally anisotropic effects on tangent bundles on Lorentz manifolds. This is one of the scopes of our future works.

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[16] There were elaborated various geometric and physical models of Finsler spacetimes and applications Finsler methods in modern gravity and cosmology, see [13–22, 33, 34] and references therein. A series of works are with metric noncompatible Finsler-type connections, or without explicit assumptions and physical motivations on Finsler nonlinear and linear connection structures. The main physical problems of such works with nonmetricity fields are that we are not able to define analogues of Finsler-type spinors, Finsler–Dirac equations and well-defined 'standard' conservation laws. There are not 'simple' analytic methods for constructing physically important exact solutions for Finsler-type black holes, brane configurations and to formulate a recovering procedure from cosmological data became very problematic etc, see critical remarks in [8–12]. Perhaps, most close to GR and standard physics are the (generalized/modified) Finsler-like models constructed on tangent bundles to Lorentz manifolds when there are defined natural lifts of geometric/physical objects from Einstein spaces to Einstein–Finsler analogues. In such cases, there are canonical metric compatible Finsler connections adapted to the nonlinear connection structure and an osculating approximation can be performed. Following such an approach, a self-consistent axiomatic formalism [11] which is very similar to that for the general relativity theory was formulated. It is possible to elaborate certain renormalizable models of Finsler-like quantum gravity [32] and recovering procedures from cosmological data, to find exact solutions [20–22] etc.

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