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Cosmological equivalence between the Finsler-Randers space-time and the DGP gravity model

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We perform a detailed comparison between the Finsler-Randers cosmological model and the Dvali, Gabadadze, and Porrati (DGP) braneworld model. If we assume that the spatial curvature is strictly equal to zero then we prove the following interesting proposition: despite the fact that the current cosmological models have a completely different geometrical origin, they share exactly the same Hubble expansion. This implies that the Finsler-Randers model is cosmologically equivalent with that of the DGP model as far as the cosmic expansion is concerned. At the perturbative level we find that the Finsler-Randers growth index of matter perturbations is $\gamma_{\rm FR} \simeq 9/14$, which is somewhat lower than that of DGP gravity ($\gamma_{\rm DGP} \simeq 11/16$), implying that the growth factor of the Finsler-Randers model is slightly different (~ 0.1–2%) from the one provided by the DGP gravity model.

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I. INTRODUCTION

Geometrical dark energy models act as an important alternative to the scalar-field dark energy models, since they can explain the accelerated expansion of the universe. Such an approach is an attempt to evade the coincidence and cosmological constant problems of the standard ΛCDM model. In this framework, one may consider that the dynamical effects attributed to dark energy can be resembled by the effects of a nonstandard gravity theory, implying that the present accelerating stage of the universe can be driven only by cold dark matter under a modification of the nature of gravity.

Particular attention over the last decade has been paid to the so-called Finsler-Randers (hereafter FR) cosmological model [1]. In general, metrical extensions of Riemann geometry can provide a Finslerian geometrical structure in a manifold which leads to generalized gravitational field theories. During the last decade there has been a rapid development of applications of Finsler geometry in its FR context, mainly in the topics of general relativity, astrophysics, and cosmology [1–20]. It has been found [6] that the FR field equations provide a Hubble parameter that contains an extra geometrical term which can be used as a possible candidate for dark energy.

Of course, there are many other possibilities to explain the present accelerating stage. Indeed, in the literature one can find a large family of modified gravity models (for review see Refs. [21,22]), which include the braneworld Dvali, Gabadadze, and Porrati (hereafter DGP [23]) model, $f(R)$ gravity theories [24], scalar-tensor theories [25], and Gauss-Bonnet gravity [26]. Technically speaking, it would be interesting if we could find a way to unify (up to a certain point) the geometrical dark energy models at the cosmological level. In general, we would like to pose the following question: how many (if any) of the above geometrical dark energy models can provide exactly the same Hubble expansion? In the current work we prove that the flat FR and DGP models, respectively, share the same Hubble parameter, which means that the two geometrical models are cosmologically equivalent as far as the cosmic expansion is concerned.

The structure of the paper is as follows. Initially in Sec. II, we briefly discuss the DGP gravity model, while in Sec. III we present the main properties of the FR model. In Sec. IV, we study the linear growth of perturbations as and constrain the FR growth index. Finally, in Sec. V we summarize the basic results.

II. THE DGP COSMOLOGICAL MODEL

In this section, we briefly describe the main features of the DGP gravity model. The idea here is that the "accelerated" expansion of the universe can be explained by a modification of the gravitational interaction in which gravity itself becomes weak at very large distances (close to the Hubble scale) due to the fact that our four-dimensional brane lives on a five-dimensional manifold [27,28]. Note that the Einstein field equations are defined on the five-dimensional brane. In this framework, the modified Friedmann equation can be written as

$$H^2 + \frac{k}{a^2} - \frac{2M_{(5)}^3}{M_{(4)}^2}\left(H^2 + \frac{k}{a^2}\right)^{1/2} = \frac{8\pi G}{3}\rho_m, \quad (2.1)$$

where $a(t)$ is the scale factor of the universe, $H(t) \equiv \dot{a}/a$ is the Hubble function, and $k = 0, \pm 1$ is the spatial curvature parameter.

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Notice that $M_{(5)}$ and $M_{(4)}$ are the five-dimensional and four-dimensional Planck masses, respectively. Inserting the following present-value quantities into Eq. (2.1),

$$\Omega_{rc} = \frac{1}{4r_c^2H_0^2}, \qquad \Omega_{k0} = -\frac{k}{H_0^2}, \qquad \Omega_{m0} = \frac{8\pi G\rho_{m0}}{3H_0^2}, \tag{2.2}$$

one can write

$$E^2(a) = \left[\sqrt{\Omega_{m0}a^{-3} + \Omega_{rc}} + \sqrt{\Omega_{rc}}\right]^2 + \Omega_{k0}a^{-2}, \tag{2.3}$$

where $r_c = M_{(4)}^2/2M_{(5)}^3$ and $E(a) = H(a)/H_0$. Using a spatially flat geometry ($\Omega_{k0} = 0$) and $E(1) = 1$, the above normalized Hubble parameter takes the form

$$E^2(a) = \Omega_{m0}a^{-3} + \Delta H^2, \tag{2.4}$$

where the quantity ΔH^2 is given by

$$\Delta H^2 = 2\Omega_{rc} + 2\sqrt{\Omega_{rc}}\sqrt{\Omega_{m0}a^{-3} + \Omega_{rc}}, \tag{2.5}$$

with $\Omega_{rc} = (1 - \Omega_{m0})^2/4$. On the other hand, Linder and Jenkins [29] have shown that the corresponding effective (geometrical) dark energy equation-of-state parameter of Eq. (2.4) is written as

$$w(a) = -1 - \frac{1}{3}\frac{d\ln\Delta H^2}{d\ln a}. \tag{2.6}$$

Therefore, from Eq. (2.6) it is easily shown that the geometrical dark energy equation-of-state parameter of the flat DGP model reduces to

$$w(a) = -\frac{1}{1 + \Omega_m(a)}, \tag{2.7}$$

where

$$\Omega_m(a) = \frac{\Omega_{m0}a^{-3}}{E^2(a)}. \tag{2.8}$$

From the observational point of view the DGP gravity model has been well tested against the available cosmological data [30–33]. Although the flat DGP model was found to be consistent with Type Ia supernova data (SNIa), it is under observational pressure by including in the statistical analysis the data of the baryon acoustic oscillation (BAO) and the cosmic microwave background (CMB) shift parameter [31]. Furthermore, it has been found (cf. Ref. [33]) that the integrated Sachs-Wolfe (ISW) effect poses a significant problem for the DGP cosmology, especially at the lowest multipoles.

III. THE FINSLER-RANDERS-TYPE COSMOLOGY

The FR cosmic scenario is based on the Finslerian geometry which extends the traditional Riemannian geometry. Notice that a Riemannian geometry is also a Finslerian. Bellow we discuss only the main features of the theory (for more details see Refs. [34–37]). Generally, a Finsler space is derived from a generating differentiable function $F(x, y)$ on a tangent bundle F: $TM \longrightarrow R$, $TM = \tilde{T}(M)/\{0\}$ on a manifold M. The function F is a degree-one homogeneous function with respect to $y = \frac{dx}{dt}$ and it is continuous in the zero cross section. In other words, F introduces a structure on the space-time manifold M that is called Finsler space-time. In the case of an FR space-time we have

$$F(x, y) = \sigma(x, y) + u_\mu(x)y^\mu, \qquad \sigma(x, y) = \sqrt{a_{\mu\nu}y^\mu y^\nu}, \tag{3.1}$$

where $a_{\mu\nu}$ is a Riemannian metric and $u_\mu = (u_0, 0, 0, 0)$ is a weak primordial vector field with $|u_\mu| \ll 1$. Now the Finslerian metric tensor $f_{\mu\nu}$ is constructed by the Hessian of F:

$$f_{\mu\nu} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^\mu \partial y^\nu}. \tag{3.2}$$

It is interesting to mention that the Cartan tensor $C_{\mu\nu k} = \frac{1}{2}\frac{\partial f_{\mu\nu}}{\partial y^k}$ is a significant ingredient of the Finsler geometry. Indeed, it has been found [6] that $u_0 = 2C_{000}$.

Armed with the above, the FR field equations are given by

$$L_{\mu\nu} = 8\pi G\Big(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\Big), \qquad c \equiv 1, \tag{3.3}$$

where $L_{\mu\nu}$ is the Finslerian Ricci tensor, $g_{\mu\nu} = Fa_{\mu\nu}/\sigma$, $T_{\mu\nu}$ is the energy-momentum tensor, and T is the trace of the energy-momentum tensor. Modeling the expanding universe as a Finslerian perfect fluid that includes radiation and matter with four-velocity U_μ for comoving observers,[1] we have $T_{\mu\nu} = -Pf_{\mu\nu} + (\rho + P)U_\mu U_\nu$, where $\rho = \rho_m + \rho_r$ and $P = P_m + P_r$ are the total energy density and pressure of the cosmic fluid, respectively. Note that $\rho_m = \rho_{m0}a^{-3}$ is the matter density, $\rho_r = \rho_{r0}a^{-4}$ denotes the density of the radiation, and $P_m \equiv 0$, $P_r \equiv \rho_r/3$ are the corresponding pressures.[2] Thus the energy-momentum tensor becomes $T_{\mu\nu} = \mathrm{diag}(\rho, -Pf_{ij})$, where the Greek indices belong to 0, 1, 2, 3 and the Latin ones to 1, 2, 3.

In the context of an Friedmann-Lemaître-Robertson-Walker metric[3]

$$a_{\mu\nu} = \mathrm{diag}\Big(1, -\frac{a^2}{1 - kr^2}, -a^2r^2, -a^2r^2\sin^2\theta\Big), \tag{3.4}$$

[1]Here we use $U^\alpha = \frac{dx^\alpha}{dt} = y^\alpha = (1, 0, 0, 0)$, where t is the cosmic time.

[2]We use the fact that the radiation component is negligible in the matter-dominated era.

[3]The nonzero components of the Finslerian Ricci tensor are $L_{00} = 3(\frac{\ddot{a}}{a} + 3\frac{\dot{a}}{4a}\dot{u}_0)$ and $L_{ii} = -(a\ddot{a} + 2\dot{a}^2 + 2k + \frac{11}{4}a\dot{a}\dot{u}_0)/\Delta_{ii}$, where $(\Delta_{11}, \Delta_{22}, \Delta_{33}) = (1 - kr^2, r^2, r^2\sin^2\theta)$.

the gravitational FR field equations (3.3), for comoving observers, boil down to modified Friedmann equations [6]:

$$\frac{\ddot{a}}{a} + \frac{3}{4}\frac{\dot{a}}{a}Z_t = -\frac{4\pi G}{3}(\rho + 3P), \tag{3.5}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}}{a} + 2\frac{k}{a^2} + \frac{11}{4}\frac{\dot{a}}{a}Z_t = 4\pi G(\rho - P), \tag{3.6}$$

where the over-dot denotes a derivative with respect to the cosmic time t and $Z_t = \dot{u}_0 < 0$ (see Ref. [6]). With the aid of the Eqs. (3.5) and (3.6) we obtain—after some simple algebra—the Friedmann-like expression in the matter dominated era ($\rho = \rho_m$),

$$H^2 + \frac{k}{a^2} + HZ_t = \frac{8\pi G}{3}\rho_m, \tag{3.7}$$

which looks similar to the form of the DGP Friedmann equation [see Eq. (2.1)]. Obviously, the extra term $H(t)Z_t$ in the modified Friedmann equation (3.7) affects the dynamics of the universe. If we consider $u_0 = 0$ (or $C_{000} = 0$, $F/\sigma = 1$), which implies $Z_t = 0$, then the field equations (3.3) reduce to the nominal Einstein's equations ($L_{\mu\nu} = R_{\mu\nu}$, where $R_{\mu\nu}$ is the usual Ricci tensor) a solution of which is the usual Friedmann equation.

Therefore, utilizing the last two equalities of Eqs. (2.2) and (3.7), and $E(a) = H(a)/H_0$, one can easily show that the normalized Hubble parameter is written as

$$E^2(a) = \left[\sqrt{\Omega_{Z_t} + \Omega_{m0}a^{-3} + \Omega_{k0}a^{-2}} + \sqrt{\Omega_{Z_t}}\right]^2, \tag{3.8}$$

where $\sqrt{\Omega_{Z_t}} = -\frac{Z_t}{2H_0}$. Assuming now a spatially flat geometry $k = 0$ ($\Omega_{k0} = 0$) and $E(1) = 1$, the above expression becomes

$$E^2(a) = \Omega_{m0}a^{-3} + \Delta H^2_{\rm FR}, \tag{3.9}$$

where $\Delta H^2_{\rm FR}$ is given by

$$\Delta H^2_{\rm FR} = 2\Omega_{Z_t} + 2\sqrt{\Omega_{Z_t}}\sqrt{\Omega_{m0}a^{-3} + \Omega_{Z_t}}, \tag{3.10}$$

with $\Omega_{Z_t} = (1 - \Omega_{m0})^2/4$. Amazingly, the Hubble parameter of the FR cosmology reduces to that of the flat DGP gravity, $\Delta H^2_{\rm FR} = \Delta H^2$ [see Eqs. (2.4) and (2.5) or Eqs. (2.1) and (3.7)].

The importance of the current work is that we find that the flat FR model has exactly the same Hubble parameter as the flat DGP gravity model, despite the fact that the geometrical base of the two models is completely different. Our result implies that the flat FR and the DGP models can be seen as equivalent cosmologies as far as the Hubble expansion is concerned. Below we investigate at the perturbative level the predictions of the FR model with the DGP cosmology in order to show the extend to which they are comparable.

IV. THE LINEAR MATTER FLUCTUATIONS

In this section, we briefly present the basic equation which governs the behavior of the matter perturbations on subhorizon scales and within the context of any dark energy model, including those of modified gravity ("geometrical dark energy"), in which the dark energy is homogeneously distributed. The reason for investigating the growth analysis in this work is to give the reader the opportunity to appreciate the relative strength and similarities of the FR and DGP models at the perturbative level.

At subhorizon scales the effective (geometrical in our case) dark energy component is expected to be smooth and thus it is fair to consider perturbations only on the matter component of the cosmic fluid [38]. The evolution equation of the matter fluctuations $\delta_m \equiv \delta\rho_m/\rho_m$, for cosmological models where the dark energy fluid has a vanishing anisotropic stress and the matter fluid is not coupled to other matter species (see Refs. [39–45]), is given by

$$\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G_{\rm eff}\rho_m\delta_m, \tag{4.1}$$

where $G_{\rm eff}$ is the effective Newton's constant and ρ_m is the matter density. Transforming Eq. (4.1) from t to a ($\frac{d}{dt} = H\frac{d}{d\ln a}$), we simply obtain

$$\frac{a^2}{\delta_m}\frac{d^2\delta_m}{da^2} + \left(3 + \frac{d\ln E}{d\ln a}\right)\frac{a}{\delta_m}\frac{d\delta_m}{da} = \frac{3}{2}\Omega_m(a)\frac{G_{\rm eff}(a)}{G_N}, \tag{4.2}$$

with G_N denoting Newton's gravitational constant. It is interesting to mention that solving Eq. (4.2) for the concordance Λ cosmology,[4] we derive the well-known perturbation growth factor [46] scaled to unity at the present time:

$$\delta_m \propto D(z) = \frac{5\Omega_{m0}E(z)}{2}\int_z^{+\infty}\frac{(1+u)du}{E^3(u)}. \tag{4.3}$$

Notice that we have used $a(z) = 1/(1+z)$.

At this point we define the so-called growth rate of clustering, which is an important parametrization of the matter perturbations [46]:

$$f(a) = \frac{d\ln\delta_m}{d\ln a} \simeq \Omega_m^\gamma(a). \tag{4.4}$$

The parameter γ is the growth index, which plays a significant role in cosmological studies (see Refs. [29,39,45,47–49]).

Combining the first equality of Eq. (4.4) with Eq. (4.2), we derive (after some algebra) that

[4]For the usual ΛCDM cosmological model we have $w(a) = -1$, $\Omega_\Lambda(a) = 1 - \Omega_m(a)$, and $G_{\rm eff}(a) = G_N$.

$$\frac{df}{d\Omega_m}\frac{d\Omega_m}{d\ln a} + f^2 + \left(2 + \frac{d\ln E}{d\ln a}\right)f = \frac{3}{2}\Omega_m(a)\frac{G_{\rm eff}(a)}{G_N}. \tag{4.5}$$

In our case the basic quantities of Eq. (4.5) are (see also Refs. [39,50])

$$\frac{d\ln E}{d\ln a} = \begin{cases} -\frac{3\Omega_m(a)}{1+\Omega_m(a)} & \text{DGP or FR,} \\ -\frac{3}{2}\Omega_m(a) & \Lambda\text{CDM,} \end{cases} \tag{4.6}$$

$$\frac{d\ln\Omega_m}{d\ln a} = \begin{cases} -\frac{3\Omega_m(a)[1-\Omega_m(a)]}{1+\Omega_m(a)} & \text{DGP or FR,} \\ -3\Omega_m(a)[1-\Omega_m(a)] & \Lambda\text{CDM,} \end{cases} \tag{4.7}$$

and

$$\frac{G_{\rm eff}(a)}{G_N} = \begin{cases} 1 & \Lambda\text{CDM or FR,} \\ \frac{2+4\Omega_m^2(a)}{3+3\Omega_m^2(a)} & \text{DGP.} \end{cases} \tag{4.8}$$

Inserting the ansatz $f \simeq \Omega_m^{\gamma(\Omega_m)}$ into Eq. (4.5), using simultaneously Eqs. (4.6), (4.7), and (4.8) and performing a first-order Taylor expansion around $\Omega_m = 1$ (for a similar analysis see Refs. [45,49,51]), we find that the asymptotic value of the FR growth index to the lowest order is $\gamma_{\rm FR} \simeq 9/14$, while in the case of the DGP braneworld model we have $\gamma_{\rm DGP} \simeq 11/16$ (see also Refs. [28,45,50,52]). Notice that for the concordance ΛCDM cosmology it has been found (see Refs. [45,47–49]) that $\gamma_\Lambda \simeq 6/11$. Using the above we find the following restriction:

$$\gamma_\Lambda < \gamma_{\rm FR} < \gamma_{\rm DGP}.$$

The small difference ($\sim 7\%$) between $\gamma_{\rm FR}$ and $\gamma_{\rm DGP}$ is due to $G_{\rm eff}/G_N$ used in the growth analysis [see Eq. (4.8)].

In Fig. 1 we present the evolution of the growth rate of clustering for the current cosmological models, i.e., the FR model (solid line), the DPG (dashed line), and standard ΛCDM (long dashed line) models and the fractional difference between the first (FR model) and each of the other two models (insert panel). Notice that in order to produce the curves we utilize $\Omega_{m0} = 0.273$. The general behavior of the functional form of the FR growth rate is an intermediate case between the DGP and ΛCDM growth rates.

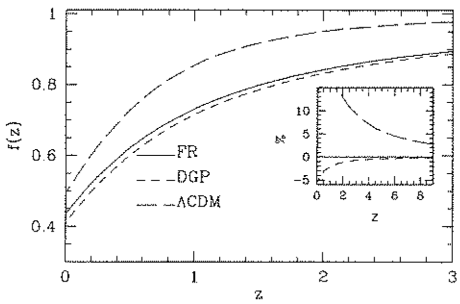

FIG. 1. The evolution of the growth rate of clustering $f(z)$. The lines correspond to the following models: FR (solid), DGP (dashed), and ΛCDM (long dashed). In the insert panel we present the corresponding fractional difference of the DGP (dashed line) and ΛCDM (long dashed line) models with respect to the FR model. To produce the curves we use $\Omega_{m0} = 0.273$.

In Fig. 2 we show the growth factor evolution, which is derived by integrating Eq. (4.4), for the FR cosmological model. In the insert panel of Fig. 2 we plot the fractional difference between the different models, similarly to Fig. 1, but now for the growth factor. Obviously, the growth factor of the flat FR model is slightly different [$\frac{\delta D}{D_{\rm FR}}(\%) \sim 0.1$–$2\%$] from the one provided by the conventional flat DGP cosmology. Concering the ΛCDM model, the expected differences are small at low redshifts, but become gradually larger for $z \gtrsim 1$, reaching variations up to $\sim -15\%$ at $z \sim 3$.

We would like to end this section with a brief discussion about the observational consequences of the FR model. Since the flat FR model shares exactly the same Hubble parameter with the flat DGP model, this implies that the flat FR model inherits all the merits and demerits of the flat DGP gravity model. Thus, it becomes obvious that the FR model is under observational pressure when we compare against the background cosmological data (SNIa, BAO, CMB shift parameter). We would like to mention that the FR model is in agreement with the SNIa data [6] (a similar situation holds also for the DGP; see Sec. II). As far as the ISW effect is concerned the situation is almost the same. In particular, the dependence of the ISW effect on the different cosmologies enters through the different behavior of $D(a)$ (the growth factor), which is affected by γ, and of $H(a)$ [see Eq. (14.16) in Ref. [22]]. Taking the above arguments into account—namely, the same $H(a)$ and a very small difference in $D(a)$ [see insert panel of Fig. 2]—we conclude that both flat FR and DGP models predict almost the same ISW effect, which of course is in disagreement with the ISW observational data. It is however possible to derive an extended version of the FR model free from the observational problems by including additional terms of the Finslerian metric $f_{\mu\nu}$ in the

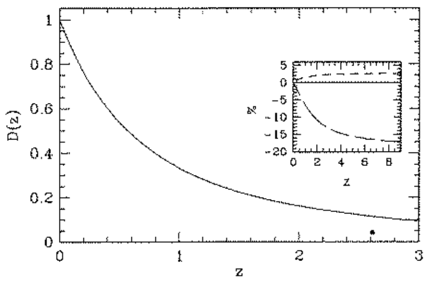

FIG. 2. The evolution of the growth factor, with that corresponding to the FR model ($\gamma_{\rm FR} = 9/14$) showing a $\sim$0.1–2% difference with respect to that of the DGP model ($\gamma_{\rm DGP} = 11/16$), especially at large redshifts ($z \gtrsim 1$). Notice that the growth factor is normalized to unity at the present time.

modified Friedmann equation. Such an analysis is in progress and will be published elsewhere.

V. CONCLUSIONS

In this paper we compared the Finsler-Randers (FR) space-time against the DGP gravity model. To our surprise, we found that the flat FR space-time is perfectly equivalent to the cosmic expansion history of the flat DGP cosmological model, despite the fact that the two models live in a completely different geometrical background. At the perturbative level we studied the linear growth of matter perturbations, and it was found that the FR growth index is $\gamma_{FR} \simeq 9/14$, which is almost $\sim 7\%$ less than the theoretically predicted value of the DGP gravity model, $\gamma_{DGP} \simeq 11/16$. The latter implies that the growth factor of the flat FR model is slightly different (~ 0.1–2%) from the one provided by the conventional flat DGP cosmology.

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