Resembling dark energy and modified gravity with Finsler-Randers cosmology

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In this article we present the cosmological equivalence between the relativistic Finsler-Randers cosmology and the dark energy and modified gravity constructions at the background level. Starting from a small deviation from the quadraticity of the Riemannian geometry, through which the local structure of general relativity is modified and the curvature theory is extended, we extract the modified Friedmann equation. The corresponding extended Finsler-Randers cosmology is very interesting, and it can mimic dark energy and modified gravity, describing a large class of scale-factor evolutions, from inflation to late-time acceleration, including the phantom regime. In this respect, the nontrivial Universe evolution is not attributed to a new scalar field, or to gravitational modification, but it arises from the modification of the geometry itself.

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I. INTRODUCTION

Since the discovery of the accelerated expansion of the Universe (see Ref. [1] and references therein), a lot of effort has been made in order to understand the physical mechanism which is responsible for such a cosmological phenomenon. There are two basic directions one can follow in order to obtain its explanation. The first is to introduce the concept of dark energy (hereafter DE) within the framework of general relativity (for reviews, see for instance Ref. [2]), while the second is to modify the gravitational sector itself (see Ref. [3] and references therein).

From the DE viewpoint, the simplest way to fit the current cosmological data is to include in the Friedmann equations the cosmological constant [1]. However, the disadvantage of the so-called concordance Λ cosmology is the fact that it suffers from the cosmological constant problem itself [4]. This intrinsic problem appears as a difficult issue which includes many aspects: not only the problem of understanding the tiny current value of the vacuum energy density ($\rho_\Lambda = c^2\Lambda/8\pi G \simeq 10^{-47}$ GeV^4) [4] in the context of quantum field theory or string theory, but also the cosmic coincidence problem, namely why the density of matter is now so close to the vacuum density [5]. Unfortunately, the alternative and more complex DE scenarios, for instance quintessence [6–8], phantom [9], quintom [10], etc., are not free from similar fine-tuning and other problems no less severe (including the presence of

extremely tiny masses and peculiar forms of the scalar-field kinetic energy).

The above problems have inspired many authors to proceed in the alternative direction of modified gravity, such as the braneworld Dvali, Gabadadze and Porrati (DGP) [11] model, $f(R)$ gravity [12], $f(T)$ gravity [13,14], scalar-tensor theories [15], Gauss-Bonnet gravity [16], Hořava-Lifshitz gravity [17], nonlinear massive gravity [18], etc. The underlying idea is that the accelerated expansion, either during inflation or at late times, can be driven by a modification of the Einstein-Hilbert action, while the matter content of the Universe remains the same (relativistic and cold dark matter). However, the majority of modified gravity models are plagued with no physical basis and/or many parameters.

On the other hand, in the last decade the Finslerian relativistic extensions have gained a lot of attention, since Finsler geometry naturally extends the traditional Riemannian geometry [19]. In this formulation, in general one starts with the Lorentz symmetry breaking, which is a common feature within quantum gravity phenomenology. Such a departure from relativistic symmetries of spacetime leads to the possibility for the underlying physical manifold to have a broader geometric structure than the simple pseudo-Riemann geometry. Along these lines, Finsler geometry is the simplest class of extensions, since it generalizes Riemann geometry. Note that the Riemannian geometry itself is a special type of the Finslerian one.

One of the most characteristic features of Finsler geometry is the dependence of the metric tensor on the position coordinates of the base manifold and on the tangent vector of a geodesic congruence, and this velocity dependence reflects the Lorentz-violating character of the kinematics. Additionally, Finsler geometry is strongly connected to the effective geometry within anisotropic

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media [20] and naturally enters the analogue gravity program [21]. These features suggest that Finsler geometry may play an important role within quantum gravity physics.

From the cosmological viewpoint, in a series of works [22,23] it was reported that in the osculating Riemannian limit, the cosmic expansion of the flat Finsler-Randers (hereafter FR) gravity is identical to that of flat DGP, despite the fact that the geometrical origin of the two cosmological models is completely different. This means that the flat FR model inherits all the advantages and disadvantages of the flat DGP gravitational construction. However, the fact that DGP gravity is under observational pressure [24] implies that the flat FR model faces the same problems [23].

Therefore, in the present work we are interested in extending the results of Refs. [22,23] in order to derive an extended version of the FR model (hereafter EFR), free from the observational inconsistencies. To achieve this, instead of using the osculating Riemannian limiting processes [22], which is a metric-based approach, we use the covariant $1 + 3$ formalism [25], that under certain conditions can be naturally extended in the Finslerian framework [26]. In this less restrictive case, we can mimic all non-interacting DE models and the majority of modified gravitational constructions, and we are able to describe a large class of cosmological evolutions.

The plan of the work is as follows: In Sec. II, we present the metrical extension of Riemannian geometry, and we discuss the evolution of the kinematical variables and the Finsler-Randers geometrical structure. In Sec. III, we focus on the isotropic expansion, and we develop the cosmological model. We prove the equivalence between the EFR and DE models, as well as some classes of modified gravity at the expansion level, and we discuss some particular examples. Finally, in Sec. IV, we draw our conclusions.

II. RELATIVISTIC FINSLER GEOMETRY

Recently, there is an increasing interest in Finsler geometry since it has been reported within different aspects of quantum gravity. The effective metric depends either on velocitylike variables or on the tangent vector field of the observers' cosmic lines. A representative example of the first case is the stochastic space-time D foam, where the effective metric depends on the velocity of D particles that recoil on the world sheet [27]. Another scenario where Finsler geometry emerges, and the metric depends on fiber coordinates, is the covariant Galilean transformations in curved space-times [28]. On the other hand, dependence of the metric on the particle's 4-velocity arises in other Lorentz-violating theories, such as the Hořava-Lifshitz gravity [29]. Additionally, a Finslerian line element has been encountered in deformations of Cohen and Glashow's very special relativity [30], as well as in holographic fluids [31]. Moreover, bimetric constructions can be naturally incorporated in the Finsler framework [32]. Finally, we

mention that Finsler geometry can be closely related to the standard model extension [33]. Before proceeding to the cosmological application of Finsler geometry, in the following subsections we briefly present its basic features.

A. Finsler congruences

The main object in Finsler geometry is the fundamental function $F(x, dx)$ that generalizes the Riemannian notion of distance (see for example Refs. [34–36]). In Riemann geometry, the latter is a quadratic function with respect to the infinitesimal increments dx^a between two neighboring points. Keeping all the postulates of Riemann geometry but accepting a nonquadratic distance measure, a metric tensor can be introduced as

$$g_{ab}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^a \partial y^b}, \quad y^a \neq 0 \qquad (1)$$

for a given connecting curve with tangent $y^a = \frac{dx^a}{d\tau}$. Note that when the generating function $F(x, y)$ is quadratic, the above definition is still valid and leads to the metric tensor of Riemann geometry. The dependence of the metric tensor on the position coordinates x^a and on the fiber coordinates y^a suggests that the geometry of Finsler spaces is a geometry on the tangent bundle TM. In other words, the Finsler manifold is a fiber space where tensor fields depend on the position and on the infinitesimal coordinate increments y^a. Therefore, the position dependence of Riemann geometry is replaced by the so called *element of support*, which is the pair (x^a, y^a).

In relativistic applications of Finsler geometry, the role of the supporting direction y^a must be explicitly given. For example, it may stand as an internal variable, as an explicit or implicit violation of Lorentz symmetry, as an aetherlike direction, or simply as the velocity of the fundamental observer. In this article, we restrain our analysis to the latter case, where the supporting direction y^a is the tangent to the cosmic flow lines. Using only variational arguments, we can arrive at the deviation equation for the supporting congruence y^a. The deviation equation directly provides all the information for the internal deformation of the timelike geodesic flow y^a. Following the same procedure with GR, we can extract the propagation formulas for the expansion, shear and vorticity of an infinitesimal cross section of the cosmological flow.

The infinitesimal distance between two neighboring points on the base manifold (position space) is given by the small displacement along the connecting curve $\gamma(\tau)$, which depends on the position x^a and on the coordinate increments dx^a:

$$d\tau = F(x, dx), \qquad (2)$$

where in Finsler geometry $F(x, dx)$ does not necessarily depend quadratically on the dx^a increments. The actual distance traveled on the base manifold along a given direction is

$$I = \int F(x, dx/d\tau) d\tau, \qquad (3)$$

where the metric function $F(x, y)$ is homogeneous of first order with respect to the displacement arguments $y^a = dx^a/d\tau$. Applying the least action principle to the previous integral, we arrive at the geodesic equation for the supporting direction:

$$\dot{y}^a \equiv y^a \nabla_a y^b = \frac{dy^a}{d\tau} + 2G^a(x, y) = 0, \qquad (4)$$

where G^a are the spray coefficients with respect to the $F(x, y)$ fundamental function, and they are given by

$$G^a = \frac{1}{4} g^{ab} \left(\frac{\partial^2 F^2}{\partial x^c \partial y^b} y^c - \frac{\partial F^2}{\partial x^b} \right). \qquad (5)$$

When y^a stands for the velocity of the fundamental observer, then the dot operator in Eq. (4) is a direct generalization of the time propagation of GR relativistic kinematics. In other words, if the supporting direction $y^a = \frac{dx^a}{d\tau}$ is the observers' 4-velocity, then the affine parameter τ is the proper time. Note that we can recast Eq. (4) in the familiar form with respect to the Christofell symbols, the only difference being that the metric will depend on the supporting element.

The relative acceleration between two neighboring observers is given by the second variation of the distance module. Focusing the analysis along the y^a direction, the second variation leads to the Jacobi equation. The relative acceleration between nearby geodesics is monitored by an infinitesimal connecting vector (the deviation vector) defined as

$$\tilde{x}^a = x^a + \xi^a, \qquad (6)$$

where the tilde stands for the neighboring reference frame. Then, substituting the previous expression into the Euler-Lagrange equations (4) and keeping up to first-order terms with respect to the deviation vector ξ^a leads to the following formula:

$$\ddot{\xi}^a + \mathcal{H}^a{}_b(x, y) \xi^b = 0, \qquad (7)$$

where $\mathcal{H}^a{}_b$ is a tensor field that incorporates the relative displacement of nearby geodesics in a Finslerian framework, given by

$$\mathcal{H}^a{}_b = 2 \frac{\partial G^a}{\partial x^b} - y^c \frac{\partial^2 G^a}{\partial y^b \partial x^c} + 2G^c \frac{\partial^2 G^a}{\partial y^b \partial y^c} - \frac{\partial G^a}{\partial y^c} \frac{\partial G^c}{\partial y^b}. \qquad (8)$$

The first-order homogeneity of the metric function leads to the constraint $\frac{\partial g_{ab}}{\partial y^c} y^c = 0$. The latter guarantees that for most connection structures (for example Chern, Cartan or Berwald), the Jacobi field in Eq. (7) remains the same (see for example Refs. [34–37]).

The coefficients of the tensor field $\mathcal{H}^a{}_b$ are directly determined by the metric function $F(x, y)$ through the least

action principle that gives back the spray coefficients [Eq. (5)]. As in Riemann geometry, expression (8) is second-order homogeneous with respect to y^a, but the dependence is nonquadratic. Its eigenvalues correspond to the sectional curvatures in the principal directions and designate the relative motion between neighboring integral curves. Equation (8) encloses all the relevant information for Finslerian tidal effects on the y^a congruence. The tensor field $\mathcal{H}^a{}_b$ is responsible for the relative acceleration between nearby observers and will generate expansion and shear on the timelike y^a congruence. Apparently, the y^a-deformable kinematics will be modified due to the nonquadratic dependence of $\mathcal{H}^a{}_b$ on the velocity of the fundamental observer.

B. Deformable kinematics

The observers' timelike congruence introduces a unidirection in the physical manifold. This asymmetry is encoded in the metric function $F(x, y)$ and induces the $1 + 3$ "threading" of space-time [25]. In the covariant $1 + 3$ formalism, the metric is not the central object, since we do not use a particular coordinate system. Instead, we use the kinematic quantities, the irreducible components of curvature and conservation arguments, while Einstein's field equations enter as simple algebraic relations between curvature and matter [25]. The deviation of geodesics is of central importance, since it monitors the internal deformation of the cosmic medium in a covariant way.

From the Finslerian perspective, the space and time decomposition is directly related to the first-order homogeneity of $F(x, y)$. In particular, the fundamental observer's velocity y^a defines a family of integral curves on the space-time manifold. With respect to this 4-velocity, we can decompose tensor fields along y^a and on the perpendicular spatial hypersurface. In fact, we can recast the metric tensor in Eq. (1) in the following form:

$$g_{ab} = F \frac{\partial^2 F}{\partial y^a \partial y^b} + \frac{\partial F}{\partial y^a} \frac{\partial F}{\partial y^b}, \qquad (9)$$

where we have split the space-time metric into two parts by using the quantities

$$l_a = \frac{\partial F}{\partial y^a}, \qquad h_{ab} = F \frac{\partial^2 F}{\partial y^a \partial y^b}. \qquad (10)$$

Using the first-order homogeneity of the metric function $F(x, y)$, we can prove that l_a is the normalized velocity of the observers' flow lines, $l_a = y_a/F$. In addition, the first-order homogeneity of the fundamental function implies that $h_{ab} l^b = 0$ and also that the rank is $(h_{ab}) = 3$. Therefore, the tensor h_{ab} stands for the projection tensor of relativistic kinematics.

With the space-time split [Eq. (9)] in hand, we can decompose tensor fields to their irreducible parts, in direct analogy to the standard gravitational physics, for example

$$X_a = g_a{}^b X_b = (h_a{}^b + l^b l_a) X_b = X l_a + \mathcal{X}_a, \quad (11)$$

where $X = l_a X^a$ is the timelike part and $\mathcal{X}_a = h_a{}^b X_b$ is the spacelike part. Using the $1 + 3$ covariant formalism, we will track the internal motion of the normalized supporting direction l^a (the congruence l^a is considered to be timelike, $l_a l^a = 1$ [38]). Restraining the analysis along the l timelike flow, the propagation equation of the deviation vector at first order is given by

$$\dot{\xi}^a = B^a{}_b \xi^b, \quad (12)$$

where it is straightforward to prove that $B^a{}_b = \nabla_b l^a$—that is, the tensor field $B^a{}_b$ is the distortion tensor of the timelike congruence. Following Eq. (10), we can decompose the distortion tensor to its irreducible parts

$$\nabla_b l_a = \frac{1}{3} \Theta h_{ab} + \sigma_{ab} + \epsilon_{abc} \omega^c, \quad (13)$$

where for the 3D spatial derivative $\mathrm{D}_a = h_a{}^b \nabla_b$, the irreducible components are the expansion $\Theta = \mathrm{D}^a l_a$ that tracks volume changes, the shear $\sigma_{ab} = \mathrm{D}_{\langle b} l_{a\rangle}$[1] that incorporates shape distortions, and the vorticity $\omega_a = \epsilon_{abc} \mathrm{D}^b l^c / 2$ that accounts for changes of the orientation of the infinitesimal spatial cross section, parallel transported along l^a.

Taking the time derivative of Eq. (12) and substituting in Eq. (7), we arrive at the evolution equation for the internal deformations of the timelike flow:

$$\dot{B}_{ab} + B_{ac} B^c{}_b = -\mathcal{H}_{ab}. \quad (14)$$

This propagation law reflects the effect of the Finslerian curvature tensor $\mathcal{H}_{ab}$ on the deformable kinematics of a timelike flow. The irreducible parts of Eq. (14) provide the evolution equation for the expansion (Raychaudhuri's equation)

$$\dot{\Theta} + \frac{1}{3} \Theta^2 = -\mathcal{K} - 2(\sigma^2 - \omega^2), \quad (15)$$

the propagation of shear (which describes kinematic anisotropies)

$$\dot{\sigma}_{\langle ab\rangle} = -\frac{2}{3} \Theta \sigma_{ab} - \sigma_{c\langle a} \sigma^c{}_{b\rangle} - \omega_{\langle a} \omega_{b\rangle} - \mathcal{H}_{\langle ab\rangle}, \quad (16)$$

and finally the propagation of vorticity

$$\dot{\omega}_a = -\frac{2}{3} \Theta \omega_a + \sigma_{ab} \omega^b, \quad (17)$$

where $\mathcal{K} = \mathcal{H}_{ab} h^{ab}$ is the scalar flag curvature of the Finslerian manifold when $\mathcal{H}_{\langle ab\rangle} = 0$ [26,34].

The above system of propagation equations is very similar to the analogous expressions in general relativity. In particular, they are the same except that in the

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[1] Angle brackets stand for the projective, symmetric and trace-free part of a second-rank tensor $X_{\langle ab\rangle} = h^c{}_{(a} h^d{}_{b)} X_{cd} - \frac{1}{3} X_{cd} h^{cd} h_{ab}$.

Riemannian limit, the tensor in Eq. (8) depends quadratically on the observers' 4-velocity. Thus, the key difference is the nontrivial dependence of the curvature tensor $\mathcal{H}_{ab}$ on the velocity of the fundamental observer, which modifies the way that curvature generates deformations on a timelike medium.

C. Finsler-Randers metric function

In Finsler geometry, the form of the metric function $F(x, y)$ is of central importance, since it generates all the other geometric quantities. One of the most simple cases after the Riemann limit is the Randers norm [39], which is given by

$$F = \alpha + \beta, \quad (18)$$

where $\alpha = \sqrt{\alpha_{ab} y^a y^b}$ is a Riemann metric function and $\beta = b_a y^a$ stands for an arbitrary 1-form. The fundamental function [Eq. (18)] interfaces a Riemann space-time with a Finslerian one in a simple way, since the Randers metric is the limiting case of a large number of Finsler space-times, when we consider small departures from GR. For example, in a large class of (α, β) metrics where $F = \alpha \phi(\beta/\alpha)$, the almost Riemannian limit $\phi \sim 1$ provides a Randers-type geometry at first order for β/α, when $\phi \sim 1 + \beta/\alpha$.

Equation (18) has the important consequence that one can separate geometric quantities into the purely Riemann part, with respect to the α metric function, and the Finsler contribution. In this case, for specific examples, we can directly inspect the effect of nonquadraticity on the space-time medium.

In particular, the geometric entity that accurately incorporates the nonquadraticity of the metric function is the indicatrix $F(x, y) = 1$, which represents an arbitrary locus on the tangent bundle [36]. This locus in the Riemann case defines a quadratic hypersurface. In the case of a Randers-type geometry [Eq. (18)], the hypersurface is still quadratic but becomes eccentric [39]. In other words, the Randers metric function in Eq. (18) assigns at each space-time point a vector b_a that describes the displacement of the center of the indicatrix. This property translates to a disformal correlation between the Finslerian metric tensor [Eq. (1)] and the Riemannian a_{ab} given in Eq. (18). In fact, substitution of Eq. (18) into Eq. (1) yields

$$g_{ab} = \frac{F}{\alpha} (\alpha_{ab} - \bar{l}_a \bar{l}_b) + l_a l_b, \quad (19)$$

where $\bar{l}_a = \frac{\partial \alpha}{\partial y^a}$ is the normalized velocity on the Riemann sector. The disformal relation [Eq. (19)] introduces an explicit dependence of the space-time metric on the velocity of the cosmic flow lines. Note that similar behavior of the effective geometry is commonly reported in investigations of anisotropic media [20].

Concerning the signature of the Finsler-Randers space-time, it is useful to introduce the nonholonomic frame on TM, namely

$$Y_a{}^b = \sqrt{\frac{\alpha}{F}}\left(\delta_a{}^b + \sqrt{\frac{\alpha}{F}}l_a l^b - \bar{l}^a \bar{l}_b\right), \qquad (20)$$

and it is straightforward to prove the identity $Y_a{}^c Y_c{}^b = \delta_a{}^b$. Then, using Eq. (20), we can recast the Finslerian metric tensor in the following form [40]:

$$g_{ab} = Y_a{}^c Y_b{}^d \alpha_{cd}. \qquad (21)$$

Thus, from the definition in Eq. (20) and the above relation, we conclude that the Finsler-Randers metric tensor and the Riemann metric α_{ab} have the same signature. Taking into account that the metric tensor [Eq. (19)] must be real, we deduce that the timelike y^a congruence is positive definite, or equivalently, the signature is $(+, -, -, -)$. This restrains the y^a bundle of geodesics to be timelike [32]; however, we can define first-order Finslerian tensor fields as spacelike $[U(x, y)^a U(x, y)_a < 0]$, null $[U(x, y)^a U(x, y)_a = 0]$, or timelike $[U(x, y)^a U(x, y)_a > 0]$ [41].

As a first attempt to examine the physical impact of nonquadraticity, we assume that the Riemann sector of Eq. (19) represents the "gravitational" geometry, while the Finsler manifold describes the "physical" geometry [42]. This provides an effective geometric setup to model possible implications of nonquadraticity on the expansion dynamics.

We consider a Randers-type (α, β) metric, and we assume that the velocity of the fundamental observer is given by the normalized vector $l^a = y^a/F$. Then, if b_a is a closed form with respect to the Riemann covariant derivative of the α metric, $b_{[a;b]} = 0$, the spray coefficients [Eq. (5)] for the normalized velocity l^a take the simplified form [34]

$$G^a = \bar{G}^a + \frac{1}{2}\Phi l^a, \qquad (22)$$

where bars denote the Riemann parts with respect to the α metric, and we define $\Phi = b_{a;b}l^a l^b$. Substituting Eq. (22) into the covariant expression [Eq. (8)], the curvature tensor takes the simplified form

$$\mathcal{H}^a{}_b = \bar{\mathcal{H}}^a{}_b + \frac{1}{4}(3\Phi^2 - 2\Psi)h^a{}_b, \qquad (23)$$

where the last two scalars are given with respect to the Riemann covariant derivative of the α metric,

$$\Psi = b_{a;b;c}l^a l^b l^c, \qquad (24)$$

and we have defined $\bar{\mathcal{H}}_{ab} = F^{-2}\bar{R}_{acbd}y^c y^d$ for the part of the curvature coming from the Riemann metric function α of Eq. (18). The second-rank tensor [Eq. (23)] clearly incorporates the relation between a part of the Riemann curvature and a part of the actual curvature of the foreground manifold. As we have already mentioned, the latter curvature generates deformations in the assumed "physical" space-time which is of Finsler type, while the Riemann curvature represents the gravitational sector. Note that Ψ—that is, the nature of the Finsler-Randers

contribution to the curvature—is defined on a geometrical basis, since it directly originates from the curvature of the Finsler-Randers geometry [Eq. (23)]. Therefore, from Eq. (23) and the deformable kinematics given in Eqs. (15)–(17), we conclude that the Riemann curvature of the gravitational sector generates deformations in the foreground space-time in a modified way.

III. FINSLER-RANDERS COSMOLOGY

In this section, we investigate the conditions under which the Finsler-Randers cosmology can provide a cosmic acceleration equivalent to the traditional scalar-field DE or classes of modified gravity. We assume that the "physical" geometry is represented by the nonquadratic metric function [Eq. (18)], while the gravitational geometry is given by its Riemannian part. Thus, in a FRW-like scenario, the Riemann curvature $\bar{R}_{abcd}$ is related to the energy-momentum tensor of a perfect fluid through the Einstein field equations

$$\bar{R}_{ab} - \frac{1}{2}\bar{R}\alpha_{ab} = T_{ab} = \rho\bar{l}_a\bar{l}_b + p\bar{h}_{ab}, \qquad (25)$$

where we define the projection tensor of the Riemannian sector as $\bar{h}_{ab} \equiv \alpha_{ab} - \bar{l}_a\bar{l}_b$, the overall energy density as measured in the $\bar{l}^a$ frame as $\rho = T_{ab}\bar{l}^a\bar{l}^b$, and the total isotropic pressure as $p = T_{ab}\bar{h}^{ab}/3$, while we impose the usual convenient units, setting $8\pi G \equiv 1$. Furthermore, for our setup it is natural to assume a homogeneous and isotropic Riemannian sector for the gravitational geometry. Hence, we can neglect the nonlocal gravitational degrees of freedom, and the Weyl curvature becomes negligible. In this case, the Riemann curvature depends only on its local parts:

$$\bar{R}_{abcd} = \frac{1}{2}(\alpha_{ac}\bar{R}_{bd} + \alpha_{bd}\bar{R}_{ac} - \alpha_{bc}\bar{R}_{ad} - \alpha_{ad}\bar{R}_{bc})$$
$$- \frac{1}{6}\bar{R}(\alpha_{ac}\alpha_{bd} - \alpha_{ad}\alpha_{bc}). \qquad (26)$$

We consider shear- and vorticity-free evolution for the cosmic fluid, which is in agreement with the tight constrains of the cosmic microwave background (CMB) anisotropies (see for instance Ref. [43]). In fact, using Eqs. (23), (25), and (26), we obtain that $\mathcal{H}_{\langle ab\rangle} \sim b_{\langle a}b_{b\rangle}$, and the source term in the propagation of shear [Eq. (16)] is negligible if b_a tends to be purely timelike. Thus, our kinematical setup is consistent with a shear- and vorticity-free bulk flow, since there are no source terms in Eqs. (16) and (17). Then, keeping up to first-order terms with respect to b_a in Eq. (23), and using the field equations [Eq. (25)] and the decomposition [Eq. (26)], Raychaudhuri's formula [Eq. (15)] acquires the simplified form

$$\dot{\Theta} + \frac{1}{3}\Theta^2 = -\frac{1}{2}(1 - \beta)(\rho + 3p) - \frac{3}{2}\Psi, \qquad (27)$$

where we have used the auxiliary relation $\bar{\mathcal{H}}_{ab}h^{ab} = \frac{1}{2} \times (1 - \beta)(\rho + 3p)$. Raychaudhuri's equation [Eq. (27)] is the

fundamental equation that describes the cosmological evolution. The crucial point is the sign of its right-hand side. In particular, negative terms align with the gravitational pull, while positive terms accelerate the expansion. The Finsler contribution in the first term of the rhs of Eq. (27) acts as an effective coupling constant. As an example, if we neglect this term, then the significant term that incorporates the effects of nonquadraticity is the last one, and when $\Psi < 0$, it can drive an accelerating phase; while for $\Psi > 0$, it increases the gravitational attraction; and for $\Psi = 0$, the current scenario reduces to the Einstein–de Sitter model in the matter era. In other words, the adoption of a nonquadratic measure affects the local structure of space-time, since the $SO(4)$ symmetry is broken, and hence it implies new kinematic effects for the bulk flow of matter by modifying the curvature theory. Apparently, by discarding the local flatness of general relativity, we acquire long-range modifications in the Finslerian geometrodynamics.

Let us discuss here the energy conservation in the scenario at hand. The energy density and the isotropic pressure as measured in the Riemannian frame $\bar{l}^a$ are related to the "physical" frame l^a by the following relations:

$$\rho = \frac{F^2}{\alpha^2}\rho^{(f)}, \qquad p = \frac{F}{\alpha}p^{(f)}, \qquad (28)$$

where we have defined $\rho^{(f)} = T_{ab}l^a l^b$ and $p^{(f)} = T_{ab}h^{ab}/3$ for the total energy density and pressure, respectively, in the Finslerian frame. Hence, taking into account that at late times $F/\alpha = 1 + \beta/\alpha \sim 1$, we obtain that at first order in the two frames, the energy density and pressure are the same, namely $\rho \sim \rho^{(f)}$ and $p \sim p^{(f)}$.[2] At early times, our first-order approximation is no longer valid, since the two frames will start to diverge, having a direct impact on the effective equation of state. Additionally, in the presence of pressure, the spatial part of the energy-momentum conservation $h_a{}^c \nabla^b T_{cb} = 0$ yields

$$(\rho + p)l^b \nabla_b l_a = -D_a p. \qquad (29)$$

However, by construction, the l^a congruence is geodesic [Eq. (4)], and the previous relation implies that $D_a p = 0$. The latter condition is valid in an isotropic and homogeneous background, but considering cosmological perturbations, nongeodesic congruences will be involved in the calculations (for a $1 + 3$ treatment of nongeodesic flows, see Ref. [26]). Hence, our model is consistent at late times of the cosmological history (for example, in dust- and radiation-dominated eras) and for vanishing gradients of pressure.

On the other hand, taking the timelike part of the energy momentum conservation, $l^a \nabla^b T_{ab} = 0$, and decomposing it to the irreducible parts with respect to the l^a congruence, we obtain

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[2] In relativistic cosmology, a similar limiting process between relative frames is used to study "peculiar" frames and the Zeldovich approximation (see for example Ref. [44]).

$$\dot{\rho} = -\Theta(\rho + p) \qquad (30)$$

for the total energy density. Here we mention that the above relation is valid for the first-order approximation, where the energy density is almost the same in the Remannian frame $\bar{l}^a$ and in the Finslerian one l^a that represents the bulk flow of matter. Introducing the characteristic length scale a (*scale factor*) of the spatial volume by $dV \propto a^3$, we extract that for the expansion we have $\Theta = (dV)^{\cdot}/dV = 3\dot{a}/a$. Using this expression, we can recast Raychaudhuri's formula [Eq. (27)] in terms of the scale factor for late times of the cosmological evolution as

$$3\frac{\ddot{a}}{a} = -\frac{1}{2}(\rho + 3p) - \frac{3}{2}\Psi, \qquad (31)$$

where the total matter fluid itself is in general a mixture of relativistic matter (i.e. radiation, ρ_r with $p_r = \rho_r/3$) and nonrelativistic matter (i.e. cold matter, ρ_m with $p_m = 0$) components, implying $\rho = \rho_m + \rho_r$ and $p = p_m + p_r = \rho_r/3$. Now, using the continuity equation [Eq. (30)] together with the Raychaudhuri's formula [Eq. (31)], we retrieve the modified Friedmann equation:

$$H^2(a) = \frac{1}{3}\rho - a^{-2}\int a\Psi(a)da - \frac{C_1}{a^2}. \qquad (32)$$

In the above expression, C_1 is an integration constant, which in the FRW limit coincides with the spatial curvature, and thus without loss of generality, in the following we set it to zero. For the rest of our analysis, we focus on the matter-dominated era (well after radiation-matter equality) in which the radiation component is considered negligible, and thus we use $\rho \equiv \rho_m$.

Equation (32) incorporates the effects of Finsler-Randers geometry in the expansion of the Universe. We remind the reader that $\Psi(a)$, which is the nature of the Finsler-Randers contribution to the curvature, is defined on a geometrical basis, since it directly originates from the curvature of the Finsler-Randers geometry [Eq. (23)]. Since from first principles the evolution of Ψ remains unconstrained (this could be achieved by relating the Finsler structure to a particular quantum gravity scenario, which lies beyond the scope of the present work), any $\Psi(a)$ profile is possible. Thus, from Eq. (32), one can deduce that a large class of scale factor evolution can be realized within the context of Finsler-Randers geometry.

Let us examine the condition of a local small departure from quadraticity, in relation to the accelerated cosmological expansion. In a first approach, we may write

$$\Psi = b_{a;b;c}l^a l^b l^c \sim \frac{\beta}{\lambda_{\mathcal{F}}^2}, \qquad (33)$$

where $\lambda_{\mathcal{F}}$ is a characteristic length scale related to the variation of β. A small value of β corresponds to a sort length scale of the modification. Using the approximation $\int_a u\Psi(u)du \sim \Psi a^2$, together with Eq. (33), the Friedmann

equation [Eq. (32)] for an accelerated phase leads to the approximate relation

$$|\beta| \sim \left(\frac{\lambda_{\mathcal{F}}}{\lambda_H}\right)^2, \tag{34}$$

where $\lambda_H = H^{-1}$ is the Hubble horizon. The above relation is a rough estimation of the β parameter, with respect to the Hubble horizon, in order to obtain an accelerated expansion. Thus, the length $\lambda_{\mathcal{F}}$ represents a characteristic scale above which the gravitational physics is affected. The condition $|\beta| \ll 1$ can be easily fulfilled if $\lambda_{\mathcal{F}}$ is orders of magnitude below the Hubble horizon. For example, if we assume that the gravitational sector is modified above galactic scales (kpc) and take into account that $\lambda_H \sim 10^{10}$ pc, Eq. (34) leads roughly to $|\beta| \sim 10^{-14}$. Hence, interestingly enough, even small departures from Lorentz invariance can lead the cosmic flow to the accelerated phase.

Let us make a comment here on the Lorentz invariance violation. The effective geometric formulation of the present work stands for the geometry of space-time as measured by the comoving observers of the self-gravitating cosmic medium. Thus, the β variable parameterizes possible departures from Lorentz invariance in the gravitational sector. The most stringent constraints of Lorentz violation in the gravitational sector arise from parametrized post-Newtonian (PPN) analysis using Solar System data [45], and the most recent results from Gravity Probe B put an upper bound at 10^{-7} [46]. Therefore, the above representative example lies far inside this window. Note that during the last few years, PPN analysis in Finsler geometry has been developed in Ref. [47]; however, to the best of our knowledge the metric functions that have been used are not of Randers type. Furthermore, the study of Lorentz violation in the gravity sector involves possible future detection of gravitational waves and possible Lorentz violation corrections (see for example Ref. [48]), constraints on the inverse square law and gravitomagnetic effects, CMB anisotropies, and black hole physics [49]. This detailed analysis eventually will also constrain the "Finslerity" of the gravitational sector but lies beyond the scope of this work.

A. Analogue to dark energy and modified gravity

In this subsection, we show that the above Finsler-Randers-modified Friedmann equation [Eq. (32)] can mimic any dark energy scenario, through a specific reconstruction of the $\Psi(a)$—that is, of the Finsler-Randers contribution to the curvature. For this sake, we write

$$\Psi(a) = \Psi_0 X(a)/3, \quad \Psi_0 < 0, \tag{35}$$

$$H(a) = H_0 E_{EFR}(a), \tag{36}$$

and using also the fact that $\rho_m = \rho_{m0} a^{-3}$, the Friedmann equation [Eq. (32)] is written as

$$E_{EFR}^2(a) = \Omega_{m0} a^{-3} + \Omega_{\Psi_0} a^{-2} Q(a). \tag{37}$$

In this expression, we have defined

$$Q(a) = \int_0^a u X(u) du, \tag{38}$$

while the density parameters read as $\Omega_{m0} = \rho_{m0}/3H_0^2$ and $\Omega_{\Psi_0} = -\Psi_0/3H_0^2$, with $\Omega_{m0} + \Omega_{\Psi_0} = 1$.[3] We mention that for mathematical convenience, $Q(a)$ is normalized to unity at the present time.

Now we can return to the aforementioned basic question: Under which circumstances can Eq. (37) resemble that of dark energy? In order to address this crucial question, we need to calculate the effective equation of state parameter (hereafter EoS) $w(a)$ for the EFR cosmology introduced above. We proceed as though we would not know that the original Hubble function is the one given by Eq. (37), and we assume that it behaves according to the typical expansion rate of the Universe where the DE is caused by a scalar field with negative pressure, namely $P_D = w(a)\rho_D(a)$. Therefore, for homogeneous and isotropic cosmologies, driven by nonrelativistic matter and a scalar field DE, the first Friedmann equation is given by

$$E_{DE}^2(a) = [\Omega_{m0} a^{-3} + \Omega_{DE0} f(a)] \tag{39}$$

with

$$f(a) = \exp\left\{-3\int_1^a \left[\frac{1 + w(u)}{u}\right] du\right\}, \tag{40}$$

where $\Omega_{DE0} = \rho_{DE0}/3H_0^2$ is the DE density parameter at the present time, which obeys $\Omega_{m0} + \Omega_{DE0} = 1$.

The next step is to require the equality of the expansion rates of the original EFR picture [Eq. (37)] and that of the DE picture [Eq. (39)], namely $E_{RF}(a) = E_{DE}(a)$ for every scale factor, and by doing so, we extract the integral equation

$$Q(a) = a^2 f(a). \tag{41}$$

Differentiating the above equation, and using Eqs. (38) and (40), we obtain the function $X(a)$ [and thus $\Psi(a)$] in terms of the EoS parameter $w(a)$, as

$$X(a) = -[1 + 3w(a)]f(a). \tag{42}$$

In this viewpoint, if we know *a priori* the effective EoS parameter, then we can obtain via Eq. (42) the Finser-Randers function $X(a)$ and vice versa. Finally, inverting Eq. (42) and utilizing again Eqs. (38) and (40), we find after some simple algebra that

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[3]Practically, defining Ω_{m0} and Ω_{r0} as the standard nonrelativistic and radiation density parameters at the present time, we can find that the complete Hubble function reads as $E_{EFR}^2(a) = \Omega_{m0} a^{-3} + \Omega_{r0} a^{-4} + \Omega_{\Psi_0} a^{-2} Q(a)$ in the limit of $F/\alpha = 1 + \beta/\alpha \sim 1$ (see Sec. III). Note that at the last scattering surface (z_{CMB}), the Hubble horizon is $\lambda_H \sim 2.5 \times 10^5$ pc, which implies that $|\beta| \ll 1$ [see Eq. (34)]. Finally, as usual, the above density parameters satisfy the extended sum rule $\Omega_{m0} + \Omega_{r0} + \Omega_{\Psi_0} = 1$.

$$w(a) = -1 - \frac{a}{3}\left[-\frac{2}{a} + \frac{d\ln Q}{da}\right]. \quad (43)$$

Equation (43) is one of the basic results of our work. It provides the relation of any EoS evolution with the necessary form of the Finsler-Randers geometry. In particular, for a given desired form of $w(a)$, we use Eq. (43) in order to find the corresponding $Q(a)$. Then, through Eqs. (41) and (42) we calculate $X(a)$, and using Eq. (35) we obtain $\Psi(a)$. Finally, with the profile of $\Psi(a)$ in hand, we determine the Friedmann equation of motion [Eq. (32)] that together with the continuity equation [Eq. (30)] fully determines the cosmological evolution.

We stress here that there is not any restriction at all—namely, the above procedure can be applied for any $w(a)$, as long as the corresponding Hubble function is given by Eq. (39), for instance including the quintessence and phantom regimes, the phantom divide crossing from both sides, etc. In the following subsection, without loss of generality, we reconstruct $X(a)$ of the Finsler-Randers metric function for the most familiar cosmological scenarios.

B. Specific examples

In order to proceed to specific examples, the precise functional form of $X(a)$ has to be determined. However, note that this is also the case for any dark energy model, as far as the equation of state (EoS) parameter is concerned. Potentially, in the current work we could phenomenologically treat $X(a)$ [and thus $Q(a)$ and $\Psi(a)$] either as a Taylor expansion around $a = 1$ [$X(a) = X_0 + X_1(1 - a)$] or as a power law $X(a) \propto a^\nu$. Instead of doing that, we have decided to mathematically investigate the conditions under which the Finsler-Randers cosmological model can produce some of the well-known DE models. Below we provide some specific examples along the above lines. In particular, we first consider some literature scalar-field DE models that emerge from FRW cosmology with general relativity, and for these models we reconstruct the functional forms of $\Psi(a) = \Psi_0 X(a)/3$ of the equivalent Finsler-Randers cosmology.

(1) Cosmological constant:
   Inserting $w_\Lambda = -1 = \text{const}$ into Eq. (43), we obtain that $Q(a) = a^2$, which leads to $f(a) = 1$, and thus to $X(a) = 2$.

(2) Quintessence and phantom models with constant w:
   In these constant-w scenarios [2,7,9], DE is attributed to a homogeneous scalar field, with a suitable potential in order to keep the EoS constant, which requires a form of fine-tuning. Specifically, the DE models with a canonical kinetic term of the scalar field lead to $-1 \leq w$, while models of phantom DE ($w < -1$) require an exotic nature, namely a scalar field with negative kinetic energy, which could lead to unstable quantum behavior [50]. Substituting $w(a) = w = \text{const}$ into Eq. (43), we find

$$Q(a) = a^{-(1+3w)} \quad (44)$$

and thus

$$X(a) = -(1 + 3w)a^{-3(1+w)}. \quad (45)$$

In other words, if we desire to construct a quintessence or phantom look-alike Hubble expansion (frequently used in cosmological studies), we need to write $X(a)$ as in Eq. (45).

(3) Chevalier-Polarski-Linder DE:
   We consider the Chevalier-Polarski-Linder parametrization [51], in which the dark energy EoS parameter is defined as a first-order Taylor expansion around the present epoch:

$$w(a) = w_0 + w_1(1 - a). \quad (46)$$

In this case, we straightforwardly obtain

$$Q(a) = a^{-(1+3w_0+3w_1)} \exp\left[-3w_1(1 - a)\right], \quad (47)$$

and therefore

$$X(a) = \frac{3w_1(a - 1) - (1 + 3w_0)}{a^2} Q(a). \quad (48)$$

Similarly to the previous example, if we want to build a CPL look-alike Hubble expansion in the context of Finsler-Randers geometry, then the corresponding functional form of $X(a)$ needs to obey Eq. (48).

(4) Pseudo-Nambu-Goldstone boson scenario:
   In the pseudo-Nambu-Goldstone boson model [52], the dark energy EoS parameter is found with the aid of the potential $V(\phi) \propto [1 + \cos(\phi/p)]$, and it reads

$$w(a) = -1 + (1 + w_0)a^p, \quad (49)$$

where p is a free parameter of the model. Based on this parametrization, the basic ERF functions are given by

$$Q(a) = a^2 \exp\left[-3\frac{1 + w_0}{p}(a^p - 1)\right] \quad (50)$$

and

$$X(a) = \frac{2 - 3(1 + w_0)a^p}{a^2} Q(a). \quad (51)$$

(5) $f(T)$ gravity:
   Let us now give an example of how we can reconstruct the functional forms of $X(a)$ and $Q(a)$ of the equivalent Finsler-Randers cosmology in the case of a modified gravitational model. As a specific case, we choose the $f(T)$ construction, which is based on the teleparallel equivalence of general relativity. In this formulation the gravitational information is included in the torsion tensor and the corresponding

torsion scalar T, and one extends the Lagrangian considering arbitrary functions $f(T)$ [13,14]. Within such a framework, the Hubble function is written as

$$H^2 = \frac{8\pi G}{3}\rho_m - \frac{f(T)}{6} - 2f_T H^2, \quad (52)$$

where $T = -6H^2$ is the torsion scalar and $f_T = \partial f(T)/\partial T$. Based on the matter epoch, defining $E_{FT}^2(a) = H^2(a)/H_0^2$ and using $\rho_m = \rho_{m0}a^{-3}$, the above equation always becomes

$$E_{FT}^2(a) = \Omega_{m0}a^{-3} + \Omega_{F0}y(a), \quad (53)$$

where $\Omega_{F0} = 1 - \Omega_{m0}$. The function $y(a)$ is scaled to unity at present time and is given by

$$y(a) = -\frac{1}{\Omega_{F0}}\left[\frac{f(T)}{6H_0^2} + 2f_T E^2(a)\right]. \quad (54)$$

Comparing Eqs. (53) and (54) with Eq. (37), we find that

$$Q(a) = \frac{y(a)}{a^2} \quad (55)$$

and

$$X(a) = \frac{\frac{dy}{d\ln a} - 2y(a)}{a^4}. \quad (56)$$

As an example, we use the power-law model of Bengochea and Ferraro [13] with

$$f(T) = \alpha(-T)^b, \qquad \alpha = (6H_0^2)^{1-b}\frac{\Omega_{F0}}{2b-1}, \quad (57)$$

where b is the free parameter of the model, which has to be less than unity in order to ensure a cosmic acceleration. Inserting Eq. (57) into Eq. (54), we arrive at $y(a) = E^{2b}(a) = a^2 Q(a)$. Obviously, for $b = 0$ the power-law $f(T)$ model reduces to the ΛCDM model, while for $b = 1/2$ it reduces to the DGP one [11], which implies that potentially we have a cosmological equivalence among the EFR, DGP, and $f(T)$ power-law gravity models. Note that, as we said in the Introduction, the equivalence of DGP with Finsler-Randers cosmology was already found by some of us in Ref. [23].

In summary, from the above analysis and the specific examples, it becomes clear that the DE scenarios (including some modified gravity models) that satisfy Eq. (39) can be seen as equivalent to the geometrical EFR cosmological model.

Finally, in a forthcoming publication, we attempt to physically derive the precise functional form of $X(a)$, as well as to provide a full perturbation analysis, which can be used in order to distinguish the Finsler-Randers scenario from other DE and modified gravity models [53].

IV. CONCLUSIONS

In the present work, we investigated an extended form of Finsler-Randers cosmology, and we showed that it can mimic any noninteracting dark energy scenario, as well as modified gravity models, at the background level. In particular, we started from a small deviation from the quadraticity of the Riemannian geometry, and we extracted the modified Friedmann equation that determines the Universe evolution.

The effect of the Finsler-Randers modification is to produce correction terms to the Friedmann equation that can lead to a large class of scale factor evolution, including the quintessence and phantom regimes, the phantom divide crossing from both sides, etc. As we showed, for a given dark energy equation of state parameter, we can reconstruct the corresponding functions of the Finsler-Randers space that indeed give rise to such a behavior, and vice versa. Therefore, the present work is a completion of the previous works of some of us [22,23], wherein we showed the equivalence of Finsler-Randers cosmology with particular modified gravitational models as the DGP one, since we now show that the extended Finsler-Randers cosmology can resemble a large class of cosmological scenarios.

In this respect, the nontrivial Universe evolution, and especially its accelerated phase either during inflation or at late times, is not attributed to a new scalar field, or to gravitational modification, but it arises from the modification of the geometry itself. In particular, even a very small nonquadraticity of the Finsler-Randers geometry, in which the local structure of general relativity is modified and the curvature theory is extended, can lead to significant implications for the cosmological evolution. One should still provide an explanation for the origin of the Finsler-Randers geometry itself, and the small departure from the Riemann one. Although there are indications that this must be related to quantum gravity effects [21,27–33], this issue lies beyond the scope of the present work and is left for a future investigation.

We close this work by making two comments. The first is that, as we discussed in the text, our analysis is valid at intermediate and late times, including the radiation era, where all the energy conservations hold as usual. The second is that the above equivalence between Finsler-Randers geometry and dark energy and modified gravity models has been obtained at the background level—that is, demanding the same scale factor evolution. However, a necessary step is to proceed to a detailed analysis of the cosmological perturbations and see whether the aforementioned equivalence breaks, which would allow us to distinguish between the various scenarios (this was indeed the case in the equivalence of the simple Finsler-Randers geometry with the DGP model [23]), or whether it is maintained, in which case the degeneracy of the above

constructions would be complete. This complicated and detailed investigation is in progress [53].

APPENDIX: THE 1 + 3 COVARIANT FORMALISM

We briefly summarize the 1 + 3 covariant formalism as developed by J. Ehlers and G. F. R. Ellis [25]. The notation in this appendix is for the Riemannian limit that will serve as guidance through our calculations to the Finslerian case. The covariant approach employs a timelike vector field u^a with $u_a u^a = 1$. With respect to this normalized vector field, we split space-time into time and space. The 1 + 3 split is a particular case of the tetrad formalism, where the u^a congruence represents the frame of comoving observers. With respect to the 4-velocity u^a, we can decompose all tensors to their irreducible parts. In particular, using the projection second-rank tensor $h_{ab} = g_{ab} - u_a u_b$, we can covariantly define the time derivative and the spatial gradient of an arbitrary tensor field, namely

$$\dot{S}_{ab..}{}^{cd..} = u^e \nabla_e S_{ab..}{}^{cd..}, \qquad (A1)$$

$$D_e S_{ab..}{}^{cd..} = h_e{}^s h_a{}^f h_b{}^p h_q{}^c h_r{}^d \ldots \nabla_s S_{fp..}{}^{qr..}. \qquad (A2)$$

Instead of writing the metric to a particular coordinate system, the geometry as measured by the u^a family of observers is described by the irreducible parts of the following tensor field:

$$D_b u_a = \frac{1}{3} \Theta h_{ab} + \sigma_{ab} + \omega_{ab}, \qquad (A3)$$

where we define the kinematic quantities as follows: the expansion $\Theta = D^a u_a$, the shear $\sigma_{ab} = D_{\langle b} u_{a\rangle}$, and the vorticity $\omega_{ab} = D_{[b} u_{a]}$. The projective symmetric and trace-free part is defined as

$$X_{\langle ab\rangle} = h^c{}_{(a} h^d{}_{b)} X_{cd} - \frac{1}{3} X_{cd} h^{cd} h_{ab}, \qquad (A4)$$

where indices in squared brackets indicate the symmetrized part.

In case of a shear- and vorticity-free expanding congruence of geodesics, the evolution of the deviation vector that connects nearby observers is $\dot{\xi}_a = \frac{1}{3} \Theta \xi_a$ [25]. Taking the time derivative of the previous expression, and substituting into the deviation of geodesics $\ddot{\xi}_a + R_{acbd} u^c u^d \xi^b = 0$, gives back Raychaudhuri's equation

$$\dot{\Theta} + \frac{1}{3} \Theta^2 = -R_{ab} u^a u^b. \qquad (A5)$$

The energy-momentum tensor of pressureless matter is $T_{ab} = \rho u_a u_b$, and the contracted Einstein's field equations along the observers' 4-velocity give back the auxiliary expression, $R_{ab} u^a u^b = \frac{1}{2} \rho$. Moreover, an important geometric entity is the characteristic length scale of the expanding 3D cross section, namely the scale factor a given by $dV \propto a^3$. The reader should notice that the scale factor is covariantly defined, in contrast to the metric-based approach, where it is introduced through a particular coordinate system. Thus, for the expansion we obtain $\Theta = (dV)\dot{}/dV = 3\dot{a}/a$, and therefore we can rewrite Eq. (A5) in the form

$$3 \frac{\ddot{a}}{a} = -\frac{1}{2} \rho. \qquad (A6)$$

The physical requirement of pressureless matter implies that for a conservative system, the matter energy density scales with the volume element—that is, $\rho dV = $ const. Furthermore, using the definition for the scale factor, we acquire $(\rho a^3)\dot{} = 0$. (Alternatively, one may decompose the energy momentum conservation law $\nabla^b T_{ab} = 0$ to its irreducible parts [25].) The latter, together with Eq. (A6), fully determines the evolution of the dustlike medium (for further details, see for example Ref. [43]).

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