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FINSLERIAN DEVIATIONS OF GEODESICS OVER TANGENT BUNDLE

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The geodesic deviation equation is derived in the framework of the fibered Finslerian gauge approach, under the general conditions when all the Finslerian curvature and torsion tensors are taken into account. Just as the Riemannian curvature tensor enters the equation of deviation of Riemannian geodesics, the Yang–Mills-type gauge tensor proves to govern the behaviour of the vertical part of the Finslerian deviations, thereby exhibiting its observable nature.

1. Introduction

The profound role of the equation of deviations of Riemannian geodesics has been recognized in the general relativity for a long time (see, e.g., [1]). In contrast to the equations of geodesics proper, the deviation equation relates immediately to the observation conditions, for any real observer follows one or another trajectory traced in the gravitational field.

In the framework of the classical Finsler geometry the deviation equation was treated in Rund's monograph [2] and was checked, partly, for possible observable consequences in [3]. However, to synthesize the general relativity with the Yang–Mills field theory one must use the fibered Finslerian approach [4], in contrast to the classical Finslerian approach [2] which was constructed to be the theory for concomitants of the Finslerian metric function $F(x, y)$ prescribing the length of the tangent vector y^i, and dealt with a single metric tensor, namely,

$$g_{ij}^{\text{Classical}} = \tfrac{1}{2}\partial^2 F^2/\partial y^i \partial y^j .$$

The classical approach was constructed directly by generalizing the Riemannian definition of the length of vector, whereas the fibered Finslerian approach proceeds from the difference between the horizontal and vertical geometrical entities. The present work, like its precursors [4, 5], is based on the concepts of the fibered approach because it is in terms of the latter that the feasibility of unifying the relativity and the theory of Yang–Mills gauge fields in a geometrical way proves to be directly evident. The deviation equation derived in Section 2 indicates explicitly, although qualitatively, how the geometry of a fibre influences the behaviour of the background geodesics, namely new additional terms appear, which play the role of additional forces.

2. Derivation of generalized equation of deviations of geodesics

The notes given in Introduction motivate the study of deviations of geodesic equations introduced in terms of the fibered Finslerian approach. The geodesics of such type were derived in [5]. Following this reference, we consider a fibration related to local coordinates (x^i, z^P), where x^i and z^P are coordinates of points of the background manifold and of the fibre, respectively $(i, j, \ldots = 1, \ldots, N$ and $P, Q, \ldots = 1, \ldots, M)$. We shall follow the notation adopted in Appendix B of the book [5]. So, we introduce two (nondegenerate and symmetric) metric tensors, namely, $a_{ij}(x,z)$ in the background manifold and $g_{PQ}(x,z)$ in the fibre.

The covariant derivatives are given by

$$D_i a_{mn} = d_i a_{mn} - L_m{}^k{}_i a_{kn} - L_n{}^k{}_i a_{mk}\,, \tag{1}$$

$$S_R a_{mn} = d_R a_{mn}\,, \tag{2}$$

$$D_i g_{PQ} = d_i g_{PQ} - D_P{}^R{}_i g_{RQ} - D_Q{}^R{}_i g_{PR}\,, \tag{3}$$

$$S_R g_{PQ} = d_R g_{PQ} - Q_P{}^S{}_R g_{SQ} - Q_Q{}^S{}_R g_{PS}\,, \tag{4}$$

where

$$d_i = \partial_i + N_i^P d_P\,, \quad \partial_i = \partial/\partial x^i\,, \quad d_P = \partial/\partial z^P\,. \tag{5}$$

Generally, the connection coefficients N, L, D, Q are functions of x^i and z^P. The associated torsion and curvature tensors are given by the formulae,

$$S_i{}^k{}_j = L_i{}^k{}_j - L_j{}^k{}_i\,, \quad Y_P{}^T{}_R = Q_P{}^T{}_R - Q_R{}^T{}_P\,, \tag{6}$$

$$A_j{}^Q{}_R = -d_R N^Q{}_j - D_R{}^Q{}_j\,, \quad M_j{}^R{}_i = d_j N_i^R - d_i N_j^R \tag{7}$$

and

$$L_n{}^k{}_{ij} = d_j L_n{}^k{}_i - d_i L_n{}^k{}_j - L_n{}^m{}_j L_m{}^k{}_i + L_n{}^m{}_i L_m{}^k{}_j\,, \tag{8}$$

$$B_n{}^k{}_{jR} = d_R L_n{}^k{}_j\,, \tag{9}$$

$$E_P{}^Q{}_{ij} = d_j D_P{}^Q{}_i - d_i D_P{}^Q{}_j - D_P{}^R{}_j D_R{}^Q{}_i + D_P{}^R{}_i D_R{}^Q{}_j + Q_P{}^Q{}_R M_i{}^R{}_j\,, \tag{10}$$

$$Q_P{}^T{}_{Rj} = d_j Q_P{}^T{}_R + Q_P{}^T{}_S d_R N_j^S - D_P{}^S{}_j Q_S{}^T{}_R + D_S{}^T{}_j Q_P{}^S{}_R - d_R D_P{}^T{}_j\,, \tag{11}$$

$$Z_P{}^Q{}_{MN} = d_N Q_P{}^Q{}_M - d_M Q_P{}^Q{}_N - Q_P{}^S{}_N Q_S{}^Q{}_M + Q_P{}^S{}_M Q_S{}^Q{}_N\,, \tag{12}$$

respectively.

The geodesics over the vibration under consideration can be defined in a natural way by means of the Lagrangian

$$L = \alpha L_{(h)} + \beta L_{(v)}\,, \tag{13}$$

$$L_{(h)} = (a_{ij}(x,z)\dot x^i \dot x^j)^{1/2}\,, \quad L(v) = (g_{PQ}(x,z)a^P a^Q)^{1/2} \tag{14}$$

via the variational principle

$$\delta \int_C L dt = 0\,, \tag{15}$$

where $C = (x^i(t), z^P(t))$ is a curve on the fibration parametrized by some parameter t;

$$\dot x^i = dx^i/dt$$

and

$$a^P = \dot z^P - N_i^P(x,z)\dot x^i \tag{16}$$

is the covariant velocity of the element z^P of the fibre; $\dot z^P = dz^P/dt$; α and β are constants. Using the formulae presented on p. 298 of [5], we can conclude that the Euler–Lagrange equations associated with (13)–(15) read

$$D\dot x^i/dt = F^i \tag{17}$$

and

$$Da^P/dt = F^P \tag{18}$$

with

$$F^i = \dot x^i d\ln L_{(h)}/dt - \dot x^n a^P a^{im} S_P a_{mn} - a^{ik}\dot x^m \dot x^n (D_n a_{km} - \tfrac{1}{2}D_k a_{mn} + S_{kmn}) - \alpha^{-1}\beta L_{(h)} L_{(v)}^{-1}\left[a^P(\dot x^n M^i{}_{Pn} + a^Q A^i{}_{PQ}) - \tfrac{1}{2}a^P a^Q D^i g_{PQ}\right]\,, \tag{19}$$

and

$$F^P = a^P d\ln L_{(v)}/dt + a^Q \dot x^i A_{iQ}{}^P - \dot x^i a^R g^{PT} D_i g_{TR} - g^{PT} a^Q a^R (S_R g_{TQ} - \tfrac{1}{2}S_T g_{QR} + Y_{TQR}) + \tfrac{1}{2}\alpha\beta^{-1} L_{(v)} L_{(h)}^{-1} \dot x^i \dot x^j S_P a_{ij} \tag{20}$$

playing the role of forces produced by the geometry of the fibre;

$$D\dot x^i/dt = d\dot x^i/dt + L_k{}^i{}_j(x(t), z(t))\dot x^k \dot x^j\,, \tag{21}$$

$$Da^P/dt = da^P/dt + Q_Q{}^P{}_R(x(t), z(t))a^Q a^R + D_Q{}^P{}_i(x(t), z(t))a^Q \dot x^i\,. \tag{22}$$

The equations (17) and (18) represent, respectively, the horizontal and vertical parts of the equation of geodesics.

Now, we extend the method used in Chap. IV, § 4 of the Rund's book [2]. Namely, we shall consider a two-parametric family of geodesics. The first parameter will be the t used in the previous formulae, while the second parameter, hereafter denoted by v, will enumerate members of the family. So we have,

$$x^i = x^i(t,v)\,, \quad z^P = z^P(t.v)\,. \tag{23}$$

Denoting

$$\xi^i = \partial x^i/\partial t\,, \quad \eta^i = \partial x^i/\partial v\,, \tag{24}$$

we have

$$\partial\xi^i/\partial v = \partial\eta^i/\partial t \tag{25}$$

and further

$$\delta_t\xi^i \overset{\text{def}}{=} \xi^j D_j\xi^i = \partial_t\xi^i + L_k{}^i{}_j\xi^k\xi^j\,, \tag{26}$$

$$\delta_v\xi^i \overset{\text{def}}{=} \eta^j D_j\xi^i = \partial_v\xi^i + L_k{}^i{}_j\xi^k\eta^j\,. \tag{27}$$

Similarly,

$$\delta_t\eta^i = \partial_t\eta^i + L_k{}^i{}_j\eta^k\xi^j\,, \tag{28}$$

$$\delta_v\eta^i = \partial_v\eta^i + L_k{}^i{}_j\eta^k\eta^j\,. \tag{29}$$

Comparing (27) with (28) shows that

$$\delta_v\xi^i = \delta_t\eta^i + S_k{}^i{}_j\xi^k\eta^j\,. \tag{30}$$

η^i is the deviation vector. Because of (30), we have

$$(\delta_v\delta_t - \delta_t\delta_v)\xi^i = \eta^m\xi^n(\delta_m\delta_n - \delta_n\delta_m)\xi^i + S_k{}^j{}_m\xi^k\eta^m\delta_j\xi^i\,, \tag{31}$$

where (as in the equation (4.5), p. 113 of [2]) the ξ^i are regarded as functions of x (the first member of (23) can be inverted to give t and v as functions of x^i) and δ_m is the covariant derivative taken with the osculating connection coefficients

$$\Gamma_i{}^m{}_n(x) = L_i{}^m{}_n(x, z(x))\,. \tag{32}$$

Therefore,

$$(\delta_m\delta_n - \delta_n\delta_m)\xi^i = S_m{}^k{}_n\delta_k\xi^i - K_k{}^i{}_{mn}\xi^k \tag{33}$$

with $K_k{}^i{}_{mn}(x)$ being the Riemannian curvature tensor constructed from the connection coefficients (32):

$$K_k{}^i{}_{mn} = \partial_n\Gamma_k{}^i{}_m - \partial_m\Gamma_k{}^i{}_n - \Gamma_k{}^l{}_n\Gamma_l{}^i{}_m + \Gamma_k{}^l{}_m\Gamma_l{}^i{}_n\,. \tag{34}$$

Comparing (34) with the definitions (8) and (9) shows that,

$$L_k{}^i{}_{mn}(x, z(x)) = K_k{}^i{}_{mn} - z_n^P B_k{}^i{}_{mP}(x, z(x)) + z_m^P B_k{}^i{}_{nP}(x, z(x))\,, \tag{35}$$

where

$$z_n^P = \partial z^P(x)/\partial x^n - N_n^P(x, z(x))\,. \tag{36}$$

Since the right-hand sides of the definitions (26) and (21) are equal, we can use (17) to get

$$\delta_v\delta_t\xi^i = \delta_v F^i\,, \tag{37}$$

Finally, using (30) in (31), we find the *horizontal part of the deviation equation* in the following form:

$$\delta_t(\delta_t\eta^i + S_k{}^i{}_j\xi^k\eta^j) = K_k{}^i{}_{mn}\xi^k\xi^n\eta^m + \delta_v F^i\,. \tag{38}$$

This procedure can be repeated for deriving the equation for the vertical part of deviations. With this purpose we put

$$\lambda^P = \partial_t z^P - N_i^P\xi^i\,, \tag{39}$$

$$\nu^P = \partial_v z^P - N_i^P\eta^i \tag{40}$$

(cf. (16) and (24)). Like (16), the definitions (39) and (40) will be covariant under the gauge transformations

$$z^P = Z^P(x, z'^Q) \tag{41}$$

(the formula (2) on p. 236 of the book [5]), namely, they will be contravariant vectors under (41)

$$\lambda^P = Z_Q^P\lambda'^Q\,, \quad \nu^P = Z_Q^P\nu'^Q \tag{42}$$

(because $dz^P - N_i^P dx^i$ is the covariant differential of z^P); $Z_Q^P = \partial z^P/\partial z'^Q$. The action of the covariant derivative δ_m on λ^P and ν^P will read:

$$\delta_m\lambda^P = \partial_m\lambda^P + d_Q{}^P{}_m\lambda^Q\,, \tag{43}$$

$$\delta_m\nu^P = \partial_m\nu^P + d_Q{}^P{}_m\nu^Q\,, \tag{44}$$

where

$$d_Q{}^P{}_m(x) = D_Q{}^P{}_m(x, z(x)) + z_m^R Q_Q{}^P{}_R(x, z(x)) \tag{45}$$

(cf. (32)). Further,

$$\delta_i\delta_j\lambda^Q = \partial_i\delta_j\lambda^Q - \Gamma_j{}^k{}_i\delta_k\lambda^Q + d_R{}^Q{}_i\delta_j\lambda^R \tag{46}$$

which entails

$$(\delta_i\delta_j - \delta_j\delta_i)\lambda^Q = S_i{}^k{}_j\delta_k\lambda^Q - \varepsilon_P{}^Q{}_{ij}\lambda^P\,, \tag{47}$$

where

$$\varepsilon_P{}^Q{}_{ij} = \partial_j d_P{}^Q{}_i - \partial_i d_P{}^Q{}_j - d_P{}^R{}_j d_R{}^Q{}_i + d_P{}^R{}_i d_R{}^Q{}_j = E_P{}^Q{}_{ij}(x, z(x)) - z_j^R Q_P{}^Q{}_{Ri}(x, z(x)) + z_i^R Q_P{}^Q{}_{Rj}(x, z(x)) + z_i^R z_j^S Z_P{}^Q{}_{RS}(x, z(x))\,. \tag{48}$$

The definitions (43)–(45) can be explained as follows. Let $W^P(x, z)$ be a vector under the transformation (41), that is , $W^P(x, z) = Z_Q^P W'^Q(x, z')$. Then, as was explained in Section B.2 of [5], the covariant derivative of W^P with respect to x^m must read

$$D_m W^P = d_m W^P + D_Q{}^P{}_m W^Q \equiv W_m^P\,.$$

If we consider a field $z^P(x)$ and put $w^P(x) = W^P(x, z(x))$, we shall have

$$W_m^P(x, z(x)) = \delta_m w^P - z_m^Q(S_Q w^P)|z = z(x) \tag{49}$$

with

$$\delta_m w^P = \partial w^P/\partial x^m + d_Q{}^P{}_m w^Q\,, \tag{50}$$

where $d_Q{}^P{}_i$ are given by (45). Obviously, since the left-hand side as well as the second term on the right-hand side of (49) are covariant under (41), the definition (50) also will be covariant under (41). In (50) the field $w^P(x)$ obeys transformation law just of the type (42). Therefore, the definition (50) is applicable to our λ^P and ν^P.

Thus,

$$\eta^i \xi^j (\delta_i \delta_j - \delta_j \delta_i) \lambda^Q = \eta^i \xi^j [S_i{}^k{}_j \delta_k \lambda^Q - \varepsilon_P{}^Q{}_{ij} \lambda^P] . \tag{51}$$

At the same time, because of (45) and the identity $z^R_m \dot{x}^m = a^R$, the right-hand side of (22) is equal to $\xi^m \delta_m \lambda^P$, whence

$$\delta_t \lambda^P \stackrel{\text{def}}{=} \xi^m \delta_m \lambda^P = F^P \tag{52}$$

in accordance with (18). So, recollecting (30), we find that

$$\eta^i \xi^j (\delta_i \delta_j - \delta_j \delta_i) = \eta^i \delta_i \delta_t - \eta^j \delta_j \delta_v - S_k{}^j{}_i \xi^\kappa \eta^i \delta_j , \tag{53}$$

which reduces (51) to

$$\xi^j \delta_j (\delta_v \lambda^Q) = \varepsilon_P{}^Q{}_{ij} \eta^i \xi^j \lambda^P + \delta_v F^Q . \tag{54}$$

Finally, simple calculations show that

$$\delta_v \lambda^Q = \delta_t \nu^Q - A_m{}^Q{}_R (\lambda^R \eta^m - \nu^R \xi^m) + \xi^n \eta^i M_n{}^Q{}_i + Y_R{}^Q{}_S \lambda^R \nu^S \tag{55}$$

(cf. (30)), where $\delta_t \nu^Q = \xi^m \delta_m \nu^Q$. Inserting (55) in the left-hand side of (54), we find eventually that the *vertical part of the deviation equations reads:*

$$\delta_t [\delta_t \nu^Q - A_m{}^Q{}_R (\lambda^R \eta^m - \nu^R \xi^m) + \xi^n \eta^i M_n{}^R{}_i + Y_R{}^Q{}_S \lambda^R \nu^S] = \varepsilon_P{}^Q{}_{ij} \eta^i \xi^j \lambda^P + \delta_v F^Q . \tag{56}$$

The Yang–Mills-type gauge tensor $\varepsilon_P{}^Q{}_{ij}$ enters (56) instead of the curvature tensor $K_k{}^i{}_{mn}$ in the horizontal part (38).

3. Conclusions

In the Riemannian case proper, the equation of deviations of geodesics plays an important role because the Riemannian curvature tensor $R_i{}^j{}_{mn}(x)$ enters the equation in full. This circumstance enables the tensor $R_i{}^j{}_{mn}$ to be interpreted physically as an observable, and the way of measuring all of its components to be proposed.

May we belive that a similar interpretation is admitted by the Yang–Mills-type gauge tensor? The positive answer is given explicitly by the equation (56), for the Yang–Mills-type gauge tensor $\varepsilon_P{}^Q{}_{ij}$ enters (56) (as compared with the background curvature tensor $K_k{}^i{}_{mn}$ in the horizontal part (38)). In particular, this conclusion completes the comparison between the Yang–Mills-theory and the gravitation made in [6]. Of course, the Yang–Mills gauge tensor enters the Wong equation [7] for particles with isotopic spin. However, the latter equation is not of any geometrical meaning (cf. also [8]), describing merely a quasiclassical limit of the relevant Heisenberg-type equation.

Obviously, the magnitude of the influence of the tensor $\varepsilon_P{}^Q{}_{ij}$ on the behaviour of the geodesic deviations depends on the estimations. These could be proposed only after finding sufficiently exact solutions for the associated Finslerian field equations in order that the form of all the torsion tensors entering the deviation equations (38) and (56) should be specified. This must be the subject of further investigations.

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