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TIDAL FORCES IN VERTICAL SPACES OF FINSLERIAN SPACE-TIME

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The classical equation of geodesic deviation is extended in the case of vertical geodesics associated with a Finslerian space-time. It is shown that the deviations can appear only if the vertical component of the energy-momentum tensor differs from zero.

1. Introduction

The equation of geodesic deviation plays an important part in the Riemannian geometrization of the gravitational field because the equation describes observable phenomena and involves the full curvature tensor [4, 6, 8, 10]. The deviations of geodesically-moving particles are caused by the curvature of the space-time and can describe the tidal forces. These forces can vary the relative velocity, giving rise to approaching or diverging of the particles. In general, the relative accelerations D^2n^i/ds^2 will not be zero, unless the Riemannian curvature tensor vanishes.

In the Finslerian approach, there are various ways that can be used in order to derive the equation of deviations from the background geodesics [7, 9]. Following the convenient method described in Section 4.4 of [7], the derivation starts by considering a two-parameter family $x^i = x^i(u,v)$ of geodesics, where u is the parameter of the arc-length. Then, if z^i denotes a vector joining two corresponding points on nearby geodesics, the deviation equation can be found in the form

$$\delta^2 z^i/\delta u^2 + R^i_{j\,hk}(x,\xi)\xi^j\xi^h z^k = 0 \tag{1.1}$$

(see p. 116 of [7]). Here, $\xi^i = \partial x^i/\partial u$, $\delta z^i/\delta u = z^i_{|h}\xi^h$, where

$$z^i_{|h} = \partial z^i/\partial x^h + F^i_{h\,k}(x,\xi)z^k;$$

F^i_{hk} are the Cartan connection coefficient, and $R^i_{j\,hk}$ is the Cartan curvature tensor of the background manifold. Thus, $R^i_{j\,hk} = 0$ would imply the vanishing of the Finslerian tidal forces represented by $\delta^2 z^i/\delta u^2$. In this respect, it will be noted that in (1.1) the extension of the Einstein equations to the Finslerian case was proposed in the form,

$$R_{mn} - \tfrac{1}{2}g_{mn}R = kT_{mn}, \tag{1.2}$$

where $R_{mn} = R^i_{mni}$ and $R = g^{mn}R_{mn}$. Accordingly, the solutions of such equations would govern the behaviour of the Finslerian tidal forces through equation (1.1). Interesting physical applications of the deviation equation (1.1) were considered in [1].

However, the curvature of a Finsler space is characterized not only by the tensor $R_{i\ mn}^{j}$ but also by the tensors $S_{i\ mn}^{j}$, $P_{i\ mn}^{j}$ and $K_{i\ mn}^{j}$ [2, 5, 7]. Thus, the question arises if it is possible to find a full interpretation of the curvature of a Finsler space in terms of geodesic deviations. To this end it will be noted that from a general viewpoint the Finsler space is a fibered space, so that, on introducing the generalized element of length in accordance with,

$$d\sigma^2 = g_{ij}(x,y)dx^i dx^j + g_{ij}(x,y)Dy^i Dy^j, \tag{1.3}$$

where $Dy^i = dy^j + N_i^j(x,y)dx^i$, the paths of free particles are primarily defined on all the tangent bundle. Deviations of such curves $(x^i(t), y^i(t))$ will be examined in a separate paper. Besides, the geodesics inside the tangent spaces can be introduced in a natural way, and our aim will be to derive the deviation equation for such vertical geodesics.

2. Deviation of vertical geodesics

If we consider the tangent Riemannian space $M(x)$ defined by the metric

$$ds^2 = g_{ij}(x,y)dy^i dy^j, \tag{2.1}$$

where x^i are assumed to be fixed and y^i are arbitrary, then we can introduce the geodesics on $M(x)$ by means of the variational principle

$$\delta \int ds = 0 \tag{2.2}$$

which entails the geodesic equation

$$d^2y^i/ds^2 + C_{mn}^i(x,y)dy^m/dsdy^n/ds = 0, \tag{2.3}$$

where

$$C_{mjn} = \tfrac{1}{2}\partial g_{mj}/\partial y^n$$

is the so-called Cartan torsion tensor (cf. p. 29 of [2]). Let $F_x^{(2)}$ be a two-dimensional geodesic surface in $M(x)$, so that any point of $F_x^{(2)}$ can be represented parametrically by the equation

$$y^i = y^i(u,v)$$

(of class C^4). These u and v are Gaussian parameters of the surface. The vectors tangent to the parameter lines u =const and v =const will be defined as

$$\xi^i = \partial y^i(u,v)/\partial u, \qquad \eta^i = \partial y^i(u,v)/\partial v \tag{2.4}$$

so that

$$\partial\xi^i/\partial v = \partial\eta^i/\partial u. \tag{2.5}$$

The infinitesimal deviation vector between two nearby geodesics will be given by

$$z^k = (\partial y^k/\partial u)du + (\partial y^k/\partial v)dv \tag{2.6}$$

according to the definition of the covariant derivative in $M(x)$, we have

$$D_j\xi^i = \partial\xi^i/\partial y^j + C^i_{jk}\xi^\kappa \tag{2.7}$$

which entails directly that

$$(D_k\xi^i)\eta^k = (D_k\eta^i)\xi^k. \tag{2.8}$$

Using the commutation relation

$$D_jD_m\xi^i - D_mD_j\xi^i = F^{-2}S^i_{kmj}\xi^k, \tag{2.9}$$

where

$$F^{-2}S^i_{kmn} = C^i_{hm}C^{hn}_k - C^i_{hn}C^h_{km} \tag{2.10}$$

is the curvature tensor of $M(x)$, we get

$$(D_jD_m\xi^i - D_mD_j\xi^i)\xi^j\eta^m = F^{-2}S^i_{kmn}\xi^k\xi^j\eta^m. \tag{2.11}$$

From the last relation it can readily be concluded that, using the arc-length s defined by (2.1) to be the parameter u and choosing the deviation vector z^k to be related to the case $du = 0$, so that $\eta^k || z^k$, the sought equation of geodesic deviations reads

$$D^2\eta^i/ds^2 + F^{-2}S^i_{jkm}\xi^j\xi^k\eta^m = 0. \tag{2.12}$$

It will be noted that this equation can be related to the so-called Einstein equations in the tangent space

$$S_{ij} - \tfrac{1}{2}Sg_{ij} = kt_{ij} \tag{2.13}$$

discussed in [3, 11], where t_{ij} plays the role of the internal component of the energy-momentum tensor. In the physical four-dimensional case, the vanishing $t_{ij} = 0$ would imply $S^j_{imn} = 0$, for $S_{ij} = 0$ is equivalent to $S^j_{imn} = 0$ because of the special structure (1.129) of [2] derived for the tensor S^j_{imn}.

3. Discussion

Similar to the Riemannian approaches, the Finslerian curvatures of the space-time reveal themselves in the behaviour of deviations of test particles following nearby geodesics, giving rise to the relative accelerations towards or away from each other. At the same time, the deviations of the vertical geodesics are peculiar in that they imply the existence of the vertical component of the energy-momentum tensor, so that in case of "the empty vertical space", the vertical geodesics are free of any deviations.

An interpretation of the horizontal-vertical curvature tensor in terms of geodesic deviations will be presented elsewhere.

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