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MOTIONS IN A CONTINUOUSLY DEFORMED BACKGROUND.

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Introduction. The purpose of this paper is to relate ideas on differential geometry to the Kinematical concepts which are traditionally used in classical and relativistic particle mechanics. This means that instead of hanging uncritically physical names on mathematical objects we search for a suitable framework in which the various Kinematical entities of classical and relativistic physics have a precise and explicit meaning. In this framework the ambiguity which is inherent in the use of many physical terms, when they are applied to curved space-time, completely disappears. In addition, it turns out, that the very concept of an *inertial* frame as well as that of a non inertial one can be also given a precise and natural meaning. Clearly, the notion of an arbitrary frame of reference, introduced here, is much more restrictive compared to the traditional one, [**5**][1] [**7**], where the notion of an observer is mixed with the choice of a particular, more or less, system of coordinates. However, this is not a disadvantage. On the contrary, the mathematical clarification of the aforementioned concept demonstrates explicity the subtle relation which exists between the underlying manifold and the frames of reference (inertial and noninertial) which are admitted by this particular manifold. In this sense our approach is essentially different from that of Wei-Tou Ni and M. Zimmermann, [**7**] as well as from that of B. Defacio et al, [**2**], [**3**] although terms of common origin can be identified in all cases.

In this paper a general formula is obtained which relates accelerations and deformation in various permissible frames of reference[2]. The implicit content of this formula is further explored by Theorem 1, which enable us to define the curved spaces for which inertial backgrounds exist.

Throughout this paper, all data such as manifolds, mappings tensor fields etc. are assumed to be differentiable up to necessary times and "space" always means a connected paracompact manifold endowed with an affine connection.

The more interesting case where the underlying manifold is a Finsler space is also discussed at the end of § **3**.

§ 1. Preliminaries. Let M be any connected paracompact manifold endowed with a symmetric linear connection (which is not necessarily a Riemannian one) and let G be a Lie group acting effectively on M,

$$\varphi : G \times M \to M,$$

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1) Numbers in brackets refer to the references at the end of the paper.

2) This formula has been considered previously by the first of the authors in a different context and by a slightly different approach [**1**].

as a Lie Group of transformations of M. According to our definitions φ is differentiable and the restriction map $\varphi_g: M \to M$ is a transformation of M, i.e. a diffeomorphism of M into itself, for any $g \in G$. For our purposes it is convenient to consider a coordinate system (x) in M and a suitable neighborhood $U \subseteq M$ such that both U and $\varphi_g(U)$ are covered by (x), for the action of the elements of G which are sufficiently close to the identity of G.

Suppose that $g \in G$ is an arbitrary but otherwise fixed element of the aforementioned kind, then the transformation φ_g deforms, in general, the geometry of M i.e. the connection which is defined in M. Strictly speaking we have a *dragging along* of the coordinate system (x) as well as of the connection by the transformation φ_g[3]. This dragging along procedure can be repeated for any geometric object which is defined at $m \in U$. Hence, if $c: \boldsymbol{R} \to M$, is the orbit of a hypothetical pointlike constituent passing through the point $m \in U$, then for the section of the transformed orbit $\varphi_g \circ c$, which lies in the coordinate systems (x) we have the equations.

$$(1.1) \qquad x^i_{\varphi(m)}(t) = \varphi^i(g, x_m(t)) \qquad g \in G \qquad (i = 1, 2, \cdots, n)$$

where n is the dimension of M. In addition we have

$$(1.2) \quad (a) \quad x^{i'}_{\varphi(m)} \equiv \delta^{i'}_i x^i_m \qquad (b) \quad \tilde{u}^{i'}_{\varphi(m)} \equiv \delta^{i'}_i u_m{}^i, \left(u^i_m \equiv \frac{dx^i_m}{dt} \quad m \in M \right)$$

which define the dragged along coordinate system (x') as well as the dragged along velocity at the point $\varphi(m) \in M$. Using (1.2)(a), (b) we can immediately verify the relations

$$(1.3) \qquad \tilde{u}^i_{\varphi(m)} = u^i_{\varphi(m)} = {}^* u^i_{\varphi(m)},$$

where

$$(1.3) \quad (a) \qquad {}^* u_{\varphi(m)} = \varphi_{g*}(u_m)$$

is the derived velocity at $\varphi(m)$ induced by the isomorphism $\varphi_{g*}|_m$ of $T_m(M)$ onto $T_{\varphi(m)}(M)$. The first of (1.3) restates globally the fact that the Lie derivative of the velocity vanishes while the third one illustrates that the dragged velocity and the derived velocity are simply a change of the point of view adopted. Here, $u^i_{\varphi(m)}$ is the velocity, the hypothetical particle should have at $\varphi(m)$, if it traced the deformed orbit $\varphi_g \circ c$ instead of c.

The situation becomes more delicate if we attempt to reconsider the acceleration of the hypothetical particle since now, the *coefficients* of the connection are involved in the definition,

$$(1.4) \qquad \gamma^i_m \equiv \frac{d^2 x^i_m}{dt^2} + \Gamma^i_{jk}(x_m) \frac{dx^j_m}{dt} \frac{dx^k_m}{dt},$$

while the connection itself is also dragged along by the transformation φ_g. Clearly, we may introduce, in complete analogy to (1.2)(b) the dragged along acceleration and the dragged along connection

$$(1.5) \quad (a) \quad \tilde{\gamma}^{i'}_{\varphi(m)} = \delta^{i'}_i \gamma^i_m, \qquad (b) \quad \tilde{\Gamma}^{i'}_{j'k'}(\varphi(m)) = \delta^{i'}_i \delta^j_{j'} \delta^k_{k'} \Gamma^i_{jk}(m).$$

Changing from the coordinate system (x') to the coordinate system (x) we have

3) see also, [**8**], [**9**].

(1.6) $$\Gamma^l_{jk}(x_m)\frac{\partial\varphi^i_g}{\partial x^l}=\tilde{\Gamma}^i_{\alpha\tau}(x_{\varphi(m)})\frac{\partial\varphi^\alpha_g}{\partial x^j}\frac{\partial\varphi^\tau_g}{\partial x^k}+\frac{\partial^2\varphi^i_g}{\partial x^j\,\partial x^k}$$

which relates the dragged connection to the original one. We may now consider the possibility of defining a new vector at the point $\varphi(m)$, after the action of the transformation φ_g, namely

(1.7) $$\tilde{\gamma}^i_{\varphi(m)}=\frac{d^2x^i_{\varphi(m)}}{dt^2}+\tilde{\Gamma}^i_{jk}(x_{\varphi(m)})\frac{dx^j_{\varphi(m)}}{dt}\frac{dx^k_{\varphi(m)}}{dt},$$

as an alternative definition of the "deformed" acceleration at the aforementioned point. However, using (1.6), (1.7) and differentiating (1.1) twice on the one hand, while using the definition (1.5) and changing from the coordinate system (x') to (x) on the other hand, we can easily verify the relations

(1.8) $$*\tilde{\gamma}^i_{\varphi(m)}=*\tilde{\gamma}^i_{\varphi(m)}=\tilde{\gamma}^i_{\varphi(m)},$$

where

(1.8) (a) $$*\gamma_{\varphi(m)}=\varphi_{g*}(\gamma_m)$$

is the derived acceleration at $\varphi(m)$, induced by the isomorphism $\varphi_{g*}|_m$ of $T_m(M)$ onto $T_{\varphi(m)}(M)$. Equations (1.8) are important in the sense that they demonstrate the self-consistency of the definiton (1.4), as the acceleration of our hypothetical particle, during the dragging prosess.

Differentiating twice (1.1) and using the definition (1.4) and the relation (1.6) we have

(1.9) $$\gamma^i_{\varphi(m)}=\tilde{\gamma}^i_{\varphi(m)}-\alpha^i_{jk}(x_{\varphi(m)})u^j_{\varphi(m)}u^k_{\varphi(m)},$$

where,

(1.9) (a) $$\alpha^i_{jk}(x_m,g)\equiv\tilde{\Gamma}^i_{jk}(x_m,g)-\Gamma^i_{jk}(x_m)$$

is a tensor which involves the deformation of the geometry due to the transformation φ_g, $g\in G$. Equation (1.9) relates the deformation of the acceleration and the deformation of the geometry, caused by the action of the element $g\in G$ on M, through the deformation tensor α^i_{jk}.

§ 2. Kinematics in a continuously deformed background. We consider now the action of a 1-dimensional subgroup of G, $G_1\equiv\{g(s)\in G,\ s\in R\}$. We may always choose the parameter s in a way such that $s=0$ corresponds to the identity of the group [**4**]. For this particular choice of the parameter the group operator reduces to

(2.1) $$g(s_2)\cdot g(s_1)=g(s_2+s_1)$$

while (1.1) can be now replaced by the more general expression

(2.2) $$x^i_{\varphi(m)}(s,t)=\varphi^i(g(s),x_m(t)),$$

for every $m\in U$ and $s\in V=R$. Now, the fundamental differential equation for the aforementioned subgroup can be written in the form (see also [**4**])

(2.3) $$\frac{\partial x^i_{\varphi(m)}}{\partial s}=\xi^i(x_{\varphi(m)}),$$

where ξ^i is the generator of the particular subgroup under consideration, namely

$$\xi^i(x)=\omega^\alpha \xi^i_{(\alpha)}(x) \quad (i=1, \cdots, n, \quad \alpha=1, \cdots, k). \tag{2.3 (a)}$$

Here $\xi^i_{(\alpha)}(x)$ are the generators and k is the dimension of the Lie group G while ω^α are arbitrary constants.

So far the group-parameter s and the time-evolution parameter t are quite independent. However, we are mainly interested in the case where there is a one-to-one relation

$$s=\sigma(t), \qquad t\in V\subseteq \boldsymbol{R} \tag{2.4}$$

between the aforementioned parameters such that $\sigma(0)=0$. In this particular case differentiating (2.2) with respect to time and using (2.3) we have

$$u^i_{\varphi(m)}=\bar{u}^i_{\varphi(m)}+\dot{\sigma}\xi^i(x_{\varphi(m)}). \tag{2.5}$$

Equation (2.5) can be interpreted as describing the total velocity at the point $\varphi(m)$. This velocity is nothing more than the velocity of the particle, plus the velocity of the background manifold at the aforementioned point. Here the motion of the background is well defined being the result of the action of the subgroup G_1, generated by (2.3)(a), on M.

In complete analogy substituting (2.2) in the definition (1.4) and using equations (1.6), (2.3) and (2.5) we obtain

$$\begin{aligned}
\gamma^i_{\varphi(m)}(t)=&\frac{\partial\varphi^i}{\partial x^j}\frac{d^2x^j_{(m)}}{dt^2}+\frac{\partial^2\varphi^i}{\partial x^j\partial x^k}\frac{dx^j_{(m)}}{dt}\frac{dx^k_{(m)}}{dt}+\Gamma^i_{jk}(x_{\varphi(m)})u^j_{\varphi(m)}u^k_{\varphi(m)}\\
&+2\frac{\partial^2\varphi^i}{\partial x^k\partial s}\frac{dx^k_{(m)}}{dt}\dot{\sigma}+\frac{\partial^2\varphi^i}{\partial s^2}\dot{\sigma}^2+\frac{\partial\varphi^i}{\partial s}\ddot{\sigma}\\
=&\bar{\gamma}^i_{\varphi(m)}(t)-\alpha^i_{jk}(x_{\varphi(m)},\sigma)\bar{u}^j_{\varphi(m)}\bar{u}^k_{\varphi(m)}\\
&+2\dot{\sigma}\left(\frac{\partial\xi^i}{\partial x^k_{\varphi(m)}}+\Gamma^i_{jk}(x_{\varphi(m)})\xi^j(x_{\varphi(m)})\right)\bar{u}^k_{\varphi(m)}\\
&+\dot{\sigma}^2\left(\frac{\partial\xi^i}{\partial x^k_{(m)}}\frac{\partial\varphi^k}{\partial s}+\Gamma^i_{jk}(x_{\varphi(m)})\xi^j(x_{\varphi(m)})\frac{\partial\varphi^k}{\partial s}\right)+\ddot{\sigma}\xi^i(x_{\varphi(m)})\\
=&\bar{\gamma}^i_{\varphi(m)}(t)-\alpha^i_{jk}(x_{\varphi(m)},\sigma)\bar{u}^j_{\varphi(m)}\bar{u}^k_{\varphi(m)}\\
&+2\dot{\sigma}\xi^i_{;j}(x_{\varphi(m)})\bar{u}^j_{\varphi(m)}+\dot{\sigma}^2\xi^i_{;j}(x_{\varphi(m)})\xi^j(x_{\varphi(m)})+\ddot{\sigma}\xi^i(x_{\varphi(m)}),
\end{aligned} \tag{2.6}$$

where '; ' indicates the covariant derivative with respect to the arbitrary, but otherwise fixed, connection of the manifold M.

Equation (2.6) is our main result. It is quite general including a *deformation* term a generalized *Coriolis* term and a generalized *centrifugal* one. Besides in the oversimplified case where M is the three-dimensional Euclidean space and G is the group of rotations equation (2.6) reduces to the familiar form

$$\gamma_m=\bar{\gamma}_m+2\boldsymbol{\Omega}\times\bar{\boldsymbol{u}}+\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{r}_m+\dot{\boldsymbol{\Omega}}\times\boldsymbol{r}_m, \qquad m\in E^3, \tag{2.6 (a)}$$

where the scalar factor $\dot{\sigma}$ has been absorbed in the definition of the vector

$$\boldsymbol{\Omega}=\dot{\sigma}(\omega^1, \omega^2, \omega^3), \qquad (\boldsymbol{r}_m=(x^1, x^2, x^3)). \tag{2.6 (b)}$$

In physical terms, $\bar{\gamma}$ and $\bar{\boldsymbol{u}}$ in equation (2.6)(a) are the acceleration and the velocity of the particle relative to the rotating frame of reference while $\boldsymbol{\Omega}$ given by (2.6)(b) is the angular velocity of the aforementioned frame.

§ 3. **Inertial and non-inertial backgrounds.** So far the group G and the geometry of M are quite arbitrary. Now, we specialize to the case where G is the group of *affine* motions of M. Although this assumption is rather severe, restricting considerably the possible geometries on M, it includes all the cases of interest and in particular the Riemannian geometries which admit a non trivial group of motions. In all these cases the deformation term vanishes and the righthand side of (2.6) depends only on the generator ξ and on the derivatives of σ. For an arbitrary but otherwise fixed choice of the pair (ξ, σ) the 1-dimensional subgroup $G_1(\xi)$, generated by ξ, prescribes a well-defined motion of the manifold M which can be considered as a virtual kinematical background for any possible (Newtonian) type of dynamics in M. We shall call the pair $(G_1(\xi), \sigma)$, in short, a *background for* M. Clearly, if $m>2$, where m is the dimension of G, there is an infinity of essentially different backgrounds for M.

In the light of the above the condition

$$\gamma_m = \bar{\gamma}_m, \qquad (\xi \neq 0) \tag{3.1}$$

becomes important as a criterion for the discrimination between the various backgrounds which are possible for M. In particular a background such that (3.1) holds is said to be an *inertial background for* M. Since $\dot{\sigma} \neq 0$, (3.1) implies the necessary conditions

$$\text{(3.2)} \quad \text{(a)} \quad \xi^i_{;j} = 0, \qquad \text{(b)} \quad \ddot{\sigma} = 0.$$

The integrability conditions of (3.2)(a) can be written in the form (see also [**5**])

$$\text{(3.3)} \quad \text{(a)} \quad \xi^l R_{lkj}{}^i = 0, \qquad \text{(b)} \quad \xi^l R_{lkj}{}^i{}_{;h_1} = 0, \cdots,$$

where $R_{lkj}{}^i$ is the curvature tensor. On the other hand the condition that ξ generates an affine motion,

$$\mathcal{L}\Gamma^i_{jk} = \xi^i_{;jk} - \xi^l R_{lkj}{}^i = 0 \tag{3.4}$$

is automatically satisfied as a result of (3.2)(a) and (3.3). Consequently we have proved the following:

Theorem 1. *The pair* (ξ, σ) *is an inertial background for the space* M, *if and only if the conditions* (3.2)(a), (b) *and* (3.3) *are satisfied.*

Clearly, Theorem 1 covers also the Riemannian case, the vector ξ now being a killing vector for the space M.

All the steps of § **3** can be repeated *mutatis-mutandis* in the case where M is a Finsler space assuming only that the coefficients of the connections $\Gamma^i_{jk}(x_m)$ are replaced by the Finslerian coefficients $\Gamma^{*i}_{jk}(x_m, v_m)$. This means that equations (1.8), (1.9) and (2.6) can be naturally generalised for a Finslerian background.

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