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Fuzzy equivalence and the resulting topology

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Abstract: In this paper we show how a topology can be generated by a fuzzy binary relation of indifference, i.e. a fuzzy equivalence relation, on a set of objects. This construction generalizes the classical construction of the topology of metric spaces, thus showing that the idea of fuzziness is already present in classical mathematical objects. An application related to Poincaré's conception of continuum is also given.

Keywords: Indifference relation; fuzzy equivalence; topology; metric space; distance; triangle inequality; transitivity; fuzzy transitivity; non-transitivity; indistinguishability.

Introduction

The meaning of the concept of *distance*, as introduced by Fréchet [3] for the study of general spaces, includes the idea of resemblance or indifference between two (mathematical) objects. Chevallard and Johsua [1] have pointed out that the distance between two objects (e.g. two functions, two curves or surfaces etc.) is apparently a measure of how different these objects are, or how different are their mathematical properties (their integrals, lengths, areas, etc.). This *special character,* of the concept of distance introduced by Fréchet, does not exist in the usual meaning of the notion of distance in elementary geometry or mechanics. Two points on the plane do not resemble each other more if their distance is getting smaller!

A generalization of the notions of distance and metric space, which has more or less a statistical or probabilistic character, was provided by Menger [5, 6], Schweizer and Sklar [9, 10], Fritsche [4] and others. But there are some difficulties in the construction of a topology for a probabilistic metric space (for example, it is not always possible to construct a closure operator in the sense of Kuratowski but only in the sense of Čech (see R.M. Tardiff [11]).

In this paper we follow a direction which is close to the 'degree of difference' character of the Fréchet's concept of distance. We generalize this concept — and the construction of the resulting topology — in a setting that uses notions and techniques from the theory of fuzzy binary relations; namely we use a fuzzy equivalence relation (satisfying some natural conditions) as a fuzzy set theoretic analogue of distance, and we prove that such a relation generates a topology, whose properties we investigate. We also give an application of these notions and results to Poincaré's conception of physical continuum.

1. Triangle operations on the interval [0, 1]

In this section we define a notion of triangle operation (or norm) on the interval [0, 1], as a technical tool for the generalization of the Triangle Inequality. This notion is similar to that defined by Schweizer and Sklar, who worked towards the direction of probabilistic metric spaces, but it will be used in a totally different context.

Definition. We put $I=[0, 1]$ and $I^\circ=(0, 1)$. A mapping

$\triangle: I\times I\to I$

is called a *triangle operation* (or *triangle norm*) *on I* iff the following conditions are satisfied:

(i) $\triangle$ is commutative, i.e.

$s\triangle t=t\triangle s$ for every $s, t\in I$.

(ii) $t\triangle 1=t$ for every $t\in I$; also, $0\triangle 0=0$.

(iii) For every $t \in I^\circ$ there is a $t_1 \in I^\circ$ such that $t_1 \Delta t_1 \geq t$.

(iv) Δ is monotonic, i.e. if $s_1 \leq s_2$ and $t_1 \leq t_2$, then $s_1 \Delta t_1 \leq s_2 \Delta t_2$.

Example. The operations

$$s_1 \Delta_1 t = \min\{s, t\}, \qquad s \Delta_2 t = s \cdot t$$

satisfy conditions (i)–(iv). These operations are also associative, but we shall not need associativity throughout this paper.

Actually we can immediately see that any triangle operation Δ (satisfying (i)–(iv)) also satisfies:

$$s \Delta t \leq \min\{s, t\} \quad \text{for every } s, t \in I. \tag{1.1}$$

Because from (ii) and (iv),

$$s \Delta t \leq s \Delta 1 = s$$

and by the same argument $s \Delta t \leq t$.

From the triangle operations $s \Delta_1 t = \min\{s, t\}$ and $s \Delta_2 t = s \cdot t$ we can produce 'dual' operations by using the involution $s \to s' = 1 - s$ as follows:

$$s \nabla_1 t = (\min\{s', t'\})' = 1 - \min\{1-s, 1-t\} = \max\{s, t\}$$

and

$$s \nabla_2 t = (s' \cdot t')' = 1 - (1-s)(1-t) = s + t - st.$$

Motivated by these examples, we define, by the same procedure, the *dual of a triangle operation* Δ *on* I, as follows:

$$s \nabla t := (s' \Delta t')' = 1 - (1-s) \Delta (1-t).$$

It can be easily proved that the dual operation ∇ of a triangle operation Δ on I has the following properties:

(i') ∇ is commutative;

(ii') $t \nabla 0 = t$ for every $t \in I$; also, $1 \nabla 1 = 1$;

(iii') for every $t \in I^\circ$ thre is a $t_1 \in I^\circ$ such that $t_1 \nabla t_1 \leq t$;

(iv') if $s_1 \leq s_2$ and $t_1 \leq t_2$, then $s_1 \nabla t_1 \leq s_2 \nabla t_2$.

Also, we have

$$s \nabla t \geq \max\{s, t\} \quad \text{for every } s, t \in I.$$

The proof of these statements is direct from the definitions and from the fact that $(s')' = s$, and will be omitted. The following kind of De Morgan laws is also immediate:

$$(s \nabla t)' = s' \Delta t', \qquad (s \Delta t)' = s' \nabla t' \tag{1.2}$$

for every $s, t \in I$.

We also note that from (iv), (iv'), (iii') and (1.1), it follows that the functions Δ and ∇ are *continuous at the point* (0, 0) with respect to the Cartesian topology on $I \times I$.

2. Δ-Fuzzy equivalence relations on a set of objects

We recall that a *fuzzy binary relation* on a set E is a mapping $\varphi: E \times E \to I$. Zadeh [13], Yeh and Bang [12] and others, have defined and studied this notion and its applications to cognitive and decision processes. Ovchinnikov [7] has studied the structure of fuzzy binary relations — in particular, fuzzy equivalence relations — in a more general setting, using, instead of $I = [0, 1]$, a complete distributive lattice with universal bounds. Our approach will be similar to the approach of this author, although we shall use special properties of I which are necessary for our purpose (as the existence of triangle operations and their duals on I and the density of the elements of I with respect to their natural order).

Definitions. We say that a fuzzy binary relation φ on E is:

- *reflexive* iff $\varphi(x, x) = 1$ for every $x \in E$;
- *symmetric* iff $\varphi(x, y) = \varphi(y, x)$ for every $x, y \in E$.

Now let Δ be a triangle operation on I. We say that φ is Δ-*transitive* iff

$$\varphi(x, y) \Delta \varphi(y, z) \leq \varphi(x, z) \quad \text{for every } x, y, z \in E.$$

We call φ *a* Δ-*(fuzzy) equivalence on* E iff it is a reflexive, symmetric and Δ-transitive fuzzy binary relation on E.

Example 2.1. Take Δ to be the usual multiplication in I. Let E be a set equipped with a *pseudo-distance* d, i.e. a mapping

$$d: E \times E \to [0, +\infty)$$

satisfying $d(x, x) = 0$, $d(x, y) = d(y, x)$ and the triangle inequality

$$d(x, y) + d(y, z) \geq d(x, z).$$

Consider on E the fuzzy binary relation defined by the formula (first introduced by Menger [5]):

$$\varepsilon(x, y) = \exp(-d(x, y)). \tag{2.1a}$$

Since $d(x, y) \geq 0$, it is obvious that ε takes values in $(0, 1]$. This fuzzy binary relation is clearly reflexive and symmetric. Also, from the Triangle Inequality, which is satisfied by d, it follows that

$$\varepsilon(x, y) \cdot \varepsilon(y, z) \leq \varepsilon(x, z) \quad \text{for every } x, y, z \in E,$$

i.e. that ε is $\triangle$-transitive with the usual multiplication as $\triangle$, and thus it is a $\triangle$-equivalence on E. Conversely, if we suppose that ε is a $\triangle$-equivalence relation on a (arbitrary) set E, with values on $(0, 1]$, it follows that the function d defined on $E \times E$ by the rule

$$d(x, y) = -\log \varepsilon(x, y) \tag{2.1b}$$

is a pseudo-distance on E. Thus the transformations (2.1a) and (2.1b) establish a one-to-one correspondence between pseudo-distances and $\triangle$-equivalences with nonzero values and such that the triangle operation $\triangle$ is the usual multiplication.

Now let φ be any $\triangle$-equivalence on a set E. Following Ovchinnikov [7], we define the *φ-class* of an element $a \in E$ as the fuzzy set $[a]_\varphi$ with membership function

$$[a]_\varphi(x) = \varphi(a, x).$$

We then have

Lemma.

$$[a]_\varphi = [b]_\varphi \text{ iff } \varphi(a, b) = 1. \tag{2.2}$$

Proof. If $[a]_\varphi = [b]_\varphi$ then $\varphi(a, b) = [a]_\varphi(b) = [b]_\varphi(b) = \varphi(b, b) = 1$, since φ is reflexive. Conversely, if $\varphi(a, b) = 1$ then for every $x \in E$,

$$[a]_\varphi(x) = \varphi(a, x) \geq \varphi(a, b) \triangle \varphi(b, x)$$
$$= 1 \triangle [b]_\varphi(x) = [b]_\varphi(x),$$

since φ is $\triangle$-transitive and $\triangle$ satisfies (ii) (Section 1). □

Returning to Example 2.1, we remark that for every point x of a pseudo-metric space (E, d), the set of points of a zero distance from x (i.e. those points y with $d(x, y) = 0$) coincides with the set

$$\bar{x} = \{y \mid [x]_\varepsilon = [y]_\varepsilon\},$$

where ε is the fuzzy equivalence corresponding to d according to the rules (2.1a) and (2.1b). Thus a necessary and sufficient condition that the pseudo-distance d is a *distance* on E, i.e. that it also satisfies the condition

$$d(x, y) = 0 \Rightarrow x = y,$$

is that for every $x \in E$ the set $\bar{x}$ as above contains only x.

3. The Kuratowski closure operator generated by a $\triangle$-fuzzy equivalence

The topology of a pseudo-metric (in particular metric) space (E, d) can be generated as follows: for every subset A of E, the closure $\bar{A}$ is the set of all points x such that $d(x, A) = 0$, where

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

The resulting topological space is a T_1 space iff the pseudo-distance d is a distance.

We will now generalize these classical facts by proving:

Theorem 3.1. *Let φ be a $\triangle$-fuzzy equivalence relation on a set E, where $\triangle$ is a given triangle operation on $I = [0, 1]$. For every element x and every subset A of E we put*

$$\varphi(x, A) = \sup\{\varphi(x, a) \mid a \in A\}.$$

Then the mapping $2^E \to 2^E$ (where 2^E is the power-set of E) defined by

$$A \to A^\varphi := \{x \mid \varphi(x, A) = 1\}$$

is a Kuratowski closure operator on E, such that for every $x \in E$ the closure of the one-element set $\{x\}$ is

$$\{x\}^\varphi = \{y \mid [x]_\varphi = [y]_\varphi\}.$$

Proof. At first we have

(1) $A \subseteq A^\varphi$ for every $A \subseteq E$. Indeed, for every $a \in A$ we have $\varphi(a, A) = \varphi(a, a) = 1$.

We next prove

(2) $(A \cup B)^\varphi = A^\varphi \cup B^\varphi$ for every $A, B \subseteq E$,

in two steps: at first we show in general the implication

(2a) $X \subseteq Y \Rightarrow X^{\varphi} \subseteq Y^{\varphi}$ (from which it follows that $A^{\varphi} \subseteq (A \cup B)^{\varphi}$ and $B^{\varphi} \subseteq (A \cup B)^{\varphi}$). Suppose that $X \subseteq Y$ and let $x_1 \in X^{\varphi}$; then

$$\varphi(x_1, Y) = \sup\{\varphi(x_1, y) \mid y \in Y\}$$
$$\geqslant \sup\{\varphi(x_1, x) \mid x \in X\} = 1$$

so that $x_1 \in Y^{\varphi}$. The second step for (2) is to show that

(2b) $(A \cup B)^{\varphi} \subseteq A^{\varphi} \cup B^{\varphi}$. Let $x \in (A \cup B)^{\varphi}$ and suppose that $x \notin A^{\varphi}$; we shall show then that $x \in B^{\varphi}$. Since $x \notin A^{\varphi}$ we have

$$\varphi(x, A) = 1 - \vartheta < 1 \quad (\text{for some } \vartheta > 0).$$

Let δ be any positive number with $\delta < \vartheta$. Since

$$\varphi(x, A \cup B) = 1,$$

there is a $c \in A \cup B$ with

$$1 - \delta \leqslant \varphi(x, c) \leqslant 1.$$

Then

$$\varphi(x, c) \geqslant 1 - \delta > 1 - \vartheta = \varphi(x, A),$$

from which it follows that $c \notin A$ (otherwise we would have $\varphi(x, A) \geqslant \varphi(x, c) > \varphi(x, A)$, which is absurd); so $c \in B$. Thus for any sufficiently small $\delta > 0$ there is a $c \in B$ with

$$1 - \delta \leqslant \varphi(x, c) \leqslant 1$$

which means that $x \in B^{\varphi}$. Thus (2) is established.

We have now to prove:

(3) $(A^{\varphi})^{\varphi} \subseteq A^{\varphi}$ for every $A \subseteq E$. Let $x \in (A^{\varphi})^{\varphi}$. Then $\varphi(x, A^{\varphi}) = 1$. We shall show that also $\varphi(x, A) = 1$. Let δ be any positive number less than 1. It suffices to find an element $a \in A$ wtih $\varphi(x, a) \geqslant 1 - \delta$. From the property (iii′) of the dual operation $\triangledown$ (Section 1) there is a number δ_1, $0 < \delta_1 < 1$, such that

$$\delta_1 \triangledown \delta_1 \leqslant \delta.$$

Since $\varphi(x, A^{\varphi}) = 1$, there exists $x_1 \in A^{\varphi}$ with $\varphi(x, x_1) \geqslant 1 - \delta_1$. But then $\varphi(x_1, A) = 1$, so there also exists an $a_1 \in A$ with $\varphi(x_1, a_1) \geqslant 1 - \delta_1$. Combining these inequalities and using the $\triangle$-transitivity of φ, the monotonicity of $\triangle$ and (1.2), we get

$$\begin{aligned} \varphi(x, a_1) &\geqslant \varphi(x, x_1) \triangle (x_1, a_1) \\ &\geqslant (1 - \delta_1) \triangle (1 - \delta_1) \\ &= \delta_1' \triangle \delta_1' = (\delta_1 \triangledown \delta_1)' \\ &= 1 - (\delta_1 \triangledown \delta_1) \geqslant 1 - \delta. \end{aligned}$$

Thus (3) has been proved.

From (1), (2) and (3) the operator $A \to A^{\varphi}$ is a Kuratowski closure operator. The truth of the last statement of the Theorem is obvious from (2.2). □

Corollary. *With the premises of Theorem* 3.1, *the resulting topology on E has the property* T_1 *iff*

$$[x]_{\varphi} = [y]_{\varphi} \Rightarrow x = y$$

or, equivalently, iff

$$\varphi(x, y) = 1 \Rightarrow x = y.$$

4. Open sets and limits with respect to the topology generated by a fuzzy equivalence

Proposition 4.1. *Let* $A \subseteq E$ *be open with respect to the topology generated by a* $\triangle$*-fuzzy equivalence relation* φ *in* E *(according to Theorem* 3.1), *and let* $\alpha \in A$. *Then*:

(i) $\{\alpha\}^{\varphi} \subseteq A$;

(ii) *there is a* $\delta \in (0, 1)$ *such that the set (subset of E)*

$$B(\alpha; \delta) = \{x \mid \varphi(\alpha, x) > 1 - \delta\}$$

is included in A.

Proof. (i) Since A is open, its complement $E - A$ is closed. Let $\alpha' \in \{\alpha\}^{\varphi}$; we show that $\alpha' \in A$ by contradiction. If $\alpha' \in E - A$, then the relation $\varphi(\alpha, \alpha') = 1$ would imply

$$\alpha \in (E - A)^{\varphi} = E - A$$

contrary to our assumption.

(ii) Since $\alpha \notin (E - A)^{\varphi}$, we have $\varphi(\alpha, E - A) < 1$. So we can put

$$\varphi(\alpha, E - A) = 1 - \delta \quad (0 < \delta < 1).$$

If x is any point of E such that

$$\varphi(\alpha, x) > 1 - \delta,$$

then clearly $x \notin E - A$, i.e. $x \in A$. □

The sets

$B(\alpha;\delta) =: \{x \mid \varphi(\alpha, x) > 1-\delta\}$
$(\alpha \in E, 0<\delta<1)$

appearing in Proposition 4.1 are the analogues of open balls in metric spaces. In order to formulate our next result about these sets, we need the following.

Definition. A triangle operation $\triangle$ on I is called *dense* iff for every $\delta \in I^\circ$ and $\delta_1 \in I^\circ$ with $\delta_1 < \delta$ there is a $\delta_2 \in I^\circ$ such that $\delta_1 \triangledown \delta_2 < \delta$. (Recall that $\triangledown$ is the dual operation of $\triangle$.) It is easy to show that the operations

$s \triangle_1 t = \min\{s, t\}, \qquad s \triangle_2 t = s \cdot t$

are dense.

Proposition 4.2. *If $\triangle$ is a dense triangle operation on I, then every set $B(\alpha;\delta)$ defined as above is* open *in E with respect to the topology generated by the $\triangle$-fuzzy equivalence φ; therefore $B(\alpha;\delta)$ is a neighborhood of the point α.*

Proof. We shall prove that $E - B(\alpha;\delta)$ is closed, i.e. that

$(E - B(\alpha;\delta))^\varphi \subseteq E - B(\alpha;\delta)$.

Let $y \in (E - B(\alpha;\delta))^\varphi$, which means that

$\varphi(y, E - B(\alpha;\delta)) = 1$.

We have to show that $y \notin B(\alpha;\delta)$. Suppose the contrary; then

$\varphi(\alpha, y) > 1-\delta$

and we can also find a $\delta_1 < \delta$ with

$\varphi(\alpha, y) > 1-\delta_1 > 1-\delta$.

Since $\triangle$ is dense, there exists $\delta_2 > 0$, $\delta_2 < 1$, such that

$\delta_1 \triangledown \delta_2 < \delta$;

and since

$$\varphi(y, E - B(\alpha;\delta)) = \sup\{\varphi(y,z) \mid z \in E - B(\alpha;\delta)\} = 1,$$

there exists $z \in E - B(\alpha, \delta)$ with

$1-\delta_2 \leqslant \varphi(y, z) \leqslant 1$.

But then

$$\begin{aligned}\varphi(\alpha, z) &\geqslant \varphi(\alpha, y) \triangle \varphi(y, z)\\ &\geqslant (1-\delta_1)\triangle(1-\delta_2)\\ &= 1-(\delta_1 \triangledown \delta_2)\\ &> 1-\delta,\end{aligned}$$

which is a contradiction since $z \notin B(\alpha;\delta)$. □

Corollary. *With the same premises as in Proposition 4.2, if a net $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ converges in E to the points x_1 and x_2, then these points are of the same 'φ – class', i.e. $\varphi(x_1, x_2) = 1$.*

Proof. Suppose the contrary, i.e.

$\varphi(x_1, x_2) = 1-\delta, \quad 0<\delta<1.$

Then there is a $\delta_1 > 0$ with $\delta_1 < \delta$ and a $\delta_2 > 0$, $\delta_2 < 1$, with $\delta_1 \triangledown \delta_2 < \delta$. By Proposition 4.2 the sets $B(x_1;\delta_1)$ and $B(x_1;\delta_2)$ are neighborhoods of x_1 and x_2 respectively. Since the net $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ converges to both x_1 and x_2, there exists $\lambda \in \Lambda$ such that

$\alpha_\lambda \in B(x_1;\delta_1) \cap B(x_2;\delta_2)$.

Therefore

$$\begin{aligned}\varphi(x_1, x_2) &\geqslant \varphi(x_1, \alpha_\lambda) \triangle \varphi(x_2, \alpha_\lambda)\\ &\geqslant (1-\delta_1)\triangle(1-\delta_2) > 1-\delta,\end{aligned}$$

which is a contradiction. □

Example. A natural fuzzy equivalence φ on the space $^*\mathbb{R}$ of nonstandard real numbers (the triangle operation being the usual multiplication on I) is defined as follows:

$$\varphi(x, y) := \begin{cases} \exp(-|\mathrm{st}(x-y)|) & \text{if } x-y \text{ is finite (not infinitely large)},\\ 0 & \text{if } x-y \text{ is infinitely large,}\end{cases}$$

where $\mathrm{st}(x)$ denotes the standard part of x. It is clear that the relation $\varphi(x_1, x_2) = 1$ holds iff x_2 belongs to the 'monad' of x_1. Suppose that a net $\{\alpha_\lambda\}$ converges in the space $^*\mathbb{R}$ with respect to the topology generated by the fuzzy equivalence φ. Then, according to the above corollary, any two limits of $\{\alpha_\lambda\}$ belong to the same 'monad' of $^*\mathbb{R}$. In particular, if there is a real x among these limits, then x is uniquely determined by $\{\alpha_\lambda\}$.

5. An application related to Poincaré's conception of continuum

As an application of our general constructions, we shall present a rigorous formulation of the continuum of physical quantities, conceived by Poincaré [8]. For another approach based on Boolean Fuzzy Sets see Drossos and Markakis [2]. Poincaré's conception of physical continuum can be described vaguely in terms of a relation of 'closeness', concerning physical quantities or events. The essential point, here, is *nontransitivity*: while the quantity or event A may be 'close' to the quantity or event B and B 'close' to C (with respect to the difference of their magnitudes or the time they happened), we cannot conclude that A is 'close' to C.

We suppose that the 'continuum of physical quantities' has the structure of an *ordered field* Σ. This implies, in particular, that Σ is dense with respect to its ordering, since it contains a copy of the rational numbers. Let $d>0$ be a fixed, for the moment, element in Σ. We say that d is a *limit of distinguishability*, with respect to some particular observer or some 'microscope'. For two quantities $a, b \in \Sigma$ we say that *a is (very) close to b with respect to d* or that *a and b are indistinguishable with respect to d*, in symbols

$$a \approx_0 (d)\, b$$

or simply

$$a \approx_0 b,$$

iff

$$|a-b|<d$$

where $|\cdot|$ denotes the usual absolute value in Σ (i.e. $|a|=a$ if $a \geqslant 0$ and $|a|=-a$ if $a<0$). The relation $\approx_0(d)$ is clearly reflexive and symmetric, but it is *not transitive,* as it can easily be shown by taking e.g. $c=a+d$ and $b=a+e$, where $0<e<d$.

Although the above relation of closeness or indistinguishability (with respect to some fixed element d of Σ) is not transitive, we may consider chains of indistinguishable terms between two elements of Σ.

Let $a, b \in \Sigma$ and $n \in \mathbb{N}$ $(n=0, 1, 2, \ldots)$. We write

$$a \approx_n (d)\, b$$

iff there is a chain of $n+2$ elements of Σ

$$a_0, a_1, \ldots, a_{n+1}$$

such that $a_0=a$, $a_{n+1}=b$, and

$$|a_i-a_{i+1}|<d \quad \text{for } i=0, 1, \ldots, n.$$

This implies that

$$|a-b| \leqslant |a-a_1|+\cdots+|a_n-b|<(n+1)d$$

and thus that α, β are indistinguishable with respect to $(n+1)d$. This suggests that an 'order' of distinguishability between two quantities in Σ should be defined, as we are now going to do.

For any pair of quantities $a, b \in \Sigma$ and any $d \in \Sigma$, $d>0$, two cases are possible:

(a) there exists $n \in \mathbb{N}$ such that $a \approx_n (d)\, b$;

(b) such an $n \in \mathbb{N}$ does not exist.

In case (a) we define the *order of distinguishability between a, b with respect to d*, in symbols

$$N_d(a, b),$$

as the *least* natural number n with the property $a \approx_n (d)\, b$. In case (b) we say that $N_d(a, b)$ is not finite and we write $N_d(a, b)=+\infty$. We omit the easy proof of the following.

Lemma 5.1. *The function* $(a, b, d) \rightarrow N_d(a, b)$ *has the following properties*:

(i) $N_d(a, a)=0$;

(ii) $N_d(a, b)=N_d(b, a)=N_d(a-b, 0)$;

(iii) $N_d(a, c) \leqslant N_{d/2}(a, b)+N_{d/2}(b, c)$;

(iv) *if* $d_1 \geqslant d_2$ *then* $N_{d_1}(a, b) \leqslant N_{d_2}(a, b)$.

Let $d>0$ be fixed in Σ, and consider the sequence

$$d/k=\frac{1}{k} \cdot d \quad (k=1, 2, \ldots)$$

which is obviously decreasing in Σ. From the last property in the above lemma, the sequence of corresponding orders of distinguishability

$$N_{d/k}(a, b) \quad (k=1, 2, \ldots)$$

is increasing. It follows that the limit

$$\lim_{k \rightarrow \infty} \exp(-N_{d/k}(\alpha, b))$$

always exist in $I=[0, 1]$, where the expression $\exp(-t)$ means zero in case $t=+\infty$. We define the function

$$\Phi_d : \Sigma \times \Sigma \to I$$

by the rule

$$\Phi_d(\alpha, b) := \lim_{k\to\infty} \exp(-N_{d/k}(\alpha, b)).$$

Proposition 5.1. *Φ_d is a $\triangle$-fuzzy equivalence in Σ, where $\triangle$ is the usual multiplication operation in I.*

Proof. What we have to prove is $\triangle$-transitivity, i.e. that

$$\Phi_d(a, b) \cdot \Phi_d(b, c) \leqslant \Phi_d(a, c).$$

This inequality is obtained directly from

$$N_{d/2k}(a, b) + N_{d/2k}(b, c) \geqslant N_{d/k}(a, c)$$

which holds for every $k = 1, 2, \ldots$ by Lemma 5.1. □

Corollary. *For every $d > 0$ in Σ we have a topology in Σ resulting from the fuzzy equivalence Φ_d.*

In fact this topology describes Σ with respect to the limit of distinguishability d. For example, if $\Sigma = {}^*\mathbb{R}$ and d is real, then for every $a, b \in \Sigma$ we have $\Phi_d(a, b) = 1$ iff b belongs to the 'monad' of a (i.e. $a - b$ is infinitesimal) and $\Phi_d(a, b) = 0$ otherwise.

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