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GENERALIZED-FINSLERIAN EQUATION OF GEODESIC DEVIATIONS

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We indicate that the main term in the geodesic deviation equation is given by the generalized torsion tensor. In the static spherically-symmetric case, the explicit representation for the main term is found in the first-order low-velocity approximation.

1. Introduction

The equation of geodesic deviation was studied in general relativity from various points of view (see, e.g. [1–3]). The equation can be extended to the case of the classical Finsler geometry [4] as well as to the gauge-Finslerian approach [5]. In the generalized-Finslerian framework [6, 7] which drops the classical condition (2.6), the equation can also be derived [7].

In the Finsler geometry [4], the feasibility of scrupulous considerations stems from the fact that the Finslerian connection coefficients can be constructed in an explicit and lucid manner. However, the equations of the generalized-Finslerian spaces cannot be resolved for the connection coefficients in the general form (see [6, 7]), which hinders investigation of the generalized-Finslerian deviation equation.

On the other hand, the generally-relativistic analysis of gravitational observations made in our solar system is founded on the particular case when the gravitational field metric is assumed to be static and spherically-symmetric [1]. The corresponding extension of the metric tensor to the generalized-Finslerian case was proposed in [8, 9], without assuming the condition (2.6). Even in this sufficiently simple case, however, the possibility of solving the metric condition (2.8) for the connection coefficients subject to the natural condition (2.13) is not obvious.

The deviation equation under study can be investigated in the low-velocity approximation approach. We start with the observation that the main term of the equation is expressible in terms of the torsion tensor which involves only the contracted coefficients $L_j{}^i$, thereby avoiding the use of the full connection coefficients $L_i{}^j{}_k$ and, hence, simplifying matters (Section 2). Following this procedure, we calculate the main term up to the nearest generalized-Finslerian corrections with respect to motion velocity (Section 3). These are the corrections that are of experimental significance, for all related observations (such as the tidal forces behaviour or the deviation of trajectories of nearby spacecrafts) involve rather small velocities. In principle, subjecting the representations (3.5) and (3.6) to experimental verifications could give valuable estimations for the generalized-Finslerian parameters b_A.

2. Geodesic deviation equation

Suppose we are given an N-dimensional differentiable manifold M endowed with a symmetric non-singular metric tensor $a_{ij}(x,y)$, where x^n are local coordinates of M and y^n are tangent vectors supported by x^n. The covariant derivative D_n will be defined in accordance with

$$D_n a_{ij} = d_n a_{ij} - L_i{}^k{}_n a_{kj} - L_j{}^k{}_n a_{ik}\,, \tag{2.1}$$

where $L_i{}^k{}_n = L_i{}^k{}_n(x,y)$ and

$$d_n = \partial_n - L_n{}^m(x,y)\frac{\partial}{\partial y^m} \tag{2.2}$$

($\partial_n = \partial/\partial x^n$). The symmetry $L_i{}^k{}_n = L_n{}^k{}_i$ will be implied. The associated curvature tensor reads

$$L_m{}^n{}_{ij} = d_j L_m{}^n{}_i - d_i L_m{}^n{}_j + L_m{}^k{}_i L_k{}^n{}_j - L_m{}^k{}_j L_k{}^n{}_i \tag{2.3}$$

and the associated torsion tensor reads

$$M_j{}^n{}_i = d_j L_i{}^n - d_i L_j{}^n\,. \tag{2.4}$$

We also introduce the notation

$$S_{nij} = \frac{1}{2}\frac{\partial a_{ij}}{\partial y^n}\,. \tag{2.5}$$

The classical Finsler geometry [4, 10] is characterized by the condition

$$S_{ijn}y^j = 0\,. \tag{2.6}$$

The condition (2.6) will not be implied below, for we shall follow the generalized-Finslerian framework developed in [6, 7].

If we introduce the associated Christoffel symbols

$$r_j{}^i{}_k = \tfrac{1}{2}a^{in}(\partial_j a_{nk} + \partial_k a_{nj} - \partial_n a_{jk})\,, \tag{2.7}$$

we can resolve formally the metric condition

$$D_n a_{ij} = 0 \tag{2.8}$$

for the connection coefficients $L_j{}^i{}_n$, obtaining

$$L_j{}^i{}_k = r_j{}^i{}_k - a^{in}(L_j{}^m S_{mkn} + L_k{}^m S_{mjn} - L_n{}^m S_{mjk})\,. \tag{2.9}$$

In the Finslerian extension of the Riemannian approach the equation of geodesic deviations can be written in the form

$$\frac{D^2 z^i}{ds^2} = -K_j{}^i{}_{hk}(x,y)y^j y^h z^k \tag{2.10}$$

(eq. (4.4.16) of [4]), where $K_j{}^i{}_{hk}$ is a Finslerian curvature tensor, and D/ds is a covariant derivative along a geodesic. This equation can be extended to the generalized-Finslerian framework (see [7]) such that the curvature tensor (2.3) will enter the right-hand side of equation (2.10) in the combination

$$F^i = F_k{}^i z^k \tag{2.11}$$

with

$$F_k{}^i = L_j{}^i{}_{hk}(x,y) y^j y^h \,. \tag{2.12}$$

As in [6, 7], we put

$$L_i{}^j = L_i{}^j{}_k y^k \,. \tag{2.13}$$

Now, if we contract (2.3) by y^m and use (2.13), we find the identity

$$y^m L_m{}^n{}_{ij} = M_j{}^n{}_i \tag{2.14}$$

which shows that *the main term* (2.11) *in the generalized-Finslerian geodesic deviation equation is determined by the torsion tensor* (2.4), namely,

$$F_k{}^i = M_k{}^i{}_j y^j \,. \tag{2.15}$$

Since the torsion tensor $M_k{}^i{}_j$ is constructed solely from $L_i{}^j$, as this follows from (2.2) and (2.4), the last observation enables us to avoid examination of the form of the full connection coefficients $L_i{}^j{}_k$, which enter the curvature tensor (2.3).

3. Static spherically-symmetric case

Following Section 4 of [8], we fix a coordinate system $x^i = (x^0, x^a)$, introduce the notation

$$r = (\delta_{ab} x^a x^b)^{1/2}, \qquad n_a = \frac{x^a}{r} \equiv \frac{\partial r}{\partial x^a},$$

and put

$$a_{00} = A_1(r,q) r_{00}(r), \qquad a_{ab} = A_2(r,q) r_{ab}(r), \qquad a_{0a} = 0 \tag{3.1}$$

together with

$$r_{ab} = -W(r)\delta_{ab}, \tag{3.2}$$

where

$$q = -\frac{r_{ab} y^a y^b}{r_{00}(y^0)^2} \tag{3.3}$$

and δ stands for the Kronecker symbol. The indices $a, b, \ldots$ run from 1 to $N-1$. The space defined in this way presents the static spherically-symmetric generalized-Finslerian case.

The low-velocity approximation technique was proposed in [9] to analyse concomitants of the metric tensor (3.1). Such a technique can successfully be applied to our case to estimate the tensor (2.14). Indeed, the equation obtained after contracting

(2.9) by y^k can be examined carefully to find the first-order approximations for the coefficients $L_i{}^j$. On doing so, we obtain the following approximate representations:

$$L_0{}^a = y^0 b_1 n^a + b_2 n_y v^a, \qquad L_a{}^b = b_3 n_y \delta_a^b + b_4(y^b n_a - y_a n^b), \tag{3.4}$$

where $n_y = n_a y^a$, $v^a = y^a/y^0$, and $n^a = r^{ab} n_b$; the coefficients $b_A, A = 1,2,3,4$, depend on only r. Insertion of (3.4) in (2.14) yields the following result after straightforward calculations:

$$\frac{F_0{}^b}{(y^0)^2} = \frac{1}{W}\left[-\left(b_3 + \frac{1}{r}\right) b_1 v^b + \left(b_{1r} - \frac{b_1}{r} - 2t_{21} b_1\right)\right] n_v n_b + o(q) \tag{3.5}$$

and

$$\frac{F_a{}^b}{y^0} = \left(b_{3r} - \frac{b_3}{r} - b_3^2\right) n_v (n_v \delta_a^b - n_a v^b) + \left(-b_{4r} + \frac{b_4}{r} + b_4^2\right) n_b (n_v v^a - W r_{00} q n_a) + \\ + \frac{1}{W}\left(b_{30} + \frac{b_1}{r}\right)\delta_a^b - \frac{1}{W}\left(b_{1r} - \frac{b_1}{r} - 2t_{21} b_1\right) n_a n_b + o(q), \tag{3.6}$$

where $b_{Ar} = \partial b_A/\partial r$, $n_v = n_a v^a$, and $t_{21} = \frac{1}{2}\partial \ln W/\partial r$.

Thus, we have proved the following

THEOREM. *In the generalized-Finslerian static spherically-symmetric approach, the first-order low-velocity approximation to the main term of the geodesic deviation equation is given explicitly by* (3.5) *and* (3.6).

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