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CURVATURE AND LORENTZ TRANSFORMATIONS OF SPACES WHOSE METRIC TENSOR DEPENDS ON VECTOR AND SPINOR VARIABLES.

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§ 1. Introduction. An interesting study of differential geometry of spaces whose metric tensor $g_{\mu\nu}$ depends on spinor variables ξ and $\bar{\xi}$ (its adjoint) as well as coordinates x^λ, has been proposed by Y. Takano [**1**][1].

Recently Y. Takano and T. Ono ([**2**], [**3**], [**4**]) have studied the above-mentioned spaces and they have given a generalization of these spaces in the case of the metric tensor $g_{\mu\nu}$ depending on spinors variables ξ and $\bar{\xi}$ and vector variables y^λ as well as coordinates x^λ. Such spaces are considered as a generalization of Finsler spaces.

In the present paper the authors study Lorentz transformations and the curvature in the above generalized spaces with metric tensor $g_{\mu\nu}(x, y, \xi, \bar{\xi})$.

Greek letters $\lambda, \mu, \nu, \cdots$ and $\alpha, \beta, \gamma, \cdots$ are used for spacetime indices and spinors indices, respectively and Latin letters $a, b, c, \cdots$ for Lorentz indices.

§ 2. Lorentz transformation. We shall now present the transformation character of the connection, the non-linear connection and the spin connection coefficients with respect to local Lorentz transformations which depend on spinor variables, vector variables as well as coordinates.

For any quantities which transform as:

$$f(x, y, \xi, \bar{\xi}) \to f'(x, y, \xi', \bar{\xi}') = U(x, y, \xi, \bar{\xi}) \tag{2.1}$$

their derivatives with respect to x^λ, y^λ, ξ_α and $\bar{\xi}^\alpha$ under Lorentz transformations:

$$x'^\lambda = x^\lambda, \qquad y'^\lambda = y^\lambda, \qquad \xi_\alpha' = \Lambda^\beta_\alpha \xi_\beta, \qquad \bar{\xi}'^\alpha = \Lambda_\beta^{-1\,\alpha} \bar{\xi}^\beta \tag{2.2}$$

will be given as follows:

$$\begin{aligned}
&\text{a)} \quad \frac{\partial U}{\partial x^\lambda} = \frac{\partial f'}{\partial x^\lambda} + \frac{\partial f'}{\partial \xi_\alpha'} \frac{\partial \Lambda^\beta_\alpha}{\partial x^\lambda} \xi_\beta + \frac{\partial f'}{\partial \bar{\xi}'^\alpha} \frac{\partial \Lambda^{-1\alpha}_{\ \beta}}{\partial x^\lambda} \bar{\xi}^\beta, \\
&\text{b)} \quad \frac{\partial U}{\partial \xi_\alpha} = \frac{\partial f'}{\partial \xi_\beta'} \Lambda^\alpha_\beta + \frac{\partial f'}{\partial \xi_\beta'} \frac{\partial \Lambda^\gamma_\beta}{\partial \xi_\alpha} \xi_\gamma + \frac{\partial f'}{\partial \bar{\xi}'^\beta} \frac{\partial \Lambda^{-1\beta}_{\ \gamma}}{\partial \xi_\alpha} \bar{\xi}^\gamma, \\
&\text{c)} \quad \frac{\partial U}{\partial \bar{\xi}^\alpha} = \frac{\partial f'}{\partial \bar{\xi}'^\beta} \Lambda^{-1\beta}_{\ \alpha} + \frac{\partial f'}{\partial \xi_\beta'} \frac{\partial \Lambda^\gamma_\beta}{\partial \bar{\xi}^\alpha} \xi_\gamma + \frac{\partial f'}{\partial \bar{\xi}'^\beta} \frac{\partial \Lambda^{-1\beta}_{\ \gamma}}{\partial \bar{\xi}^\alpha} \bar{\xi}^\gamma,
\end{aligned} \tag{2.3}$$

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1) Numbers in brackets refer to the references at the end of the paper.

$$\text{d)}\quad \frac{\partial U}{\partial y^\lambda}=\frac{\partial f'}{\partial y^\lambda}+\frac{\partial f'}{\partial \xi'_\beta}\frac{\partial \Lambda^\gamma_\beta}{\partial y^\lambda}\xi_\gamma+\frac{\partial f'}{\partial \bar{\xi}'^\beta}\frac{\partial \Lambda^{-1\beta}_{\ \ \gamma}}{\partial y^\lambda}\bar{\xi}^\gamma.$$

Taking into account that (2.23) of [**4**], namely:

$$(2.4)\qquad \begin{aligned}\frac{\partial^{[*]}}{\partial x^\lambda}&=\left(\frac{\partial}{\partial x^\lambda}+N_{\alpha\lambda}\frac{\partial}{\partial \xi_\alpha}+\bar{N}^\alpha_\lambda\frac{\partial}{\partial \bar{\xi}^\alpha}\right)-(\Gamma^{\ k}_{\tau\lambda}+\bar{C}^{k\alpha}_\tau N_{\alpha\lambda}+\bar{N}^\alpha_\lambda C^{\ k}_{\tau\alpha})y^\tau\frac{\partial}{\partial y^k}\\ &=\frac{\partial^{(*)}}{\partial x^\lambda}-(\Gamma_{\tau\lambda}^{\ (*)k}y^\tau)\frac{\partial}{\partial y^k},\end{aligned}$$

(where, the non-linear connections coefficients $N_{\alpha\lambda}$, $\bar{N}^\alpha_\lambda$, are given in [**2**]), we substitute (2.3) in (2.4), then, the non linear-connection coefficients have to be transformed for Lorentz scalar quantities as:

$$(2.5)\qquad \begin{aligned}&\text{a)}\quad N'_{\alpha\lambda}=N_{\beta\lambda}\Lambda^\beta_\alpha+\frac{\partial^{[*]}\Lambda^\beta_\alpha}{\partial x^\lambda}\xi_\beta,\\ &\text{a')}\quad \bar{N}'^\alpha_\lambda=\bar{N}^\beta_\lambda\Lambda_\beta^{-1\,\alpha}+\frac{\partial^{[*]}\Lambda^{-1\alpha}_{\ \ \beta}}{\partial x^\lambda}\bar{\xi}^\beta.\end{aligned}$$

In the above mentioned (2.5) a), a') the relation $\frac{\partial^{[*]}}{\partial x^\lambda}=\frac{\partial^{[*]'}}{\partial x^\lambda}$ was used for [*]-differential operators. For the calculation of the transformation character of non-linear connection coefficients $n_{\alpha\lambda}$, $\bar{n}^\alpha_\lambda$, $\bar{n}^{0\alpha}_\beta$, $\bar{n}^{\beta\alpha}_0$, $n^0_{\beta\alpha}$, $n^\beta_{0\alpha}$ are used the relations:

$$\frac{\partial^{[*]}}{\partial \xi_\alpha}=\Lambda^\alpha_\beta\frac{\partial^{[*]'}}{\partial \xi_\beta},\qquad \frac{\partial^{[*]}}{\partial \bar{\xi}^\beta}=\Lambda_\beta^{-1\,\alpha}\frac{\partial^{[*]'}}{\partial \bar{\xi}^\alpha},\qquad \frac{\partial^{[*]}}{\partial y^\lambda}=\frac{\partial^{[*]'}}{\partial y'^\lambda}.$$

Also by means of (2.23) b), c), d) of [**4**] and (2.3) we oftain

$$(2.5)\qquad \begin{aligned}&\text{b)}\quad n'_{\beta\lambda}=\Lambda^\alpha_\beta n_{\alpha\lambda}+\frac{\partial^{[*]}\Lambda^\gamma_\beta}{\partial y^\lambda}\xi_\gamma,\\ &\text{b')}\quad \bar{n}'^\beta_\lambda=\Lambda^{-1\beta}_\alpha\bar{n}^\alpha_\lambda+\frac{\partial^{[*]}\Lambda^{-1\beta}_{\ \ \gamma}}{\partial y^\lambda}\bar{\xi}^\gamma,\\ &\text{c)}\quad \bar{n}'^{0\beta}_\delta=\Lambda^{-1\beta}_\alpha\left(\Lambda^\varepsilon_\delta\bar{n}^{0\alpha}_\varepsilon+\frac{\partial^{[*]}\Lambda^\gamma_\delta}{\partial \xi_\alpha}\xi_\gamma\right),\\ &\text{c')}\quad \bar{n}'^{\delta\beta}_0=\Lambda^{-1\beta}_\alpha\left(\Lambda^{-1\delta}_{\ \ \varepsilon}\bar{n}^{\varepsilon\alpha}_0+\frac{\partial^{[*]}\Lambda^{-1\delta}_{\ \ \gamma}}{\partial \xi_\alpha}\bar{\xi}^\gamma\right),\\ &\text{d)}\quad n'^{\ 0}_{\beta\alpha}=\Lambda^\delta_\alpha\left(\Lambda^\gamma_\beta n^0_{\gamma\delta}+\frac{\partial^{[*]}\Lambda^\gamma_\beta}{\partial \bar{\xi}^\delta}\xi_\gamma\right),\\ &\text{d')}\quad n'^{\ \beta}_{0\alpha}=\Lambda^\delta_\alpha\left(n_0{}^\gamma{}_\delta\Lambda^{-1\beta}_{\ \ \gamma}+\bar{\xi}^\gamma\frac{\partial^{[*]}\Lambda^{-1\beta}_{\ \ \gamma}}{\partial \bar{\xi}^\delta}\right).\end{aligned}$$

Consequently, [*]-derivatives of the quantities (2.1) will satisfy the following relations:

$$(2.6)\qquad \begin{aligned}&\text{a)}\quad \frac{\partial^{[*]}U}{\partial x^\lambda}=\frac{\partial f'}{\partial x^\lambda}+N'_{\alpha\lambda}\frac{\partial f'}{\partial \xi'_\alpha}+\bar{N}'^{\ \alpha}_\lambda\frac{\partial f'}{\partial \bar{\xi}'^\alpha}-\Gamma^{(*)k}_{\tau\lambda}y'^\tau\frac{\partial f'}{\partial y'^k},\\ &\text{b)}\quad \frac{\partial^{[*]}U}{\partial y^\lambda}=\frac{\partial f'}{\partial y'^\lambda}+n'_{\alpha\lambda}\frac{\partial f'}{\partial \xi'_\alpha}+\bar{n}'^\alpha_\lambda\frac{\partial f'}{\partial \bar{\xi}'^\alpha}-C^{(*)'k}_{\tau\lambda}y'^\tau\frac{\partial f'}{\partial y'^k},\\ &\text{c)}\quad \Lambda^{-1\alpha}_{\ \ \beta}\frac{\partial^{[*]}U}{\partial \xi_\beta}=\frac{\partial f'}{\partial \xi'_\alpha}+\bar{n}'^{0\alpha}_\beta\frac{\partial f'}{\partial \xi'_\beta}+\bar{n}'^{\beta\alpha}_0\frac{\partial f'}{\partial \bar{\xi}'^\beta}-\bar{C}'^{(*)k\alpha}_\tau y'^\tau\frac{\partial f'}{\partial y'^k},\\ &\text{d)}\quad \Lambda^\beta_\alpha\frac{\partial^{[*]}U}{\partial \bar{\xi}^\beta}=\frac{\partial f'}{\partial \bar{\xi}'^\alpha}+n'^{\ 0}_{\beta\alpha}\frac{\partial f'}{\partial \xi'_\beta}+n'^{\ \beta}_{0\alpha}\frac{\partial f'}{\partial \bar{\xi}^\beta}-C^{(*)'k}_{\tau\alpha}y'^\tau\frac{\partial f'}{\partial y'^k}.\end{aligned}$$

When we have Lorentz-scalar quantities

$$f'(x, y, \xi', \bar{\xi}')=f(x, y, \xi, \bar{\xi}), \tag{2.7}$$

then, the $\frac{\partial^{[*]}f}{\partial x^{\lambda}}, \frac{\partial^{[*]}f}{\partial y^{\lambda}}, \frac{\partial^{[*]}f}{\partial \xi_{\alpha}}, \frac{\partial^{[*]}f}{\partial \bar{\xi}_{\alpha}}$ are transformed as Lorentz-scalar and spinors adjoint to each other, respectively. Consequently [*]-differentiations are covariant differential operators for Lorentz-scalar quantities. The spin connection coefficients $\omega^{[*]}_{ab\lambda}$, $\theta^{[*]}_{ab\lambda}$, $\bar{\theta}^{[*]\beta}_{ab}$, $\theta^{[*]}_{ab\beta}$ will be transformed by Lorentz transformations as follows:

We consider the relation (3.23) a) of [4], namely,

$$\omega^{[*]}_{ab\lambda}=\left(\frac{\partial^{[*]}h^{\mu}_{a}}{\partial x^{\lambda}}+\Gamma^{[*]\mu}_{\nu\lambda}h^{\nu}_{a}\right)h_{\mu b} \tag{2.8}$$

and respectively,

$$\omega^{[*]\prime}_{ab\lambda}=\left(\frac{\partial^{[*]}h'^{\mu}_{a}}{\partial x^{\lambda}}+\Gamma^{[*]\prime\mu}_{\nu\lambda}h'^{\nu}_{a}\right)h'_{\mu b}, \tag{2.9}$$

$$\Gamma^{[*]\mu}_{\nu\lambda}=\Gamma^{[*]}_{\nu k\lambda}g^{k\mu} \tag{2.10}$$

also for the tetrads h'^{μ}_{a} and h'^{μ}_{b} valid the relation $h'^{\mu}_{a}=L^{b}_{a}h^{\mu}_{b}$ ((4.1) of [2]), then taking into account that (2.8), (2.9) and (2.10) we take the transformation formula of spin connection coefficients $\omega^{[*]}_{ab\lambda}$,

$$\text{a)}\quad \omega^{[*]\prime}_{ab\lambda}=L^{c}_{a}L^{d}_{b}\omega^{[*]}_{cd\lambda}+\frac{\partial^{[*]}L^{c}_{a}}{\partial x^{\lambda}}h_{cd}L^{d}_{b}, \tag{2.11}$$

where the connection coefficients $\Gamma^{[*]\mu}_{\nu\lambda}$, $\Gamma'^{[*]\mu}_{\nu\lambda}$ are Lorentz-scalar. With similar procedure we can take the transformed connection coefficients of $\theta^{[*]}_{ab\lambda}$, $\bar{\theta}^{[*]\beta}_{ab}$, $\theta^{[*]}_{ab\beta}$, using the relations (3.23)b), c), d), of [4]. Therefore the transformations formulae are given by the expressions:

$$\text{b)}\quad \theta^{[*]\prime}_{ab\lambda}=L^{c}_{a}L^{d}_{b}\theta^{[*]}_{cd\lambda}+\frac{\partial^{[*]}L^{c}_{a}}{\partial y^{\lambda}}n_{cd}L^{d}_{b},$$

$$\text{c)}\quad \bar{\theta}^{[*]\prime\beta}_{ab}=\Lambda^{-1\beta}_{\ \ \gamma}\left[\bar{\theta}^{[*]\gamma}_{cd}L^{c}_{a}L^{d}_{b}+\frac{\partial^{[*]}L^{c}_{a}}{\partial \bar{\xi}_{\gamma}}L^{d}_{b}n_{dc}\right], \tag{2.11}$$

$$\text{d)}\quad \theta^{[*]\prime}_{ab\beta}=\Lambda^{\gamma}_{\beta}\left[\theta^{[*]}_{cd\gamma}L^{c}_{a}L^{d}_{b}+\frac{\partial^{[*]}L^{c}_{a}}{\partial \xi^{\gamma}}L^{d}_{b}n_{dc}\right].$$

Next, we shall derive the transformation character of the spin connection coefficients $\Gamma^{[*]\beta}_{\gamma\lambda}$, $C^{[*]\beta}_{\gamma\alpha}$, $C^{[*]\beta\alpha}_{\gamma}$ and $\bar{C}^{[*]\beta}_{\gamma\alpha}$ under Lorentz transformations. If we take the relation (3.6) a) of [4], namely,

$$N_{\alpha\lambda}=\Gamma^{[*]\gamma}_{\alpha\lambda}\xi_{\gamma} \tag{2.12}$$

and

$$N'_{\alpha\lambda}=\Gamma^{[*]\prime\gamma}_{\alpha\lambda}\xi'_{\gamma} \tag{2.13}$$

and we substitute (2.5) a) in (2.13), then, we obviously take the required transformation formula,

$$\text{a)}\quad \Gamma^{[*]\prime\delta}_{\alpha\lambda}=\Lambda^{-1\delta}_{\varepsilon}\Lambda^{\beta}_{\alpha}\Gamma^{[*]\varepsilon}_{\beta\lambda}+\frac{\partial^{[*]}\Lambda^{\varepsilon}_{\alpha}}{\partial x^{\lambda}}\Lambda^{-1\delta}_{\varepsilon}. \tag{2.14}$$

With similar procedure we can find the transformation character of the spin connections,

$C_{\beta\lambda}^{[*]\varepsilon}$, $\tilde{C}_{\gamma}^{[*]\beta\alpha}$, $C_{\gamma\alpha}^{[*]\beta}$.

$$\text{(2.14)}\quad \begin{aligned} &\text{b)}\quad C_{\alpha\lambda}^{[*]'\delta}=\Lambda_{\varepsilon}^{-1\delta}\Lambda_{\alpha}^{\beta}C_{\beta\lambda}^{[*]\varepsilon}+\frac{\partial^{[*]}\Lambda_{\alpha}^{\varepsilon}}{\partial y^{\lambda}}\Lambda_{\varepsilon}^{-1\delta},\\ &\text{c)}\quad \tilde{C}'^{[*]\delta\rho}=\left[\Lambda_{\varepsilon}^{\gamma}\tilde{C}_{\gamma}^{[*]\beta\alpha}\Lambda_{\beta}^{-1\delta}+\frac{\partial^{[*]}\Lambda_{\varepsilon}^{\gamma}}{\partial\xi_{\alpha}}\Lambda_{\gamma}^{-1\delta}\right]\Lambda_{\alpha}^{-1\rho},\\ &\text{d)}\quad C_{\varepsilon\rho}^{[*]'\delta}=\Lambda_{\rho}^{\alpha}\left[\Lambda_{\varepsilon}^{\gamma}C_{\gamma\alpha}^{[*]\beta}\Lambda_{\beta}^{-1\delta}+\frac{\partial^{[*]}\Lambda_{\varepsilon}^{\gamma}}{\partial\bar{\xi}^{\alpha}}\Lambda_{\gamma}^{-1\delta}\right]. \end{aligned}$$

Finally, from (3.20) of [4] and (2.5), (2.14), arbitrary terms a_{λ}, b_{λ}, $\bar{\beta}^{\alpha}$, β_{α}, are transformed as follows:

$$\begin{aligned} a_{\lambda}&=a'_{\lambda}+\bar{\beta}'^{\alpha}\left(\frac{\partial\Lambda_{\alpha}^{\beta}}{\partial x^{\lambda}}\right)\xi_{\beta}+\bar{\xi}^{\beta}\left(\frac{\partial\Lambda_{\beta}^{-1\alpha}}{\partial x^{\lambda}}\right)\beta'_{\alpha},\\ b_{\lambda}&=b'_{\lambda}+\bar{\beta}'^{\beta}\left(\frac{\partial\Lambda_{\beta}^{\gamma}}{\partial y^{\lambda}}\right)\xi_{\gamma}+\bar{\xi}^{\gamma}\left(\frac{\partial\Lambda_{\gamma}^{-1\beta}}{\partial y^{\lambda}}\right)\beta'_{\beta},\\ \bar{\beta}^{\alpha}&=\bar{\beta}'^{\gamma}\Lambda_{\gamma}^{\alpha}+\bar{\beta}'^{\gamma}\left(\frac{\partial\Lambda_{\gamma}^{\varepsilon}}{\partial\xi_{\alpha}}\right)\xi_{\varepsilon}+\bar{\xi}^{\varepsilon}\left(\frac{\partial\Lambda_{\varepsilon}^{-1\gamma}}{\partial\xi_{\alpha}}\right)\beta'_{\gamma},\\ \beta_{\alpha}&=\Lambda_{\alpha}^{-1\gamma}\beta'_{\gamma}+\bar{\beta}'^{\gamma}\left(\frac{\partial\Lambda_{\gamma}^{\varepsilon}}{\partial\bar{\xi}^{\alpha}}\right)\xi_{\varepsilon}+\bar{\xi}^{\varepsilon}\left(\frac{\partial\Lambda_{\varepsilon}^{-1\gamma}}{\partial\bar{\xi}^{\alpha}}\right)\beta'_{\gamma}. \end{aligned}$$

§ 3. Curvature. In this section we shall present the form of the curvature of the above-mentioned spaces. There must exist ten kinds of curvature tensors corresponding to four kinds of covariant derivatives with respect to x^{λ}, y^{λ}, ξ_{α}, $\bar{\xi}^{\alpha}$, (coordinates, vector variables, spinor variables).

If we denote with N, n the number of curvatures and the kinds of covariant derivatives, then we have generally, $N=\frac{n(n+1)}{2}$. In our case $N=10$, $n=4$. Like in [2] §5, here, they appear three different expressions of the above-mentioned ten curvature tensors, which are closely related to each other. The relation between ten curvature tensors $T_{\nu XY}^{\mu}$ and ten spin-curvature tensors T_{abXY}, will be the following:

$$T_{abXY}=T_{\nu XY}^{\mu}h_{a}^{\nu}h_{\mu b}, \tag{3.1}$$

which arises from integrability conditions of the partial differential equations (cf. (3.22) of [2]).

The curvature tensors which are calculated below, come from the Ricci identities [5], [6], as well as the commutation formulae of the [*]-differential operators $\frac{\partial^{[*]}}{\partial X}$, $\frac{\partial^{[*]}}{\partial Y}$.

The curvature tensors $T_{\nu XY}^{\mu}$ are defined as follows:

$$\begin{aligned} R_{\nu\lambda\kappa}^{\mu}=&\frac{\partial^{[*]}\Gamma_{\nu\lambda}^{[*]\mu}}{\partial x^{\kappa}}-\frac{\partial^{[*]}\Gamma_{\nu\kappa}^{[*]\mu}}{\partial x^{\lambda}}+\Gamma_{\nu\lambda}^{[*]\tau}\Gamma_{\tau\kappa}^{[*]\mu}-\Gamma_{\nu\kappa}^{[*]\tau}\Gamma_{\tau\lambda}^{[*]\mu}\\ &-(A_{\gamma\lambda\kappa}^{[*]}\bar{C}_{\nu}^{[*]\mu\gamma}+\hat{A}_{\lambda\kappa}^{[*]\gamma}C_{\nu\gamma}^{[*]\mu}+\bar{\bar{A}}_{\lambda\kappa}^{\tau}C_{\nu\tau}^{[*]\mu}), \end{aligned}$$

where $A_{\gamma\lambda\kappa}^{[*]}$, $\hat{A}_{\lambda\kappa}^{[*]\gamma}$, $\bar{\bar{A}}_{\lambda\kappa}^{\rho}$, $A_{\gamma\lambda\kappa}$ are given by

$$\begin{aligned} A_{\gamma\lambda\kappa}^{[*]}&=A_{\gamma\lambda\kappa}-C_{\gamma}^{0\xi}A_{\xi\lambda\kappa}-\hat{A}_{\lambda\kappa}^{\xi}C_{\gamma\xi}^{0}-\bar{\bar{A}}_{\lambda\kappa}^{k}C_{\gamma k}^{0},\\ \hat{A}_{\lambda\kappa}^{[*]\gamma}&=\hat{A}_{\lambda\kappa}^{\gamma}+\bar{C}_{0}^{\gamma\varrho}A_{\varrho\lambda\kappa}-\hat{A}_{\lambda\kappa}^{\varrho}C_{0\varrho}^{\gamma}+\bar{\bar{A}}_{\lambda\kappa}^{k}C_{0k}^{\gamma},\\ \bar{\bar{A}}_{\lambda\kappa}^{[*]\rho}&=\bar{\bar{A}}_{\lambda\kappa}^{\rho}+(\bar{C}_{\tau}^{\rho\xi}y^{\tau})A_{\xi\lambda\kappa}+\hat{A}_{\lambda\kappa}^{\xi}(C_{\tau\xi}^{\rho}y^{\tau}), \end{aligned}$$

$$A_{\gamma\lambda k}=\frac{\partial^{[*]}N_{\gamma\lambda}}{\partial x^{k}}-\frac{\partial^{[*]}N_{\gamma k}}{\partial x^{\lambda}},\qquad \hat{A}^{\gamma}_{\lambda k}=\frac{\partial^{[*]}\hat{N}^{\gamma}_{\lambda}}{\partial x^{k}}-\frac{\partial^{[*]}\hat{N}^{\gamma}_{k}}{\partial x^{\lambda}},$$

$$\bar{\bar{A}}^{\rho}_{\lambda k}=\frac{\partial^{[*]}}{\partial x^{k}}[-(\Gamma^{\rho}_{\tau\lambda}+\bar{C}^{\rho\alpha}_{\tau}N_{\alpha\lambda}+\bar{\bar{N}}^{\alpha}_{\lambda}C^{\rho}_{\tau\alpha})y^{\tau}]$$
$$-\frac{\partial^{[*]}}{\partial x^{\lambda}}[-(\Gamma^{\rho}_{\tau k}+\bar{C}^{\rho\alpha}_{\tau}N_{\alpha k}+\bar{\bar{N}}^{\alpha}_{k}C^{\rho}_{\tau\alpha})y^{\tau}].$$

Similarly, the curvatures $P^{\mu}_{\nu\lambda\alpha}$ and $W^{\mu}_{\nu\lambda k}$ can be defined as follows:

$$P^{\mu}_{\nu\lambda\alpha}=\frac{\partial^{[*]}\Gamma^{[*]\mu}_{\nu\lambda}}{\partial\xi^{\alpha}}-\frac{\partial^{[*]}C^{[*]\mu}_{\gamma\alpha}}{\partial x^{\lambda}}+\Gamma^{[*]k}_{\nu\lambda}C^{[*]\mu}_{k\alpha}-\Gamma^{[*]\mu}_{k\lambda}C^{[*]k}_{\nu\alpha}$$
$$-(E^{[*]}_{\gamma\lambda\alpha}\bar{C}^{[*]\mu\gamma}_{\nu}+\hat{E}^{[*]\gamma}_{\lambda\alpha}C^{[*]\mu}_{\nu\gamma}+\bar{\bar{E}}^{[*]k}_{\lambda\alpha}C^{[*]\mu}_{\nu k}),$$

$$W^{\mu}_{\nu\lambda k}=\frac{\partial^{[*]}C^{[*]\mu}_{\nu\lambda}}{\partial x^{k}}-\frac{\partial^{[*]}\Gamma^{[*]\mu}_{\nu k}}{\partial y^{\lambda}}+\Gamma^{[*]i}_{\nu k}C^{[*]\mu}_{i\lambda}-\Gamma^{[*]\mu}_{ik}C^{[*]i}_{\nu\lambda}$$
$$-(D^{[*]}_{\gamma\lambda k}C^{[*]\mu\gamma}_{\nu}+\hat{D}^{[*]\gamma}_{\lambda k}C^{[*]\mu}_{\nu\gamma}+\bar{\bar{D}}^{\tau}_{\lambda k}C^{[*]\mu}_{\nu\tau}).$$

The quantities $E^{[*]}$, $\hat{E}^{[*]}$, $\bar{\bar{E}}^{[*]}$, $D^{[*]}$, $\hat{D}^{[*]}$, $\bar{\bar{D}}^{[*]}$, are defined respectively to $A^{[*]}$, $\hat{A}^{[*]}$, and $\bar{\bar{A}}^{[*]}$. As a matter of fact the expressions are too big to be presented for all ten curvature tensors, we prefer to give an algorithm for the general case, presenting the following table of symbols.

Table 1.

coordinate vector spinors	connection coefficients	Non linear connections		
		$N_{\gamma X}(Y)$	$\hat{N}^{\gamma}_{X}(Y)$	$\bar{\bar{N}}^{k}_{X}(Y)$
x^{λ}	$\Gamma^{[*]\mu}_{\nu\lambda}$	$N_{\alpha\lambda}$	$\bar{\bar{N}}^{\alpha}_{\lambda}$	$-(\Gamma^{k}_{\tau\lambda}+\bar{C}^{k\alpha}_{\tau}N_{\alpha\lambda}+\bar{\bar{N}}^{\alpha}_{\lambda}C^{k}_{\tau\alpha})y^{\tau}$
y^{λ}	$C^{[*]\mu}_{\nu\lambda}$	$n_{\alpha\lambda}$	$\bar{n}^{\alpha}_{\lambda}$	$-(C^{k}_{\tau\lambda}+\bar{C}^{k\alpha}_{\tau}n_{\alpha\lambda}+\bar{n}^{\alpha}_{\lambda}C^{k}_{\tau\alpha})y^{\tau}$
ξ_{α}	$\bar{C}^{[*]\mu\alpha}_{\nu}$	$n^{0\alpha}_{\beta}$	$\bar{n}^{\beta\alpha}_{0}$	$-(\bar{C}^{k\alpha}_{\tau}+\bar{C}^{k\beta}_{\tau}\bar{n}^{0\alpha}_{\beta}+\bar{n}^{\beta\alpha}_{0}C^{k}_{\tau\beta})y^{\tau}$
$\bar{\xi}^{\alpha}$	$C^{[*]\mu}_{\nu\alpha}$	$n^{0}_{\beta\alpha}$	$n^{\beta}_{0\alpha}$	$-(C^{k}_{\tau\alpha}+\bar{C}^{k\beta}_{\tau}n^{0}_{\beta\alpha}+n^{\beta}_{0\alpha}C^{k}_{\tau\beta})y^{\tau}$

In general for each of the ten curvature tensors, we have

$$T^{\mu}_{\nu XY}=\frac{\partial^{[*]}\mathrm{Con}\,X^{\mu}_{\nu X}}{\partial Y}-\frac{\partial^{[*]}\mathrm{Con}\,Y^{\mu}_{\nu Y}}{\partial X}+\mathrm{Con}\,X^{\tau}_{\nu X}\mathrm{Con}\,Y^{\mu}_{\tau Y}-\mathrm{Con}\,Y^{\tau}_{\nu Y}\mathrm{Con}\,X^{\mu}_{\tau X}\tag{3.2}$$
$$-(A^{[*]}_{\gamma XY}\bar{C}^{[*]\mu\gamma}_{\nu}+\hat{A}^{[*]\gamma}_{XY}C^{[*]\mu}_{\nu\gamma}+\bar{\bar{A}}^{[*]\tau}_{XY}C^{[*]\mu}_{\nu\tau}),$$

where $A^{[*]}_{\gamma XY}$, $\hat{A}^{[*]\gamma}_{XY}$, $\bar{\bar{A}}^{[*]\rho}_{XY}$, $A_{\gamma XY}$, $\hat{A}^{\gamma}_{XY}$ and $\bar{\bar{A}}^{\rho}_{XY}$ are given by,

$$A^{[*]}_{\gamma XY}=A_{\gamma XY}-C^{0\xi}_{\gamma}A_{\xi XY}-\hat{A}^{\xi}_{XY}C^{0}_{\gamma\xi}-\bar{\bar{A}}^{k}_{XY}C^{0}_{\gamma k},$$
$$\hat{A}^{\gamma}_{XY}=\hat{A}^{\gamma}_{XY}+\tilde{C}^{\gamma\xi}_{0}A_{\xi XY}+\hat{A}^{\xi}_{XY}C^{\gamma}_{0\xi}+\bar{\bar{A}}^{k}_{XY}C^{\gamma}_{0k},$$
$$\bar{\bar{A}}^{[*]\rho}_{XY}=\bar{\bar{A}}^{\rho}_{XY}+(\bar{C}^{\rho\xi}_{\tau}y^{\tau})A_{\xi XY}+A^{\xi}_{XY}(C^{\rho}_{\tau\xi}y^{\tau}),$$
$$A_{\gamma XY}=\frac{\partial^{[*]}N_{\gamma X}}{\partial Y}-\frac{\partial^{[*]}N_{\gamma Y}}{\partial X},\qquad \hat{A}^{\gamma}_{XY}=\frac{\partial^{[*]}\hat{N}^{\gamma}_{X}}{\partial Y}-\frac{\partial^{[*]}\hat{N}^{\gamma}_{Y}}{\partial X},$$
$$\bar{\bar{A}}^{\rho}_{XY}=\frac{\partial^{[*]}\bar{\bar{N}}^{\rho}_{X}}{\partial Y}-\frac{\partial^{[*]}\bar{\bar{N}}^{\rho}_{Y}}{\partial X},$$

where $\mathrm{Con}\,X^{\mu}_{\nu X}$ represent the connection coefficients $\Gamma^{[*]\mu}_{\nu\lambda}$, $C^{[*]\mu}_{\nu\lambda}$, $\bar{C}^{[*]\mu\alpha}_{\nu}$, $C^{[*]\mu}_{\nu\alpha}$. We write down all ten curvatures using the algorithm presented the above and adopt the following symbolism:

Now, the spin curvature tensors T_{abXY} are defined in (3.3). The table 3 is useful. According to the table 3 our general formula becomes

Table 2.

X-Y	$T^{\mu}_{\nu XY}$	$A^{[*]}$	$\hat{A}^{[*]}$	$\tilde{A}^{[*]}$
x-x	R	A	$\hat{A}$	$\tilde{A}$
x-ξ	P	E	$\hat{E}$	$\tilde{E}$
x-$\bar{\xi}$	$\bar{P}$	F	$\hat{F}$	$\tilde{F}$
x-y	W	D	$\hat{D}$	$\tilde{D}$
ξ-ξ	$\tilde{Q}$	B	$\hat{B}$	$\tilde{B}$
ξ-$\bar{\xi}$	S	V	$\hat{V}$	$\tilde{V}$
ξ-y	Ω	G	$\hat{G}$	$\tilde{G}$
$\bar{\xi}$-$\bar{\xi}$	Q	J	$\hat{J}$	$\tilde{J}$
$\bar{\xi}$-y	$\bar{\Omega}$	Φ	$\hat{\Phi}$	$\tilde{\Phi}$
y-y	Z	H	$\hat{H}$	$\tilde{H}$

Table 3.

coordinates/ vector/spinors	Spin connection coefficients I
x^{λ}	$\omega^{[*]}_{ab\lambda}$
y^{λ}	$\theta^{[*]}_{ab\lambda}$
ξ_{α}	$\bar{\theta}^{[*]\alpha}_{ab}$
$\bar{\xi}^{\alpha}$	$\theta^{[*]}_{ab\alpha}$

(3.3)
$$T_{abXY}=\frac{\partial^{[*]}\mathrm{sp.\ Con}\,X_{abX}}{\partial Y}-\frac{\partial^{[*]}\mathrm{sp.\ Con}\,Y_{abY}}{\partial X}$$
$$+\mathrm{sp.\ Con}\,X_{acX}\,\mathrm{sp.\ Con}\,Y^{c}_{bY}-\mathrm{sp.\ Con}\,Y_{acY}\,\mathrm{sp.\ Con}\,X^{c}_{bX}$$
$$-(A^{[*]}_{\gamma XY}\,\theta^{[*]\gamma}_{ab}+\hat{A}^{\gamma}_{XY}\,\theta^{[*]}_{ab\gamma}+\tilde{A}^{[*]\tau}_{XY}\,\theta^{[*]}_{ab\tau}),$$

where sp. Con X_{abX} represent the spin connection coefficients $\omega^{[*]}_{ab\lambda}$, $\theta^{[*]}_{ab\lambda}$, $\bar{\theta}^{[*]\alpha}_{ab}$, $\theta^{[*]}_{ab\alpha}$ and $A^{[*]}$ are defined as before. We can write down all the spin-curvature tensors using the symbolism of table 2 with appropriate indices.

These spin-curvature tensors will also appear in Ricci' formulae for a Lorentz vector field. To examine the transformation character of the curvature tensors, it is convenient to divide them into the parts $T^{(0)\mu}_{\nu XY}$ and $T^{(1)\mu}_{\nu XY}$:

$$T^{\mu}_{\nu XY}=T^{(0)\mu}_{\nu XY}-T^{(1)\mu}_{\nu XY},$$

where

$$T^{(0)\mu}_{\nu XY}=\frac{\partial^{[*]}\mathrm{Con}\,X^{\mu}_{\nu X}}{\partial Y}-\frac{\partial^{[*]}\mathrm{Con}\,X^{\mu}_{\nu Y}}{\partial X}+\mathrm{Con}\,X^{\tau}_{\nu X}\mathrm{Con}\,Y^{\mu}_{\tau Y}-\mathrm{Con}\,Y^{\tau}_{\nu Y}\mathrm{Con}\,X^{\mu}_{\tau X},$$
$$T^{(1)\mu}_{\nu XY}=A^{[*]}_{\gamma XY}\,\bar{C}^{[*]\mu\gamma}_{\nu}+\hat{A}^{[*]\gamma}_{XY}\,C^{[*]\mu}_{\nu\gamma}+\tilde{A}^{[*]\tau}_{XY}\,C^{[*]\mu}_{\nu\tau}.$$

The curvature tensors $T^{(1)\mu}_{\nu XY}$ are expected to have the same transformation character as $T^{(0)\mu}_{\nu XY}$ and $T_{\nu}{}^{\mu}_{XY}$, and are confirmed to transform as tensors or spinors under general coordinate transformations and local Lorentz transformations by formulae (2.3), (2.5) and (2.14). The arbitrary terms of spin connection coefficients are contained only in the parts $T^{(1)\mu}_{\nu XY}$, the arbitrarines disappear completely by virtue of the homogeneity of $\Gamma^{[*]}_{\lambda}$, $C^{[*]}_{\lambda}$, $\tilde{C}^{[*]\alpha}$, $C^{[*]}_{\alpha}$. Therefore, $T^{\mu}_{\nu XY}$ as well as $T^{(1)\mu}_{\nu XY}$ are defined unambiguously. The following conditions are imposed on $T^{(0)\mu}_{\nu XY}$ and $T^{(1)\mu}_{\nu XY}$ and, therefore on $T^{\mu}_{\nu XY}$.

Contractions of $\bar{\xi}^{\alpha}$, ξ_{α}, y^{λ}, with the curvature tensors give the following :

(3.4) $$\bar{\xi}^{\alpha} T^{(\ )}_{(\bar{\xi}^{\alpha},\, x^{\lambda})} = 0, \qquad \bar{\xi}^{\alpha} T^{(\ )}_{(\bar{\xi}^{\alpha},\, \xi_{\alpha})} = 0,$$
$$\bar{\xi}^{\alpha} T^{(\ )}_{(\bar{\xi}^{\alpha},\, y^{\lambda})} = 0, \qquad \bar{\xi}^{\alpha} T^{(\ )}_{(\bar{\xi}^{\alpha},\, \bar{\xi}^{\alpha})} = 0.$$

The above mentioned structures and properties of curvature tensors $T^{\mu}_{\nu XY}$ are transformed to those of spin-curvature tensors T_{abXY} through the relations (3.1). Also, the integrability conditions of the partial differential equations or Ricci formulae for a spinor field, lead to another spin-curvature tensors $T^{\delta}_{\varepsilon XY}$ which are related to T_{abXY} by the relation of

$$T^{\delta}_{\varepsilon XY} = \frac{1}{2} T_{abXY} (S^{ab})^{\delta}_{\varepsilon} + i T_{XY} I^{\delta}_{\varepsilon},$$

where I^{δ}_{ε} is the unit matrix, $T^{\delta}_{\varepsilon XY}$, T_{XY} are defined below by (3.5) and (3.6) respectivly, T_{abXY} are given by (3.3) and s^{ab} by (3.18) of [4].

Again, in order to present the spin-curvature tensors $T^{\delta}_{\varepsilon XY}$ we are going to use an algorithm along with appropriate table.

The general formula is

Table 4.

coordinates/ vectors/spinors	Spin connection coefficients 2
x^{λ}	$\Gamma^{[*]\beta}_{\gamma\lambda}$
y^{λ}	$C^{[*]\beta}_{\gamma\lambda}$
ξ_{α}	$\tilde{C}^{[*]\beta\alpha}_{\gamma}$
$\bar{\xi}^{\alpha}$	$C^{[*]\beta}_{\gamma\alpha}$

(3.5) $$T^{\delta}_{\varepsilon XY} = \frac{\partial^{[*]} \mathrm{sp.\ Con}\, X^{\delta}_{\varepsilon X}}{\partial Y} - \frac{\partial^{[*]} \mathrm{sp.\ Con}\, Y^{\delta}_{\varepsilon Y}}{\partial X} + \mathrm{sp.\ Con}\, X^{j}_{\varepsilon X} \cdot \mathrm{sp.\ Con}\, Y^{\delta}_{jY}$$
$$- \mathrm{sp.\ Con}\, Y^{j}_{\varepsilon Y} \cdot \mathrm{sp.\ Con}\, X^{\delta}_{jX} - (A^{[*]}_{jXY} \tilde{C}^{[*]\delta j}_{\varepsilon} + \tilde{A}^{[*]j}_{XY} C^{[*]\delta}_{\varepsilon j} + \bar{A}^{[*]\tau}_{XY} C^{[*]\delta}_{\varepsilon\tau}),$$

where sp. Con $X^{i}_{\varepsilon X}$ represent the spin connection coefficients and $A^{[*]}$ are defined as before.

To represent all the spin-curvature tensors, we may use the following table 5 for the symbolism.

The spin-curvature tensors T_{XY} consisting of the arbitrary terms of $\Gamma^{[*]}_{\lambda}$, $C^{[*]}_{\lambda}$, $\tilde{C}^{[*]\alpha}$, $C^{[*]}_{\alpha}$ are defined as follows:

Table 5.

XY	$T^{\delta}_{\varepsilon XY}$	$A^{[*]}$	$\hat{A}^{[*]}$	$\bar{A}^{[*]}$
x-x	X	A	$\hat{A}$	$\bar{A}$
x-y	Ψ	D	$\hat{D}$	$\bar{D}$
x-ξ	$\bar{\Xi}$	E	$\hat{E}$	$\bar{E}$
x-$\bar{\xi}$	Ξ	F	$\hat{F}$	$\bar{F}$
y-y	Y	H	$\hat{H}$	$\bar{H}$
y-ξ	$\check{U}$	G	$\hat{G}$	$\bar{G}$
y-$\bar{\xi}$	U	Φ	$\hat{\Phi}$	$\bar{\Phi}$
ξ-ξ	O	B	$\hat{B}$	$\bar{B}$
ξ-$\bar{\xi}$	K	V	$\hat{V}$	$\bar{V}$
$\bar{\xi}$-$\bar{\xi}$	O	J	$\hat{J}$	$\bar{J}$

(3.6)
$$T_{XY}=\frac{\partial^{[*]}\text{coef}\,X_X}{\partial Y}-\frac{\partial^{[*]}\text{coef}\,Y_Y}{\partial X}+i\,(\text{coef}\,X_X\text{coef}\,Y_Y-\text{coef}\,Y_Y\text{coef}\,X_X)$$
$$-(A_{\gamma XY}^{[*]}\bar{\beta}_{XY}^{[*]\gamma}+\hat{A}_{XY}^{[*]\gamma}\beta_\gamma^{[*]}+\bar{A}_{XY}^{[*]\tau}b_\tau^{[*]}),$$

where coef X_X are defined in table 7. If we want to write down all ten spin-curvature tensors T_{XY} we may use the following table 6.

Table 6.

XY	T_{XY}
x-x	φ
x-y	χ
x-ξ	Ψ
x-$\bar{\xi}$	$\bar{\Psi}$
y-y	υ
y-ξ	ν
y-$\bar{\xi}$	$\bar{\nu}$
ξ-ξ	ρ
ξ-$\bar{\xi}$	μ
$\bar{\xi}$-$\bar{\xi}$	$\bar{\rho}$

Table 7.

coordinates/ vectors/spinors	coef. Xx
x^λ	$a_\lambda^{[*]}$
y^λ	$b_\lambda^{[*]}$
ξ_α	$\bar{\beta}^{[*]\alpha}$
$\bar{\xi}^\alpha$	$\beta_\alpha^{[*]}$

The spin-curvature tensors T_{XY} are defined uniquely on account of the conditions (3.27) or (3.28) of [4] and the homogeneity properties of $\Gamma_\lambda^{[*]}$, $C_\lambda^{[*]}$, $\bar{C}^{[*]\alpha}$ and $C_\alpha^{[*]}$ although the last parentheses contain the arbitrary terms.

By virtue of the conditions (3.26), (3.28) of [4] together with the homogeneity properties of $\Gamma_\lambda^{[*]}$, $C_\lambda^{[*]}$, $\bar{C}^{[*]\alpha}$, $C_\alpha^{[*]}$, similar conditions to (3.4) are imposed on T_{XY} : that is contractions of $\bar{\xi}^\alpha$, ξ_α, y^λ with the spin-curvatures T_{XY} that they form with each one of the set $\{x^\lambda$, y^λ, ξ_α, $\bar{\xi}^\alpha\}$,

(3.7)
$$\bar{\xi}^\alpha\,\Psi_{\lambda\alpha}=0.$$

Now, from (3.1), (3.5) together with (3.4), (3.7), it is easily shown that the similar conditions to (3.2) on $T^\mu_{\nu XY}$ must be imposed on $T^\delta_{\varepsilon XY}$.

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