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A GEOMETRICAL STRUCTURE IN THE λ-PARAMETER FAMILY OF GENERALIZED METRIC SPACES *

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Introduction. In the context of the time dependent geometric subjects of the Finslerian approach, the authors will provide a different standpoint from the other one that has already been developed for the rheonomic systems. In that case, the time is taken as an arbitrary (external) parameter ([1][1]), [**6**]).

In the present work, we study the geometry of the Finslerian space-time structure in relation with a λ-parameter family of a Finsler space-time $(F^{(4)}, \lambda)_{\lambda\in I}$ and its fiber metric structure which was proposed by one of the authors [**5**].

From a physical viewpoint, this consideration of λ-parameter as an *internal coordinate*, gives rise to an "evolution" of the Finslerian space-time $F_0^{(4)}$. For this reason, it is useful to consider *a second order deformed bundle* $\overset{(2)}{T}(DF)$ as the total space of the deformation of $F_0^{(4)}$ constructed locally by $X^A = (x^k, y^i, y^0 = \lambda)$, where the internal variable λ is considered independent of x^k and y^i, $(k, i = 1, 2, 3, 4)$.

§ 1. Connection.

The metrical structure in $\overset{(2)}{T}(DF)$ is given by the form

$$G = g_{\lambda k}dx^k dx^\lambda + g_{ij}\delta y^i \delta y^j + g_{00}\delta\lambda\delta\lambda,$$

with $\delta\lambda = d\lambda + M^0_\lambda dx^\lambda + L^0_i dy^i$, $\delta y^i = dy^i + N^i_\lambda dx^\lambda$, $\lambda = y^0$.

The coefficients $M^0_\lambda(x, y, \lambda)$, $L^0_i(x, y, \lambda)$ transform like the components of a covector field on M:

$$\tilde{M}^0_\lambda = \frac{\partial x^k}{\partial \tilde{x}^\lambda}M^0_k, \quad \tilde{L}^0_j = \frac{\partial x^j}{\partial \tilde{x}^j}L^0_i, \quad \tilde{N}^k_j\frac{\partial \tilde{x}^j}{\partial x^i} = \frac{\partial \tilde{x}^k}{\partial x^j}N^j_i - \frac{\partial^2 \tilde{x}^k}{\partial x^i \partial x^j}y^j.$$

In an adapted frame on $\overset{(2)}{T}(DF) = \left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i}, \frac{\partial}{\partial \lambda}\right)$, the form of local bases will be given, as follows:

$$\frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - N^i_\mu\frac{\partial}{\partial y^i} - M^0_\mu\frac{\partial}{\partial \lambda}, \quad \frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - L^0_i\frac{\partial}{\partial \lambda}, \quad \frac{\partial}{\partial \lambda} = \frac{\partial}{\partial y^0}$$

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1) Numbers in brackets refer to the references at the end of the paper.

The transformation of local bases yields the relations:

$$\frac{\delta}{\delta x^\lambda} = \frac{\partial \tilde{x}^k}{\partial x^\lambda}\frac{\delta}{\delta \tilde{x}^k}, \quad \frac{\delta}{\delta y^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{y}^j}$$

$$d\bar{x}^k = \frac{\partial \bar{x}^k}{\partial x^\lambda} dx^\lambda, \quad \delta \bar{y}^i = \frac{\partial \bar{x}^i}{\partial x^j}\delta y^j, \quad \lambda = \bar{\lambda}.$$

Considering that in a deformed bundle $\overset{(2)}{T}(DF)$, the D-connections $D\Gamma = \{L^\nu_{\lambda\mu}, L^j_{i\mu}, L^0_{0\mu}, C^m_{ij}, C^\nu_{\mu j}, C^0_{0j}, \Lambda^\nu_{\mu 0}, \Lambda^j_{i0}, \Lambda^0_{00}\}$ become d-connections, then we get

$$D_{\delta/\delta x^\mu}\frac{\delta}{\delta x^\lambda} = L^\nu_{\lambda\mu}\frac{\delta}{\delta x^\nu}, \quad D_{\delta/\delta x^\mu}\frac{\delta}{\delta y^i} = L^j_{i\mu}\frac{\delta}{\delta y^j}, \quad D_{\delta/\delta x^\mu}\frac{\partial}{\partial\lambda} = L^0_{0\mu}\frac{\partial}{\partial\lambda},$$

$$D_{\delta/\delta y^j}\frac{\delta}{\delta x^\mu} = C^\nu_{\mu j}\frac{\delta}{\delta x^\nu}, \quad D_{\delta/\delta y^j}\frac{\delta}{\delta y^i} = C^m_{ij}\frac{\delta}{\delta y^m}, \quad D_{\delta/\delta y^j}\frac{\partial}{\partial\lambda} = C^0_{0j}\frac{\partial}{\partial\lambda},$$

$$D_{\partial/\partial\lambda}\frac{\delta}{\delta x^\mu} = \Lambda^\nu_{\mu 0}\frac{\delta}{\delta x^\nu}, \quad D_{\partial/\partial\lambda}\frac{\delta}{\delta y^i} = \Lambda^j_{i0}\frac{\delta}{\delta y^j}, \quad D_{\partial/\partial\lambda}\frac{\partial}{\partial\lambda} = \Lambda^0_{00}\frac{\partial}{\partial\lambda}.$$

In the framework of our consideration, the differential of fundamental function $F(x, y, \lambda)$ of the family of spaces is given, for example, by the relation,

$$dF(x,y,\lambda) = \frac{\partial F}{\partial x^k}dx^k + \frac{\partial F}{\partial y^i}dy^i + \frac{\partial F}{\partial\lambda}d\lambda = \frac{\delta F}{\delta x^k}\delta x^k + \frac{\delta F}{\delta y^i}\delta y^i + \frac{\partial F}{\partial\lambda}d\lambda.$$

The absolute differential of an arbitrary vector X^i has the form:

$$DX^i = dX^i + L^i_{j\mu}(x,y,\lambda)X^j\delta x^\mu + C^i_{jk}(x,y,\lambda)X^j\delta y^k + \Lambda^i_{j0}X^j d\lambda,$$

where

$$L^i_{j\mu} = \frac{1}{2}g^{ir}\left(\frac{\delta g_{rj}}{\delta x^\mu} + \frac{\delta g_{r\mu}}{\delta x^j} - \frac{\delta g_{j\mu}}{\delta x^r}\right),$$

$$C^i_{jk} = \frac{1}{2}g^{ir}\left(\frac{\delta g_{rj}}{\delta y^k} + \frac{\delta g_{rk}}{\delta y^j} - \frac{\delta g_{jk}}{\delta y^r}\right),$$

$$\Lambda^i_{j0} = \frac{1}{2}g^{i\ell}\left(\frac{\delta g_{\ell 0}}{\delta y^j} + \frac{\partial g_{\ell j}}{\partial\lambda} - \frac{\delta g_{j0}}{\delta y^\ell}\right).$$

For the physical viewpoint, it is useful to impose the relation $g_{\ell 0} = g_{j0} = 0$. Also, the form of a curve $\tilde{C} : [0,1] \to \overset{(2)}{T}(DF)$ is given locally by $X^A(t) = (x^k(t), y^i(t), \lambda(t))$. A curve $\tilde{C}$ in $\overset{(2)}{T}(DF)$ is a geodesic with respect to the global affine connection $\tilde{\Gamma}^A_{BC}$, if and only if it satisfies the relation

$$\frac{d^2X^A}{ds^2} + \tilde{\Gamma}^A_{BC}\frac{dX^B}{ds}\frac{dX^C}{ds} = 0.$$

§ 2. Torsions and curvatures. We shall calculate the torsions and the curvatures in a $T^{(2)}(DF)$, taking into account [2], [7] in our case. The torsion tensor field T of a d-connection D is given by $T(X,Y) = D_X Y - D_Y X - [X,Y]$, $\forall X, Y$. For the adapted frame $\left(\frac{\delta}{\delta x^\lambda}, \frac{\delta}{\delta y^i}, \frac{\partial}{\partial \lambda}\right)$, we have the relations:

(2.1)
$$\begin{aligned}
&a) \quad T\left(\frac{\delta}{\delta x^\mu}, \frac{\delta}{\delta x^\nu}\right) = T^\lambda_{\nu\mu}\frac{\delta}{\delta x^\lambda} + T^i_{\nu\mu}\frac{\delta}{\delta y^i} + T^0_{\nu\mu}\frac{\partial}{\partial \lambda},\\
&b) \quad T\left(\frac{\delta}{\delta y^i}, \frac{\delta}{\delta x^\mu}\right) = T^\lambda_{\nu\mu}\frac{\delta}{\delta x^\lambda} + T^i_{\nu\mu}\frac{\delta}{\delta y^i} + T^0_{\nu\mu}\frac{\partial}{\partial \lambda},\\
&c) \quad T\left(\frac{\delta}{\delta y^i}, \frac{\delta}{\delta y^j}\right) = T^\lambda_{ji}\frac{\delta}{\delta x^\lambda} + T^\ell_{ji}\frac{\delta}{\delta y^\ell} + T^0_{ji}\frac{\partial}{\partial \lambda},\\
&d) \quad T\left(\frac{\delta}{\delta x^\mu}, \frac{\partial}{\partial \lambda}\right) = T^\lambda_{\mu 0}\frac{\delta}{\delta x^\lambda} + T^i_{\mu 0}\frac{\delta}{\delta y^i} + T^0_{\mu 0}\frac{\partial}{\partial \lambda},\\
&e) \quad T\left(\frac{\delta}{\delta y^i}, \frac{\partial}{\partial \lambda}\right) = T^\lambda_{i0}\frac{\delta}{\delta x^\lambda} + T^j_{i0}\frac{\delta}{\delta y^j} + T^0_{i0}\frac{\partial}{\partial \lambda},\\
&f) \quad T\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \lambda}\right) = T^\lambda_{00}\frac{\delta}{\delta x^\lambda} + T^j_{00}\frac{\delta}{\delta y^j} + T^0_{00}\frac{\partial}{\partial \lambda}.
\end{aligned}$$

The relation (2.1)a) is written as

(2.2)
$$\begin{aligned}
&D_{\delta/\delta x^\mu}\frac{\delta}{\delta x^\nu} - D_{\delta/\delta x^\nu}\frac{\delta}{\delta x^\mu} - \left[\frac{\delta}{\delta x^\mu}, \frac{\delta}{\delta x^\nu}\right]\\
&\qquad = L^\lambda_{\nu\mu}\frac{\delta}{\delta x^\lambda} - L^\lambda_{\mu\nu}\frac{\delta}{\delta x^\lambda} - R^i_{\mu\nu}\frac{\delta}{\delta y^i} - \tilde{U}^0_{\mu\nu}\frac{\partial}{\partial \lambda},
\end{aligned}$$

which implies via the relations

$$\left[\frac{\delta}{\delta x^\mu}, \frac{\delta}{\delta x^\nu}\right] = \left(\frac{\delta N^i_\mu}{\delta x^\nu} - \frac{\delta N^i_\nu}{\delta x^\mu}\right)\frac{\partial}{\partial y^j} + \left(\frac{\delta M^0_\mu}{\delta x^\nu} - \frac{\delta M^0_\nu}{\delta x^\mu}\right)\frac{\partial}{\partial \lambda} = R^i_{\mu\nu}\frac{\partial}{\partial y^j} + U^0_{\mu\nu}\frac{\partial}{\partial \lambda}$$

and

$$\tilde{U}^0_{\mu\nu} = U^0_{\mu\nu} + R^i_{\mu\nu}L^0_i.$$

Also comparing the relations (2.1)a) and (2.2), we obtain for the torsions

$$T^\lambda_{\nu\mu} = L^\lambda_{\nu\mu} - L^\lambda_{\mu\nu}, \quad T^i_{\nu\mu} = -R^i_{\mu\nu}, \quad T^0_{\nu\mu} = -\tilde{U}^0_{\mu\nu}.$$

Similarly, from the relation (2.1)b), we have

$$T\left(\frac{\delta}{\delta y^i}, \frac{\delta}{\delta x^\mu}\right) = D_{\delta/\delta y^i}\frac{\delta}{\delta x^\mu} - D_{\delta/\delta x^\mu}\frac{\delta}{\delta y^i} - \left[\frac{\delta}{\delta y^i}, \frac{\delta}{\delta x^\mu}\right]$$

$$= C^{\lambda}_{i\mu}\frac{\delta}{\delta x^{\lambda}} - L^{j}_{i\mu}\frac{\delta}{\delta y^{j}} + \frac{\delta N^{j}_{\mu}}{\delta y^{i}}\frac{\delta}{\delta y^{j}} + \tilde{Y}^{0}_{\mu i}\frac{\partial}{\partial \lambda},$$

$$T^{\lambda}_{i\mu} = C^{\lambda}_{i\mu}, \quad T^{j}_{i\mu} = \frac{\delta N^{j}_{\mu}}{\delta y^{i}} - L^{j}_{i\mu}, \quad T^{0}_{i\mu} = -\tilde{Y}^{0}_{\mu i},$$

where

$$\left[\frac{\delta}{\delta y^{i}}, \frac{\delta}{\delta x^{\mu}}\right] = \frac{\delta N^{j}_{\mu}}{\delta y^{i}}\frac{\delta}{\delta y^{j}} + \tilde{Y}^{0}_{\mu i}\frac{\partial}{\partial \lambda}.$$

From (2.1)c), we get

$$T\left(\frac{\delta}{\delta y^{i}}, \frac{\delta}{\delta y^{j}}\right) = D_{\delta/\delta y^{i}}\frac{\delta}{\delta y^{j}} - D_{\delta/\delta y^{j}}\frac{\delta}{\delta y^{i}} - \left[\frac{\delta}{\delta y^{i}}, \frac{\delta}{\delta y^{j}}\right] = C^{\ell}_{ji}\frac{\delta}{\delta y^{\ell}} - C^{\ell}_{ij}\frac{\delta}{\delta y^{\ell}} - R^{0}_{ij}\frac{\partial}{\partial \lambda}.$$

So we have

$$T^{\lambda}_{ji} = 0, \quad T^{\ell}_{ji} = C^{\ell}_{ji} - C^{\ell}_{ij}, \quad T^{0}_{ji} = -R^{0}_{ij},$$

where

$$\left[\frac{\delta}{\delta y^{i}}, \frac{\delta}{\delta y^{j}}\right] = R^{0}_{ij}\frac{\partial}{\partial \lambda}.$$

The relation (2.1)d) is written in the form

$$T\left(\frac{\delta}{\delta x^{\mu}}, \frac{\partial}{\partial \lambda}\right) = T^{\lambda}_{\mu 0}\frac{\delta}{\delta x^{\lambda}} + T^{i}_{\mu 0}\frac{\delta}{\delta y^{i}} + T^{0}_{\mu 0}\frac{\partial}{\partial \lambda}$$

$$= D_{\delta/\delta x^{\mu}}\frac{\partial}{\partial \lambda} - D_{\partial/\partial \lambda}\frac{\delta}{\delta x^{\mu}} - \left[\frac{\delta}{\delta x^{\mu}}, \frac{\partial}{\partial \lambda}\right]$$

$$= L^{0}_{0\mu}\frac{\partial}{\partial \lambda} - \Lambda^{\lambda}_{0\mu}\frac{\delta}{\delta x^{\lambda}} - \frac{\partial N^{i}_{\mu}}{\partial \lambda}\frac{\delta}{\delta y^{i}} - Z^{0}_{\mu}\frac{\partial}{\partial \lambda}.$$

Consequently, it follows:

$$T^{\lambda}_{\mu 0} = -\Lambda^{\lambda}_{0\mu}, \quad T^{i}_{\mu 0} = -\frac{\partial N^{i}_{\mu}}{\partial \lambda}, \quad T^{0}_{\mu 0} = L^{0}_{0\mu} - Z^{0}_{\mu},$$

where

$$\left[\frac{\delta}{\delta x^{\mu}}, \frac{\partial}{\partial \lambda}\right] = \frac{\partial N^{i}_{\mu}}{\partial \lambda}\frac{\partial}{\partial y^{i}} + Z^{0}_{\mu}\frac{\partial}{\partial \lambda} \quad \text{and} \quad Z^{0}_{\mu} = \frac{\partial M^{0}_{\mu}}{\partial \lambda} + \frac{\partial N^{i}_{\mu}}{\partial \lambda}L^{0}_{i}.$$

From (2.1)e), we have

$$T\left(\frac{\delta}{\delta y^{i}}, \frac{\partial}{\partial \lambda}\right) = T^{\lambda}_{i0}\frac{\delta}{\delta x^{\lambda}} + T^{j}_{i0}\frac{\delta}{\delta y^{j}} + T^{0}_{i0}\frac{\partial}{\partial \lambda}$$

$$= D_{\delta/\delta y^{i}}\frac{\partial}{\partial \lambda} - D_{\partial/\partial \lambda}\frac{\delta}{\delta y^{i}} - \left[\frac{\delta}{\delta y^{i}}, \frac{\partial}{\partial \lambda}\right]$$

$$= C^{0}_{0i}\frac{\partial}{\partial \lambda} - \Lambda^{j}_{0i}\frac{\delta}{\delta y^{j}} - \frac{\partial L^{0}_{i}}{\partial \lambda}\frac{\partial}{\partial \lambda}.$$

$$T^{\lambda}_{i0} = 0, \quad T^{j}_{i0} = -\Lambda^{j}_{0i}, \quad T^{0}_{i0} = C^{0}_{0i} - \frac{\partial L^{0}_{i}}{\partial \lambda},$$

where

$$\left[\frac{\delta}{\delta y^i}, \frac{\partial}{\partial \lambda}\right] = \frac{\partial L^0_i}{\partial \lambda}\frac{\partial}{\partial \lambda}.$$

Finally, from (2.1)f), we obtain $T^{\lambda}_{00} = T^{i}_{00} = T^{0}_{00} = 0$.

The curvature tensors and the λ-curvature tensors $I^{\,k}_{\nu 0\mu}$, $I^{\,j}_{i0\mu}$, $I^{\,0}_{00\mu}$, $J^{\,k}_{\nu 0i}$, $J^{\,\ell}_{j0i}$, $J^{\,0}_{00i}$ in $\overset{(2)}{T}(DF)$ are given by the relations:

$$R\left(\frac{\delta}{\delta x^\mu}, \frac{\delta}{\delta x^\nu}\right)\frac{\delta}{\delta x^\lambda} = R^{\;\kappa}_{\lambda\nu\mu}\frac{\delta}{\delta x^\kappa}, \quad R\left(\frac{\delta}{\delta x^\mu}, \frac{\delta}{\delta x^\nu}\right)\frac{\delta}{\delta y^i} = R^{\;j}_{i\nu\mu}\frac{\delta}{\delta y^j},$$

$$R\left(\frac{\delta}{\delta x^\mu}, \frac{\delta}{\delta x^\nu}\right)\frac{\partial}{\partial \lambda} = R^{\;0}_{0\nu\mu}\frac{\partial}{\partial \lambda}, \quad R\left(\frac{\delta}{\delta y^i}, \frac{\delta}{\delta x^\mu}\right)\frac{\delta}{\delta x^\nu} = P^{\;\kappa}_{\nu\mu i}\frac{\delta}{\delta x^\kappa},$$

$$R\left(\frac{\delta}{\delta y^i}, \frac{\delta}{\delta x^\mu}\right)\frac{\delta}{\delta y^j} = P^{\;\ell}_{ji\mu}\frac{\delta}{\delta y^\ell}, \quad R\left(\frac{\delta}{\delta y^i}, \frac{\delta}{\delta x^\mu}\right)\frac{\partial}{\partial \lambda} = P^{\;0}_{0i\mu}\frac{\partial}{\partial \lambda},$$

$$R\left(\frac{\delta}{\delta y^i}, \frac{\delta}{\delta y^j}\right)\frac{\delta}{\delta x^\mu} = S^{\;\kappa}_{\mu ji}\frac{\delta}{\delta x^\kappa}, \quad R\left(\frac{\delta}{\delta y^i}, \frac{\delta}{\delta y^j}\right)\frac{\delta}{\delta y^\ell} = S^{\;m}_{\ell ji}\frac{\delta}{\delta y^m},$$

$$R\left(\frac{\delta}{\delta y^i}, \frac{\delta}{\delta y^j}\right)\frac{\partial}{\partial \lambda} = S^{\;0}_{0ji}\frac{\partial}{\partial \lambda}, \quad R\left(\frac{\delta}{\delta x^\mu}, \frac{\partial}{\partial \lambda}\right)\frac{\delta}{\delta x^\nu} = I^{\;k}_{\nu 0\mu}\frac{\delta}{\delta x^k},$$

$$R\left(\frac{\delta}{\delta x^\mu}, \frac{\partial}{\partial \lambda}\right)\frac{\delta}{\delta y^i} = I^{\;j}_{i0\mu}\frac{\delta}{\delta y^j}, \quad R\left(\frac{\delta}{\delta x^\mu}, \frac{\partial}{\partial \lambda}\right)\frac{\partial}{\partial \lambda} = I^{\;0}_{00\mu}\frac{\partial}{\partial \lambda}$$

$$R\left(\frac{\delta}{\delta y^i}, \frac{\partial}{\partial \lambda}\right)\frac{\delta}{\delta x^\nu} = J^{\;k}_{\nu 0i}\frac{\delta}{\delta x^k}, \quad R\left(\frac{\delta}{\delta y^i}, \frac{\partial}{\partial \lambda}\right)\frac{\delta}{\delta y^j} = J^{\;\ell}_{j0i}\frac{\delta}{\delta y^\ell},$$

$$R\left(\frac{\delta}{\delta y^i}, \frac{\partial}{\partial \lambda}\right)\frac{\partial}{\partial \lambda} = J^{\;0}_{00i}\frac{\partial}{\partial \lambda}, \quad R\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \lambda}\right)\frac{\partial}{\partial \lambda} = 0$$

$$R\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \lambda}\right)\frac{\delta}{\delta x^\mu} = K^{\;k}_{\mu 00}\frac{\delta}{\delta x^\nu} = 0, \quad R\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \lambda}\right)\frac{\delta}{\delta y^i} = K^{\;j}_{i00}\frac{\delta}{\delta y^i} = 0.$$

The curvature tensor field R of a d-connection D has the form: $R(X,Y)Z = [D_x, D_y]Z - D_{[x,y]}Z$, $\forall X, Y, Z$. We apply this form the adapted frame, to produce:

$$R^{k}_{\lambda\nu\mu}\frac{\delta}{\delta x^k} = R\left(\frac{\delta}{\delta x^\mu}, \frac{\delta}{\delta x^\nu}\right)\frac{\delta}{\delta x^\lambda}$$

$$= D_{\delta/\delta x^\mu}\left(D_{\delta/\delta x^\nu}\frac{\delta}{\delta x^\lambda}\right) - D_{\delta/\delta x^\nu}\left(D_{\delta/\delta x^\mu}\frac{\delta}{\delta x^\lambda}\right) - D_{[\delta/\delta x^\mu, \delta/\delta x^\nu]}\frac{\delta}{\delta x^\lambda}$$

$$= D_{\delta/\delta x^\mu}\left(L^{k}_{\lambda\nu}\frac{\delta}{\delta x^k}\right) - D_{\delta/\delta x^\nu}\left(L^{k}_{\lambda\mu}\frac{\delta}{\delta x^k}\right) - D_{R^i_{\mu\nu}\delta/\delta y^i + \tilde{U}^0_{\mu\nu}\partial/\partial\lambda}\frac{\delta}{\delta x^\lambda},$$

$$\frac{\delta L^k_{\lambda\nu}}{\delta x^\mu}\frac{\delta}{\delta x^\mu} + L^k_{\lambda\nu}D_{\delta/\delta x^\mu}\frac{\delta}{\delta x^k} - \frac{\delta L^k_{\lambda\mu}}{\delta x^\nu} - L^k_{\lambda\mu}D_{\delta/\delta x^\nu}\frac{\delta}{\delta x^k} - R^i_{\mu\nu}D_{\delta/\delta y^i}\frac{\delta}{\delta x^\lambda} - \tilde{U}^0_{\mu\nu}D_{\partial/\partial\lambda}\frac{\delta}{\delta x^\lambda}$$

$$= \frac{\delta L^k_{\lambda\nu}}{\delta x^\mu}\frac{\delta}{\delta x^k} + L^\rho_{\lambda\nu}L^k_{\rho\mu}\frac{\delta}{\delta x^k} - \frac{\delta L^k_{\lambda\rho}}{\delta x^\nu}\frac{\delta}{\delta x^k} - L^\rho_{\lambda\mu}L^k_{\rho\nu}\frac{\delta}{\delta x^k} - R^i_{\mu\nu}C^k_{i\lambda}\frac{\delta}{\delta x^k} - \tilde{U}^0_{\mu\nu}\Lambda^k_{0\lambda}\frac{\delta}{\delta x^k}.$$

So, coefficients of the curvature of a d-connection D take the following form:

$$R^{\ k}_{\lambda\nu\mu} = \frac{\delta L^k_{\lambda\nu}}{\delta x^\mu} - \frac{\delta L^k_{\lambda\mu}}{\delta x^\nu} + L^\rho_{\lambda\nu}L^k_{\rho\mu} - L^\rho_{\lambda\mu}L^k_{\rho\nu} - R^i_{\mu\nu}C^k_{i\lambda} - \tilde{U}^0_{\mu\nu}\Lambda^k_{0\lambda}, \tag{2.3}$$

where $\tilde{U}^0_{\mu\nu}$ is given by the relation (2.2).

With a similar method, we can calculate the other types of curvatures.

$$R^{\ j}_{i\nu\mu} = \frac{\delta L^j_{i\nu}}{\delta x^\mu} - \frac{\delta L^j_{i\mu}}{\delta x^\nu} + L^\ell_{i\nu}L^j_{\ell\mu} - L^\ell_{i\mu}L^j_{\ell\nu} - R^\ell_{\mu\nu}C^j_{i\ell} - \tilde{U}^0_{\mu\nu}\Lambda^j_{0i}, \tag{2.4}$$

$$R^{\ 0}_{0\nu\mu} = \frac{\delta L^0_{0\nu}}{\delta x^\mu} - \frac{\delta L^0_{0\mu}}{\delta x^\nu} + L^0_{0\nu}L^0_{0\mu} - L^0_{0\mu}L^0_{0\nu} - R^i_{\mu\nu}C^0_{0i} - \tilde{U}^0_{\mu\nu}\Lambda^0_{00}, \tag{2.5}$$

$$P^{\ \ell}_{ji\mu} = \frac{\delta L^\ell_{j\mu}}{\delta y^i} - \frac{\delta C^\ell_{ji}}{\delta x^\mu} + L^n_{j\mu}C^\ell_{ni} - C^n_{ji}L^\ell_{n\mu} + \frac{\delta N^n_\mu}{\delta y^i}C^\ell_{jn} + \tilde{Y}^0_{\mu i}\Lambda^\ell_{0j}, \tag{2.6}$$

where

$$\tilde{Y}^0_{\mu i} = Y^0_{\mu i} + \frac{\delta N^j_\mu}{\delta y^i}L^0_j, \quad Y^0_{\mu i} = \frac{\delta M^0_\mu}{\delta y^i} - \frac{\delta L^0_i}{\delta x^\mu}.$$

$$P^{\ 0}_{0i\mu} = \frac{\delta L^0_{0\mu}}{\delta y^i} - \frac{\delta C^0_{0i}}{\delta x^\mu} + L^0_{0\mu}C^0_{0i} - C^0_{0i}L^0_{0\mu} + \frac{\delta N^j_\mu}{\delta y^i}C^0_{0j} + \tilde{Y}^0_{\mu i}\Lambda^0_{00}, \tag{2.7}$$

$$S^{\ k}_{\mu ji} = \frac{\delta C^k_{j\mu}}{\delta y^i} - \frac{\delta C^k_{i\mu}}{\delta y^j} + C^\lambda_{j\mu}C^k_{\lambda i} - C^\lambda_{i\mu}C^k_{\lambda j} - R^0_{ij}\Lambda^k_{0\mu}, \tag{2.8}$$

$$S^{\ m}_{\ell ji} = \frac{\delta C^m_{\ell j}}{\delta y^i} - \frac{\delta C^m_{\ell i}}{\delta y^j} + C^n_{\ell j}C^m_{ni} - C^n_{\ell i}C^m_{nj} - R^0_{ij}\Lambda^m_{0\ell}, \tag{2.9}$$

$$S^{\ 0}_{0ji} = \frac{\delta C^0_{0j}}{\delta y^i} - \frac{\delta C^0_{0i}}{\delta y^j} + C^0_{0j}C^0_{0i} - C^0_{0i}C^0_{0j} - R^0_{ij}\Lambda^0_{00}, \tag{2.10}$$

$$I^{\ k}_{\nu 0\mu} = \frac{\delta\Lambda^k_{0\nu}}{\delta x^\mu} - \frac{\partial L^k_{\nu\mu}}{\partial\lambda} + \Lambda^\rho_{0\nu}L^k_{\rho\mu} - L^\rho_{\nu\mu}\Lambda^k_{0\rho} - \frac{\partial N^i_\mu}{\partial\lambda}C^k_{i\nu} - Z^0_\mu\Lambda^k_{0\nu}, \tag{2.11}$$

where

$$Z^{\ 0}_\mu = \frac{\partial M^0_\mu}{\partial\lambda} + \frac{\partial N^i_\mu}{\partial\lambda}L^0_i.$$

$$I^{\ j}_{i0\mu} = \frac{\delta\Lambda^j_{0i}}{\delta x^\mu} - \frac{\partial L^j_{i\mu}}{\partial\lambda} + \Lambda^\ell_{0i}L^j_{\ell\mu} - L^\ell_{i\mu}\Lambda^j_{0\ell} - \frac{\partial N^\ell_\mu}{\partial\lambda}C^j_{i\ell} - Z^0_\mu\Lambda^j_{0i}, \tag{2.12}$$

$$I^{\ 0}_{00\mu} = \frac{\delta\Lambda^0_{00}}{\delta x^\mu} - \frac{\partial L^0_{0\mu}}{\partial\lambda} + \Lambda^0_{00}L^0_{0\mu} - L^0_{0\mu}\Lambda^0_{00} - \frac{\partial N^\ell_\mu}{\partial\lambda}C^0_{0\ell} - Z^0_\mu\Lambda^0_{00}, \tag{2.13}$$

$$J^{\ k}_{\nu 0i} = \frac{\delta\Lambda^k_{0\nu}}{\delta y^i} - \frac{\partial C^k_{i\nu}}{\partial\lambda} + \Lambda^\rho_{0\nu}C^k_{i\rho} - C^\rho_{i\nu}\Lambda^k_{0\rho} - \frac{\partial L^0_i}{\partial\lambda}\Lambda^k_{0\nu}, \tag{2.14}$$

$$J^{\ \ell}_{j0i} = \frac{\delta\Lambda^\ell_{0j}}{\delta y^i} - \frac{\partial C^\ell_{ji}}{\partial\lambda} + \Lambda^n_{0j}C^\ell_{ni} - C^n_{ji}\Lambda^\ell_{0n} - \frac{\partial L^0_i}{\partial\lambda}\Lambda^\ell_{0j}, \tag{2.15}$$

$$J_{00i}^{\ 0} = \frac{\delta \Lambda_{00}^{0}}{\delta y^{i}} - \frac{\partial C_{0i}^{0}}{\partial \lambda} + \Lambda_{00}^{0} C_{0i}^{0} - C_{0i}^{0} \Lambda_{00}^{0} - \frac{\partial L_{i}^{0}}{\partial \lambda} \Lambda_{00}^{0}, \tag{2.16}$$

$$K_{\mu 00}^{\ \nu} = K_{i00}^{\ j} = K_{000}^{\ 0} = 0. \tag{2.17}$$

Thus we can get the following:

Theorem 2.1. *In a space $\overset{(2)}{T}$ (DF), the relations (2.3)-(2.17) constitute the coefficients of the curvatures of a d-connection D.*

§ 3. Conclusions. From a physical point of view, in order to emphasize the physical effects caused by the λ-dependent quantities, we had better focus our attention on such terms as the non-linear connections M_{μ}^{0}, L_{i}^{0} and the connection coefficients $\Lambda_{\mu 0}^{\nu}, \Lambda_{i0}^{j}$. Therefore, we had better examine the λ-torsion tensors $T_{\mu 0}^{\lambda}$, $T_{\mu 0}^{i}$, $T_{\mu 0}^{0}$, T_{i0}^{j}, T_{io}^{0} and the λ-curvature tensors $I_{\nu 0 \mu}^{\ k}$, $I_{i0\mu}^{\ j}$, $I_{00\mu}^{\ 0}$, $J_{\nu 0i}^{\ k}$, $J_{j0i}^{\ \ell}$, $J_{00i}^{\ 0}$. The physical meaning of them, is of course, dependent on the physical meaning of λ itself. Therefore, for example, with respect to the non-linear connections M_{μ}^{0}, L_{i}^{0}, the λ-deformation field can signify the rate of change of the interaction's values, or the differentiation of the intensity of gravitational field relative to the above mentioned curvature tensors.

Consequently, the existence of λ-field will influence the intensity of tidal forces via of λ-curvature tensors.

The last consideration will be the object of a further our study.

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