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# NONLOCALIZED FIELD THEORY OVER SPINOR BUNDLES: POINCARÉ GRAVITY AND YANG–MILLS FIELDS

P. C. STAVRINOS and P. MANOUSELIS

Department of Mathematics, University of Athens
157 84 Panepistimiopolis, Athens, Greece

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In this paper we study the differential structure of a spinor bundle in spaces where the metric tensor $g_{\mu\nu}(x, \xi, \bar{\xi})$ of the base manifold depends on the position $x$ as well as on the spinor variables $\xi$ and $\bar{\xi}$. Notions such as: gauge covariant derivatives of tensors, connections, curvatures, torsions and Bianchi identities are presented in the context of a gauge approach, different than the one proposed in [11, 13], due to the introduction of a Poincaré group and the use of $d$-connections [6, 8] in the spinor bundle $S^{(2)}M$. The introduction of basic 1-form fields $\rho_\mu$ and spinors $\zeta_a$, $\bar{\zeta}^a$ with values in the Lie algebra of the Poincaré group is also essential in our study. The gauge field equations are derived by the authors [12]. Finally, we give the Yang–Mills and the Yang–Mills–Higgs equations in a form sufficiently generalized for our approach.

## 1. Introduction

The concept of the nonlocalized field theory has already been developed in recent years by Japanese authors [3, 11, 13] in order to provide a unified description of elementary particles. In this approach, the internal variable is replaced by a spinor $\omega = (\xi, \bar{\xi})$ ($\xi$ and its conjugate $\bar{\xi}$ are considered as independent variables).

The description of gravity through the introduction of variables $w_\mu^{ab}(x)$ as a gravitational potential (Lorentz connection coefficients) was proposed originally by Utiyama [14, 1]. He considered the Lorentz group as a local transformation group. The gravitational field is described by the tetrads $h_\mu^a(x)$ viewed as independent variables. With the help of these variables we may pass from a general system of coordinates to a local Lorentz ones.

The Einstein equations were derived in the context of Utiyama's approach, but this was not satisfactory because of the arbitrariness of the elements introduced. Later T. Kibble [2, 3] introduced a gauge approach which enables the introduction of all gravitational variables. To achieve this goal it is important to use the Poincaré group (i.e. a group consisting of rotations, boosts and translations).

This group first assigns an exact meaning to the terms: "momentum", "energy", "mass" and "spin" used to determine characteristics of elementary particles [2]. On the other hand, it is a gauge acting locally in the space-time. Thus, we may perform Poincaré transformations for a physical approach. Hence by treating the Poincaré group as a local group, we introduce the fundamental 1-form field $\rho_\mu$ taking values in the Lie algebra of the Poincaré group.

In our present study the basic idea is to consider a spinor bundle with a base manifold $M$ of a metric tensor $g_{\mu\nu}(x,\xi,\bar{\xi})$ that depends on the position coordinates $x^k$ and the spinor (Dirac) variables $(\xi_\alpha,\bar{\xi}^\alpha)\in\mathbb{C}^4\times\mathbb{C}^4$, where $\bar{\xi}^\alpha$ is the adjoint of $\xi_\alpha$, an independent variable, similar to the one proposed by Y. Takano [11], and Y. Takano and T. Ono [13]. The spinor bundle $S^{(1)}(M)$ is constructed from one of the principal fiber bundles with a fiber: $F=\mathbb{C}^4$.

Each fiber is diffeomorphic with one proper Lorentz group (which is produced by Lorentz transformations) and it entails a principal bundle $SL(4,\mathbb{C})$ over $M$, ($SL(4,\mathbb{C})$ consists of the group of rotations and boosts of unit determinant acting on a four-dimensional complex space, which is reducible to $SL(2,\mathbb{C})$).

The consideration of the $d$-connections that preserve the $(hv)$-distribution by the parallel translation (cf. [6,8]), in relation to the second order bundle $S^{(2)}(M)=M\times\mathbb{C}^{2\cdot 4}$ enables us to use a more general group $G^{(2)}$ called a structured group of all rotations and translations that is isomorphic to the Poincaré Lie algebra. Therefore, a *spinor* in $x\in M$ is an element of the spinor bundle $S^{(2)}(M)$

$$(x^\mu,\xi_\alpha,\bar{\xi}^\alpha)\in S^{(2)}(M)\,.$$

A *spinor field* is a section of $S^{(2)}(M)$.

Moreover, the fundamental gauge 1-form field mentioned above in connection with the spaces that possess metric tensor $g_{\mu\nu}(x,\xi,\bar{\xi})$ will take a similar but more general form than that proposed by other authors [5]. We shall define a nonlinear connection on $S^{(2)}(M)$ such as,

$$T(S^{(2)}M)=H(S^{(2)}M)\oplus\mathcal{F}^{(1)}(S^{(2)}M)\oplus\mathcal{F}^{(2)}(S^{(2)}M)\,,$$

where $H$, $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$ represent the horizontal, vertical, and normal distribution. In a local base, for the horizontal distribution $H(S^{(2)}M)$ we have:

$$\rho_\mu(x,\xi,\bar{\xi})=\frac{1}{2}\omega^{*ab}_{\ \mu}J_{ab}+h^a_\mu(x,\xi,\bar{\xi})P_a\,,\tag{1.1}$$

where $J_{ab}$, $P_a$ are the generators of the four-dimensional Poincaré group satisfying relations of the form:

$$\begin{gathered}[J_{ab},J_{cd}]=n_{bc}J_{ad}-n_{bd}J_{ac}+n_{ad}J_{bc}-n_{ac}J_{bd}\,,\\ [J_{ab},P_c]=n_{bc}P_a-n_{ac}P_b\,,\qquad [P_a,P_b]=0\,,\qquad J_{ab}+J_{ba}=0\,.\end{gathered}\tag{1.2}$$

The quantities $\omega_\mu^{(*)ab}$ represent the (Lorentz) spin connection coefficients and are considered as given, $n_{ab}$ is the metric for the local Lorentz spaces with signature $(+---)$.

These are connected with $g_{\mu\nu}$ by

$$g_{\mu\nu} h^{\mu}_{a} h^{\nu}_{b} = n_{ab} , \qquad g^{\mu\nu} = n^{ab} h^{\mu}_{a} h^{\nu}_{b} , \tag{1.3}$$

where $h^{\nu}_{a}$ represents the tetrads. Similarly, for the vertical and normal distributions $\mathcal{F}^{(1)}(S^{(2)}M)$, $\mathcal{F}^{(2)}(S^{(2)}M)$ the fundamental 1-forms $\zeta_\alpha$, $\bar{\zeta}^\alpha$ are given by

$$\zeta_\alpha = \frac{1}{2}\Theta^{(*)ab}_{\alpha} J_{ab} + \psi^{a}_{\alpha} P_a , \tag{1.4}$$

$$\bar{\zeta}^\alpha = \frac{1}{2}\bar{\Theta}^{(*)\alpha ab} J_{ab} + \bar{\psi}^{\alpha a} P_a , \tag{1.5}$$

where $\bar{\psi}^{\alpha a}$, $\psi^{a}_{\alpha}$ are the spin tetrad coefficients, and $\Theta^{(*)ab}_{\alpha}$, $\bar{\Theta}^{(*)\alpha ab}$ are the given spin connection coefficients which are determined in such a way that the absolute differential and the covariant derivatives of the metric tensor $g_{\mu\nu}(x,\xi,\bar{\xi})$ vanish identically.

We use the Greek letters $\lambda$, $\mu$, $\nu$ ... for space-time indices, $\alpha$, $\beta$, $\gamma$, for spinors, and the Latin letters $a$, $b$, $c$, ... for the Lorentz indices.

The general transformations of coordinates on $S^{(2)}(M)$ are:

$$x'^{\mu} = x'^{\mu}(x^\nu) , \qquad \xi'_{\alpha} = \xi'_{\alpha}(\xi_\beta, \bar{\xi}^\gamma) , \qquad \bar{\xi}'^{\alpha} = \bar{\xi}'^{\alpha}(\bar{\xi}^\beta, \xi_\gamma) . \tag{1.6}$$

## 2. Connections

We define the following gauge covariant derivatives

$$\begin{aligned} D^{(*)}_{\mu} &= \frac{\delta}{\delta x^\mu} + \frac{1}{2}\omega^{(*)ab}_{\mu} J_{ab} , \\ D^{(*)\alpha} &= \frac{\delta}{\delta \xi_\alpha} + \frac{1}{2}\bar{\Theta}^{(*)\alpha ab} J_{ab} , \\ D^{(*)\alpha} &= \frac{\delta}{\delta \bar{\xi}^\alpha} + \frac{1}{2}\Theta^{(*)ab}_{\alpha} J_{ab} , \end{aligned} \tag{2.1}$$

where

$$\frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} + N_{\alpha\mu}\frac{\partial}{\partial \xi_\alpha} - \bar{N}^{\alpha}_{\mu}\frac{\partial}{\partial \bar{\xi}^\alpha} , \qquad \frac{\delta}{\delta \xi_\alpha} = \frac{\partial}{\partial \xi_\alpha} - \tilde{N}^{\alpha\beta}_{0}\frac{\partial}{\partial \bar{\xi}^\beta} .$$

$N_{\alpha\lambda}$, $\bar{N}^{\alpha}_{\lambda}$, $\tilde{N}^{\alpha\beta}_{0}$ are the nonlinear connections which we shall define in (2.5).

The covariant derivatives of the metric tensor $g_{\mu\nu}$ are all zero:

$$D^{(*)}_{\mu} g_{\kappa\lambda} = 0 , \qquad D^{(*)\alpha} g_{\kappa\lambda} = 0 , \qquad D^{(*)}_{\alpha} g_{\kappa\lambda} = 0 . \tag{2.2}$$

The space-time frame $\delta/\delta x^\mu$ and the local Lorentz frame $\delta/\delta x^a$ are connected with

$$\frac{\delta}{\delta x^\mu} = h^{a}_{\mu}\frac{\delta}{\delta x^a} . \tag{2.3a}$$

Similarly, the spin-tetrad coefficients $\psi^{a}_{\alpha}$ and adjoint $\bar{\psi}^{\alpha a}$ connect the spin frames, $\partial/\partial\xi_\alpha$, $\partial/\partial\bar{\xi}^\alpha$ with $\partial/\partial x^a$:

$$\frac{\partial}{\partial \xi_\alpha} = \bar{\psi}^{\alpha a}\frac{\partial}{\partial x^a} , \tag{2.3b}$$

$$\frac{\partial}{\partial \bar{\xi}^\alpha} = \psi^a_\alpha \frac{\partial}{\partial x^a} . \tag{2.3c}$$

The absolute differential of an arbitrary contravariant vector $X^\nu$ is given by

$$DX^\nu = (D^{(*)}_\mu X^\nu) dx^\mu + (D^{(*)\alpha} X^\nu) d\xi_\alpha + (D^{(*)}_\alpha X^\nu) d\bar{\xi}^\alpha . \tag{2.4}$$

### 2.1. Nonlinear connections

We give the nonlinear connections in the framework of our consideration in the following form:

$$N = \{N_{\beta\mu}, \tilde{N}^{0\alpha}_\beta, N^0_{\alpha\beta}, \bar{N}^\beta_\mu, \tilde{N}^{\beta\alpha}_0, N^\beta_{0\alpha}\} , \tag{2.5}$$

$$N_{\beta\mu} = \tfrac{1}{2}\omega^{(*)ab}_\mu J_{ab}\xi_\beta , \quad \tilde{N}^{0\alpha}_\beta = \tfrac{1}{2}\bar{\Theta}^{(*)\alpha ab} J_{ab}\xi_\beta , \quad N^0_{\alpha\beta} = \tfrac{1}{2}\Theta^{(*)ab}_\alpha J_{ab}\xi_\beta ,$$
$$\bar{N}^\beta_\mu = -\tfrac{1}{2}\omega^{(*)ab}_\mu J_{ab}\bar{\xi}^\beta , \quad \tilde{N}^{\alpha\beta}_0 = -\tfrac{1}{2}\bar{\Theta}^{(*)\alpha ab} J_{ab}\bar{\xi}^\beta , \quad N^\beta_{0\alpha} = -\tfrac{1}{2}\Theta^{(*)ab}_\alpha J_{ab}\bar{\xi}^\beta .$$

The differentials of $D\xi_\alpha$, $D\bar{\xi}^\alpha$ can be written, after the relations (2.5), in the form:

$$D\xi_\beta = d\xi_\beta + N^0_{\alpha\beta} d\bar{\xi}^\alpha + \tilde{N}^{0\alpha}_\beta d\xi_\alpha + N_{\beta\mu} dx^\mu , \tag{2.6}$$

$$D\bar{\xi}^\beta = d\bar{\xi}^\beta + N^\beta_{0\alpha} d\bar{\xi}^\alpha - \tilde{N}^{\beta\alpha}_0 d\xi_\alpha - \bar{N}^\beta_\mu dx^\mu . \tag{2.7}$$

The metric in the second order tangent bundle is given by the relation [3,10]

$$G = g_{\kappa\lambda} dx^\kappa dx^\lambda + g_{ij}\delta y^i \delta y^j + g_{\alpha\beta}\delta u^\alpha \delta u^\beta , \tag{2.8}$$

and the adapted frame

$$\frac{\partial}{\partial Z^A} = \left( \frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - N^i_\lambda \frac{\partial}{\partial y^i} - M^\alpha_\lambda \frac{\partial}{\partial u^\alpha} , \frac{\delta}{\delta y^i} , \frac{\partial}{\partial u^\alpha} \right) \tag{2.9}$$

where $\delta/\delta y^i = \partial/\partial y^i - L^\alpha_i \, \partial/\partial u^\alpha$.

Furthermore, the dual frame is

$$\delta Z^A = \left( dx^\kappa , \ \delta y^i = dy^i + N^i_\lambda dx^\lambda , \ \delta u^\alpha = du^\alpha + L^\alpha_i dy^i + M^\alpha_\lambda dx^\lambda \right) .$$

The metrical structure in the spinor bundle will be defined as follows:

$$G = g_{\mu\nu}(x, \xi, \bar{\xi}) dx^\mu dx^\nu + g_{\alpha\beta}(x, \xi, \bar{\xi}) D\bar{\xi}^\alpha D\bar{\xi}^{*\beta} + g^{\alpha\beta} D\xi_\alpha D\xi^*_\beta . \tag{2.10}$$

In analogy with the previous adapted frame, a local adapted frame on a spinor bundle $S^{(2)}(M)$ will be defined as

$$\left( \frac{\partial}{\partial \zeta^A} \right) = \left\{ \frac{\delta}{\delta x^\lambda} , \frac{\delta}{\delta \xi_\alpha} , \frac{\delta}{\delta \bar{\xi}^\alpha} \right\} , \tag{2.11}$$

$$\frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} + N_{\alpha\lambda}\frac{\partial}{\partial \xi_\alpha} - \bar{N}^\alpha_\lambda \frac{\partial}{\partial \bar{\xi}^\alpha}\,, \qquad \frac{\delta}{\delta \xi_\alpha} = \frac{\partial}{\partial \xi_\alpha} - \tilde{N}^{\beta\alpha}_0 \frac{\partial}{\partial \bar{\xi}^\beta}\,,$$

and

$$\delta\zeta^A = \{dx^\kappa, D\xi_\beta, D\bar{\xi}^\beta\}\,,$$

where the expressions $D\xi_\beta$, $D\bar{\xi}^\beta$ are given by (2.6), (2.7). If we consider the connection coefficients $\Gamma^A_{BC}$ given in the general case, then in the total space $S^{(2)}(M)$ we have

$$\Gamma^A_{BC} = \left\{\Gamma^{(*)\mu}_{\nu\rho},\, C^\mu_{\nu\alpha},\, \bar{C}^{\mu\alpha}_\nu,\, \bar{\Gamma}^{(*)\gamma}_{\beta\lambda},\, C^{\gamma\alpha}_\beta,\, \tilde{C}^{\gamma\alpha}_\beta,\, \tilde{C}^\gamma_{\alpha\beta},\, \Gamma^{(*)\beta}_{\alpha\lambda},\, C^\gamma_{\alpha\beta}\right\}.$$

Considering that the connections are $d$-connections [6,8] in an adapted base, we get the following relations

$$D_{\partial/\partial x^C}\frac{\partial}{\partial x^B} = \Gamma^A_{BC}\frac{\partial}{\partial x^A}\,,$$

$$\begin{aligned}
D_{\delta/\delta x^\rho}\tfrac{\delta}{\delta x^\nu} &= \Gamma^{(*)\mu}_{\nu\rho}\tfrac{\delta}{\delta x^\mu}\,, & D_{\partial/\partial\bar{\xi}^\alpha}\tfrac{\delta}{\delta x^\nu} &= C^\mu_{\nu\alpha}\tfrac{\delta}{\delta x^\mu}\,,\\
D_{\delta/\delta\xi_\alpha}\tfrac{\delta}{\delta x^\nu} &= \bar{C}^{\mu\alpha}_\nu\tfrac{\delta}{\delta x^\mu}\,, & D_{\partial/\partial\bar{\xi}^\alpha}\tfrac{\delta}{\delta x^\nu} &= \Gamma^{(*)\gamma}_{\nu\alpha}\tfrac{\delta}{\delta\xi^\gamma}\,,\\
D_{\delta/\delta x^\lambda}\tfrac{\delta}{\delta\xi_\beta} &= \bar{\Gamma}^{(*)\beta}_{\lambda\gamma}\tfrac{\delta}{\delta\xi_\gamma}\,, & D_{\delta/\delta\xi_\alpha}\tfrac{\partial}{\partial\bar{\xi}^\beta} &= C^{\gamma\alpha}_\beta\tfrac{\partial}{\partial\bar{\xi}^\gamma}\,,\\
D_{\delta/\delta\xi_\alpha}\tfrac{\partial}{\partial\bar{\xi}^\beta} &= C^\gamma_{\beta\alpha}\tfrac{\partial}{\partial\bar{\xi}^\gamma}\,, & D_{\partial/\partial\bar{\xi}^\alpha}\tfrac{\delta}{\delta\xi_\beta} &= \tilde{C}^\beta_{\alpha\gamma}\tfrac{\delta}{\delta\xi_\gamma}\,,
\end{aligned} \tag{2.12}$$

$$D_{\delta/\delta\xi_\alpha}\frac{\delta}{\delta\xi_\beta} = \tilde{C}^{\beta\alpha}_\gamma\frac{\delta}{\delta\xi_\gamma}\,.$$

The covariant differentiation of tensors and spin-tensors of arbitrary rank may be classified into three types:

$$\begin{aligned}
\nabla_\lambda T^{\mu\ldots}_{\nu\ldots} &= \tfrac{\delta T^{\mu\ldots}_{\nu\ldots}}{\delta x^\lambda} + \Gamma^{(*)\mu}_{\kappa\lambda}T^{\kappa\ldots}_{\nu\ldots} + \cdots - \Gamma^{(*)\kappa}_{\nu\lambda}T^{\mu\ldots}_{\kappa\ldots}\,,\\
\nabla^\alpha T^{\mu\ldots}_{\nu\ldots} &= \tfrac{\delta T^{\mu\ldots}_{\nu\ldots}}{\delta\xi_\alpha} + \bar{C}^{(*)\mu\alpha}_\kappa T^{\kappa\ldots}_{\nu\ldots} + \ldots - \bar{C}^{(*)\kappa\alpha}_\nu T^{\mu\ldots}_{\kappa\ldots}\,,\\
\nabla_\alpha T^{\mu\ldots}_{\nu\ldots} &= \tfrac{\partial T^{\mu\ldots}_{\nu\ldots}}{\partial\bar{\xi}^\alpha} + C^{(*)\mu}_{\kappa\alpha}T^{\kappa\ldots}_{\nu\ldots} + \cdots - C^{(*)\kappa}_{\nu\alpha}T^{\mu\ldots}_{\kappa\ldots}\,,\\
\nabla_\lambda\Phi^{\alpha\ldots}_{\beta\ldots} &= \tfrac{\delta\Phi^{\alpha\ldots}_{\beta\ldots}}{\delta x^\lambda} - \Gamma^{(*)\gamma}_{\beta\lambda}\Phi^{\alpha\ldots}_{\gamma\ldots} - \cdots + \Phi^{\gamma\ldots}_{\beta\ldots}\Gamma^{(*)\alpha}_{\gamma\lambda} + \cdots\,,\\
\nabla^\delta\Phi^{\alpha\ldots}_{\beta\ldots} &= \tfrac{\delta\Phi^{\alpha\ldots}_{\beta\ldots}}{\delta\xi_\delta} - \tilde{C}^{(*)\gamma\delta}_\beta\Phi^{\alpha\ldots}_{\gamma\ldots} - \cdots + \Phi^{\gamma\ldots}_{\beta\ldots}\tilde{C}^{(*)\alpha\delta}_\gamma + \cdots\,,\\
\nabla_\delta\Phi^{\alpha\ldots}_{\beta\ldots} &= \tfrac{\partial\Phi^{\alpha\ldots}_{\beta\ldots}}{\partial\bar{\xi}^\delta} - C^{(*)\gamma}_{\beta\delta}\Phi^{\alpha\ldots}_{\gamma\ldots} - \cdots + \Phi^{\gamma\ldots}_{\beta\ldots}C^{(*)\alpha}_{\gamma\delta} + \cdots\,,\\
\nabla^{(*)}_\mu V^{a\ldots}_{c\ldots} &= \tfrac{\delta V^{a\ldots}_{c\ldots}}{\delta x^\mu} + \omega^{(*)a}_{\mu b}V^{b\ldots}_{c\ldots} + \cdots - \omega^{(*)b}_{\mu c}V^{a\ldots}_{b\ldots}\,,\\
\nabla^{(*)\alpha}V^{a\ldots}_{c\ldots} &= \tfrac{\delta V^{a\ldots}_{c\ldots}}{\delta\xi_\alpha} + \bar{\Theta}^{(*)a\alpha}_b V^{b\ldots}_{c\ldots} + \cdots - \bar{\Theta}^{(*)\alpha b}_c V^{a\ldots}_{b\ldots}\,,\\
\nabla^{(*)}_\alpha V^{a\ldots}_{c\ldots} &= \tfrac{\partial V^{a\ldots}_{c\ldots}}{\partial\bar{\xi}^\alpha} + \Theta^{(*)a}_{\alpha b}V^{b\ldots}_{c\ldots} + \cdots - \Theta^{(*)b}_{\alpha c}V^{a\ldots}_{b\ldots}\,.
\end{aligned} \tag{2.13}$$

### 2.2. Lorentz transformations

We can get the Lorentz transformations of linear connections $\omega_\nu^{(*)ab}$, $\bar{\Theta}^{(*)\beta ab}$, $\Theta_\beta^{(*)ab}$ in the following form:

$$\begin{aligned} \omega_\mu^{'(*)ab} &= L_c^a L_d^b \omega_\mu^{(*)cd} + \tfrac{\delta L_c^a}{\delta x^\mu} L_d^b n^{cd}\,, \\ \bar{\Theta}^{(*)'\alpha ab} &= \left[L_c^a L_d^b \bar{\Theta}^{(*)\beta cd} + \tfrac{\delta L_c^a}{\delta \xi_\beta} L_d^b n^{cd}\right] \Lambda_\beta^{-1\alpha}\,, \\ \Theta_\alpha^{(*)'ab} &= \Lambda_\alpha^\beta \left[L_c^a L_d^b \bar{\Theta}_\beta^{(*)cd} + \tfrac{\partial L_c^a}{\partial \bar{\xi}^\beta} L_d^b n^{cd}\right]\,, \end{aligned} \tag{2.14}$$

Similarly, the Lorentz transformation law of nonlinear connection is given by:

$$N'_{\beta\mu} = \frac{1}{2}\omega_\mu^{(*)ab} J_{ab}\xi_\alpha L_\beta^\alpha + \frac{1}{2} n^{cd}\frac{\delta L_c^a}{\delta x^\mu} L_d^b J_{ab}\Lambda_\beta^\alpha \xi_\alpha = N_{\alpha\mu}\Lambda_\beta^\alpha + \frac{1}{2} n^{cd}\frac{\delta L_c^a}{\delta x^\mu} L_d^b J'_{ab}\Lambda_\beta^\alpha \xi_\alpha\,,$$

$$\begin{aligned} \tilde{N}_\beta^{'0\alpha} &= \left[\tilde{N}_\gamma^{0\delta}\Lambda_\beta^\gamma + \tfrac{1}{2} n^{cd}\tfrac{\delta L_c^a}{\delta \xi_\delta} L_d^b J'_{ab}\Lambda_\beta^\gamma \xi_\gamma\right]\Lambda_\delta^{-1\alpha}\,, \\ \tilde{N}_{\alpha\beta}^{'0} &= \Lambda_\alpha^\delta \left[N_{\gamma\delta}^0 \Lambda_\beta^\gamma + \tfrac{1}{2} n^{cd}\tfrac{\partial L_c^a}{\partial \bar{\xi}^\delta} L_d^b J'_{ab}\Lambda_\beta^\gamma \xi_\gamma\right]\,, \\ \bar{N}_\mu^{'\beta} &= N_\mu^\alpha \Lambda_\alpha^{-1\beta} - \tfrac{1}{2} n^{cd}\tfrac{\delta L_c^a}{\delta x^\mu} L_d^b J'_{ab}\bar{\xi}^\gamma \Lambda_\gamma^{-1\beta}\,, \\ \tilde{N}_0^{'\alpha\beta} &= \left[\tilde{N}_0^{\gamma\delta}\Lambda_\gamma^{-1\beta} - \tfrac{1}{2} n^{cd}\tfrac{\delta L_c^a}{\delta \xi_\delta} L_d^b J'_{ab}\xi^\gamma \Lambda_\gamma^{-1\beta}\right]\Lambda_\delta^{-1\alpha}\,, \\ \tilde{N}_{0\alpha}^{'\beta} &= \Lambda_\alpha^\delta \left[N_{0\delta}^\gamma \Lambda_\gamma^{-1\beta} - \tfrac{1}{2} n^{cd}\tfrac{\partial L_c^a}{\partial \bar{\xi}^\delta} L_d^b J'_{ab}\bar{\xi}^\gamma \Lambda_\gamma^{-1\beta}\right]\,, \end{aligned} \tag{2.15}$$

where $J'_{ab} = L_a^c L_b^d J_{cd}$.

## 3. Curvatures and torsions

From the covariant derivatives $D_\mu^{(*)}$, $D^{(*)\alpha}$, $D_\alpha^{(*)}$ we get six curvatures and torsions:

(a)
$$\begin{aligned} \left[D_\mu^{(*)}, D_\nu^{(*)}\right] &= D_\mu^{(*)}D_\nu^{(*)} - D_\nu^{(*)}D_\mu^{(*)} = R_{\mu\nu}^a P_a + \frac{1}{2}R_{\mu\nu}^{ab}J_{ab}\,, \\ R_{\mu\nu}^a &= \frac{\delta h_\mu^a}{\delta x^\nu} - \frac{\delta h_\nu^a}{\delta x^\mu} + \omega_{\mu b}^{(*)a} h_\nu^b - \omega_{\nu b}^{(*)a} h_\mu^b\,, \\ R_{\mu\nu}^{ab} &= \frac{\delta \omega_\mu^{(*)ab}}{\delta x^\nu} - \frac{\delta \omega_\nu^{(*)ab}}{\delta x^\mu} + \omega_\mu^{(*)a\rho}\omega_{\nu\rho}^{(*)b} - \omega_\nu^{(*)\rho a}\omega_{\mu\rho}^{(*)b}\,, \end{aligned}$$

(b)
$$\begin{aligned} \left[D_\mu^{(*)}, D_\alpha^{(*)}\right] &= P_{\mu\alpha}^a P_a + \frac{1}{2}P_{\mu\alpha}^{ab}J_{ab}\,, \\ P_{\mu\alpha}^{ab} &= \frac{\delta \theta_\alpha^{(*)ab}}{\delta x^\mu} - \frac{\partial \omega_\mu^{(*)ab}}{\delta \bar{\xi}^\alpha} + \Theta_{\alpha c}^{(*)b}\omega_\mu^{(*)ac} - \Theta_{\alpha c}^{(*)a}\omega_\mu^{(*)cb}\,, \\ P_{\mu\alpha}^a &= \frac{\delta \psi_\alpha^a}{\delta x^\mu} - \frac{\partial h_\mu^a}{\delta \bar{\xi}^\alpha} + \omega_{\mu c}^{(*)a}\psi_\alpha^c - \Theta_{\alpha c}^{(*)a} h_\mu^c\,, \end{aligned} \tag{3.1}$$

(c)
$$\begin{aligned}
\left[D^{(*)}_{\mu}, D^{(*)\alpha}\right] &= \bar{P}^{a\alpha}_{\mu} P_a + \frac{1}{2}\bar{P}^{ab\alpha}_{\mu} J_{ab}\,, \\
\bar{P}^{ab\alpha}_{\mu} &= \frac{\delta\bar{\Theta}^{(*)\alpha ab}}{\delta x^{\mu}} - \frac{\delta\omega^{(*)ab}_{\mu}}{\delta\xi_{\alpha}} + \bar{\Theta}^{(*)ab}_{c}\omega^{(*)ac}_{\mu} - \bar{\Theta}^{(*)\alpha a}_{c}\omega^{(*)cb}_{\mu}\,, \\
\bar{P}^{a\alpha}_{\mu} &= \frac{\delta\bar{\psi}^{\alpha a}}{\delta x^{\mu}} - \frac{\delta h^{a}_{\mu}}{\delta\xi^{\alpha}} + \omega^{(*)a}_{\mu c}\bar{\psi}^{c\alpha} - \bar{\Theta}^{(*)\alpha a}_{c} h^{c}_{\mu}\,,
\end{aligned}$$

(d)
$$\begin{aligned}
\left[D^{(*)}_{\alpha}, D^{(*)\beta}\right] &= S^{\beta a}_{\alpha} P_a + \frac{1}{2} S^{ab\beta}_{\alpha} J_{ab}\,, \\
S^{\beta a}_{\alpha} &= \frac{\delta\bar{\psi}^{\beta a}}{\delta\bar{\xi}^{\alpha}} - \frac{\delta\psi^{a}_{\alpha}}{\delta\xi_{\beta}} + \bar{\Theta}^{(*)\beta ba}\psi_{ab} - \Theta^{(*)ab}_{\alpha}\bar{\psi}^{\beta}_{b}\,, \\
S^{ab\beta}_{\alpha} &= \frac{\partial\bar{\Theta}^{\beta ab}}{\partial\bar{\xi}^{\alpha}} - \frac{\partial\Theta^{(*)ab}_{\alpha}}{\delta\xi_{\beta}} + \Theta^{(*)a}_{\alpha c}\bar{\Theta}^{(*)\beta cb} - \Theta^{(*)b}_{ac}\bar{\Theta}^{\beta ca}\,,
\end{aligned}$$

(e)
$$\begin{aligned}
\left[D^{(*)}_{\alpha}, D^{(*)}_{\beta}\right] &= Q^{a}_{\alpha\beta} P_a + \frac{1}{2} Q^{ab}_{\alpha\beta} J_{ab}\,, \\
Q^{a}_{\alpha\beta} &= \frac{\partial\psi^{a}_{\beta}}{\partial\bar{\xi}^{\alpha}} - \frac{\partial\psi^{a}_{\alpha}}{\partial\bar{\xi}^{\beta}} + \Theta^{(*)ba}_{\beta}\psi_{ab} - \Theta^{(*)ab}_{\alpha}\psi_{\beta b}\,, \\
Q^{ab}_{\alpha\beta} &= \frac{\partial\theta^{(*)ab}_{\beta}}{\partial\bar{\xi}^{\alpha}} - \frac{\partial\theta^{(*)ab}_{\alpha}}{\partial\bar{\xi}^{\beta}} + \Theta^{(*)a}_{\alpha c}\Theta^{(*)cb}_{\beta} - \Theta^{(*)b}_{\alpha c}\Theta^{(*)ca}_{\beta}\,,
\end{aligned}$$

(f)
$$\begin{aligned}
\left[D^{(*)\alpha}, D^{(*)\beta}\right] &= \tilde{Q}^{\alpha\beta a} P_a + \frac{1}{2}\tilde{Q}^{ab\alpha\beta} J_{ab}\,, \\
\tilde{Q}^{\alpha\beta a} &= \frac{\delta\psi^{a}_{\beta}}{\delta\xi_{\alpha}} - \frac{\delta\psi^{a}_{\alpha}}{\delta\xi_{\beta}} + \bar{\Theta}^{(*)\beta ba}\bar{\psi}^{\alpha}_{b} - \bar{\Theta}^{(*)\alpha ba}\bar{\psi}^{\beta}_{b}\,, \\
\tilde{Q}^{ab\alpha\beta} &= \frac{\delta\bar{\theta}^{\beta ab}}{\delta\xi_{\alpha}} - \frac{\delta\bar{\theta}^{\alpha ab}}{\delta\xi_{\beta}} + \bar{\Theta}^{(*)\beta cb}\bar{\Theta}^{(*)\alpha a}_{c} - \bar{\Theta}^{(*)\alpha b}_{c}\bar{\Theta}^{\beta ca}.
\end{aligned}$$

## 4. Field equations

We derive the field equations using the spin-tetrads frames in the Lagrangian form: $\mathcal{L}(h, \omega^{(*)}, \psi, \Theta^{(*)}, \bar{\psi}, \bar{\Theta}^{(*)})$. The method of derivation of equations is similar to Palatini's one and it is analogous to [12].

We get the Lagrangian

$$\mathcal{L}(h, \omega^{(*)}, \psi, \Theta^{(*)}, \bar{\psi}, \bar{\Theta}^{(*)})$$

or

$$\mathcal{L}(\psi^A, \delta_M\psi^A) = h(R + P + \bar{P} + S + Q + \tilde{Q}), \tag{4.1}$$

where

$$\psi^A = \left(h^a_\mu(x,\xi,\bar{\xi}),\, \omega^{(*)ab}_\mu(x,\xi,\bar{\xi}),\, \psi^a_\alpha(x,\xi,\bar{\xi}),\, \bar{\psi}^{\alpha a}_{(..)},\, \Theta^{(*)ab}_{\alpha(..)},\, \bar{\Theta}^{(*)\alpha ab}_{(..)}\right),$$

$$\delta_M = \frac{\delta}{\delta z^M} = \left(\frac{\delta}{\delta x^\mu}, \frac{\delta}{\delta \xi_\alpha}, \frac{\delta}{\delta \bar{\xi}^\alpha}\right), \qquad z^M = (x^\mu,\, \xi_\alpha,\, \bar{\xi}^\alpha),$$

$$R = h^\mu_a h^\nu_b R^{ab}_{\mu\nu},$$

$$P = h^\mu_a \bar{\psi}^\alpha_b P^{ab}_{\mu\alpha}, \qquad \bar{P} = h^\mu_a \bar{\psi}_{\alpha b} \bar{P}^{ab\alpha}_\mu,$$

$$Q = Q^{ab}_{\alpha\beta}\bar{\psi}^\alpha_a \bar{\psi}^\beta_b, \qquad \tilde{Q} = \tilde{Q}^{ab\alpha\beta}\psi_{\alpha a}\psi_{\beta b},$$

$$S = \bar{\psi}^\alpha_a \psi_{\beta b} S^{ab\beta}_\alpha.$$

The Euler–Lagrange equations are written in the form:

$$\frac{\delta \mathcal{L}}{\delta z^M} = \frac{\partial \mathcal{L}}{\partial(\delta_M \psi^{(A)})} - \frac{\partial \mathcal{L}}{\partial \psi^{(A)}} = 0. \tag{4.2}$$

From the relation (4.1), the variation of $\mathcal{L}$ with respect to $h^\nu_b$ yields the equations

$$(R^a_\mu + P^a_\mu + \bar{P}^a_\mu) - \frac{1}{2}(R + P + \bar{P})h^a_\mu = 0, \tag{4.3}$$

$$H^a_\mu - \frac{1}{2}H h^a_\mu = 0, \tag{4.4}$$

where

$$P^a_\mu = \bar{\psi}^\alpha_b P^{ab}_{\mu\alpha}, \qquad \bar{P}^a_\mu = \psi_{ab}\bar{P}^{ab\alpha}_\mu, \qquad R^a_\mu = h^\nu_b R^{ab}_{\mu\nu},$$

and

$$H^a_\mu = R^a_\mu + P^a_\mu + \bar{P}^a_\mu, \qquad H = R + P + \bar{P}.$$

From the variation of $\mathcal{L}$ with respect to $\omega^{(*)ab}_\mu$

$$\frac{\delta}{\delta x^\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta}{\delta x^\mu}\omega^{(*)ab}_\nu\right)}\right) + \frac{\delta}{\delta \xi_\alpha}\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta \omega^{(*)ab}_\nu}{\delta \xi_\alpha}\right)} + \frac{\delta}{\delta \bar{\xi}^\alpha}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta}{\delta \bar{\xi}^\alpha}\omega^{(*)ab}_\nu\right)}\right) - \frac{\partial \mathcal{L}}{\partial \omega^{(*)ab}_\nu} = 0, \tag{4.5}$$

we get

$$D^{(*)}_\mu[h(h^\nu_a h^\mu_b - h^\nu_b h^\mu_a)] + D^{(*)}_\alpha[h(h^\nu_a \bar{\psi}^\alpha_b - h^\nu_a \bar{\psi}^\alpha_a)] + D^{(*)\alpha}[h(h^\nu_a \psi_{\alpha b} - h^\nu_b \psi^a_\alpha)] = 0. \tag{4.5'}$$

The variations with respect to $\Theta^{(*)ab}_\alpha$, $\bar{\Theta}^{(*)\alpha ab}$ yield the relation

$$\frac{\delta}{\delta x^\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta \Omega^{(*)}}{\delta x^\mu}\right)}\right) + \frac{\delta}{\delta \xi_\alpha}\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta \Omega^{(*)}}{\delta \xi_\alpha}\right)} + \frac{\delta}{\delta \bar{\xi}^\alpha}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\delta \Omega^{(*)}}{\delta \bar{\xi}^\alpha}\right)}\right) - \frac{\partial \mathcal{L}}{\partial \Omega^{(*)}} = 0 \tag{4.6}$$

with

$$\Omega^{(*)} = \left\{\Theta_{\alpha}^{(*)ab}, \bar{\Theta}^{(*)\alpha ab}\right\}$$

which gives us the equations:

$$D_{\mu}^{(*)}(hh_a^{\mu}\bar{\psi}_b^{\alpha}) - D_{\beta}^{(*)}(2h\bar{\psi}_a^{\alpha}\bar{\psi}_b^{\beta}) - 2D^{(*)\beta}(h\bar{\psi}_a^{\alpha}\psi_{\beta b}) = 0\,, \tag{4.7}$$

$$D_{\mu}^{(*)}(hh_a^{\mu}\psi_{b\alpha}) - 2D_{\beta}^{(*)}(h\psi_{a\alpha}\bar{\psi}_b^{\beta}) - D^{(*)\beta}(2h\psi_{a\alpha}\psi_{b\beta}) = 0\,. \tag{4.8}$$

Finally, the variation of $\mathcal{L}$ with respect to the spin-tetrad coefficients $\bar{\psi}_a^{\alpha}$, $\psi^{\alpha a}$ derives the equations:

$$Q_{\alpha\beta}^{ab}\bar{\psi}_b^{\beta} + \frac{1}{2}S_{\alpha}^{ab\beta}\psi_{\beta b} + \frac{1}{2}P_{\mu\alpha}^{ba}h_b^{\mu} = 0\,, \tag{4.9}$$

$$\tilde{Q}^{a\alpha} - \frac{1}{2}(S^{a\alpha} + \bar{P}^{a\alpha}) = 0\,. \tag{4.10}$$

## 5. Bianchi identities

From Jacobi identities,

$$\mathcal{Q}_{(XYZ)}\left[D_X^{(*)}, [D_Y^{(*)}, D_Z^{(*)}]\right] = 0\,,$$

we get 18(3×6) relations of different types. For each relation we derive two identities, namely 36 ones in total. Taking into account that

$$D_{\mu}^{(*)} = \frac{\delta}{\delta x^{\mu}} + \frac{1}{2}\omega_{\mu}^{(*)ab}J_{ab}\,,$$

$$\frac{\delta}{\delta x^{\mu}} = \frac{\partial}{\partial x^{\mu}} - N_{\mu\alpha}\frac{\partial}{\partial \xi_{\alpha}} - \bar{N}_{\mu}^{\alpha}\frac{\partial}{\partial \bar{\xi}^{\alpha}} = h_{\mu}^{a}P_a - N_{\mu\alpha}\bar{\Psi}^{\alpha a}P_a - \bar{N}_{\mu}^{\alpha}\Psi_{\alpha}^{a}P_a = A_{\mu}^{a}P_a\,,$$

where

$$A_{\mu}^{a} = h_{\mu}^{a} - N_{\mu\alpha}\bar{\Psi}^{\alpha a} - \bar{N}_{\mu}^{\alpha}\,, \qquad P_a = \frac{\partial}{\partial x^a}\,,$$

we can get

$$\left[D_{\mu}^{(*)}, [D_{\kappa}^{(*)}, D_{\lambda}^{(*)}]\right] = \left[A_{\mu}^{c}P_c, \frac{1}{2}R_{\kappa\lambda}^{ab}J_{ab}\right] + [A_{\mu}^{c}P_c, R_{\kappa\lambda}^{a}P_a] + \frac{1}{2}\omega_{\mu}^{(*)ab}R_{\kappa\lambda}^{cd}[J_{ab}, J_{cd}] + \frac{1}{2}\omega_{\mu}^{(*)ab}R_{\kappa\lambda}^{c}[J_{ab}, P_c]\,. \tag{5.1}$$

The first term of the right hand side of (5.1) by straightforward calculations is written in the form

$$\left[A_{\mu}^{c}P_c, \frac{1}{2}R_{\kappa\lambda}^{ab}J_{ab}\right] = \frac{1}{2}\frac{\delta R_{\kappa\lambda}^{ab}}{\delta x^{\mu}}J_{ab} + R_{b\kappa\lambda}^{a}A_{\mu}^{b}P_a\,. \tag{5.2}$$

Similarly, the second, third, and fourth terms of (5.1) yield the relations

$$[A_{\mu}^{c}P_c, R_{\kappa\lambda}^{a}P_a] = \frac{\delta R_{\kappa\lambda}^{a}}{\delta x^{\mu}}P_a + A_{\mu}^{c}R_{\kappa\lambda}^{a}[P_c, P_a] = \frac{\delta R_{\kappa\lambda}^{a}}{\delta x^{\mu}}P_a\,, \tag{5.3}$$

where we used the fact that $[P_c, P_a] = 0$. Also

$$\frac{1}{4}\omega_\mu^{(*)ab} R^{cd}_{\kappa\lambda}[J_{ab}, J_{cd}] = \omega_\mu^{(*)ac} R^b_{c\kappa\lambda} J_{ab}\,, \tag{5.4}$$

$$\frac{1}{2}\omega_\mu^{(*)ab} R^{c}_{\kappa\lambda}[J_{ab}, P_{c}] = \omega_{\mu b}^{(*)a} R^b_{\kappa\lambda} P_{a}\,, \tag{5.5}$$

so the relation (5.1) is written as

$$\Big[D_\mu^{(*)}, [D_\kappa^{(*)}, D_\lambda^{(*)}]\Big] = \left(\frac{1}{2}\frac{\delta R^{ab}_{\kappa\lambda}}{\delta x^\mu} + \omega_\mu^{(*)ac} R^b_{c\kappa\lambda}\right) + J_{ab} + \left(\frac{\delta R^a_{\kappa\lambda}}{\delta x^\mu} + R^a_{b\kappa\lambda} A^b_\mu + R^c_{\kappa\lambda}\omega_{\mu b}^{(*)a}\right) P_a\,. \tag{5.6}$$

Defining

$$D_\mu R^{ab}_{\kappa\lambda} = \frac{1}{2}\frac{\delta R^{ab}_{\kappa\lambda}}{\delta x^\mu} + \omega_\mu^{(*)ac} R^b_{c\kappa\lambda}\,, \tag{5.7}$$

$$D_\mu R^{a}_{\kappa\lambda} = \frac{1}{2}\frac{\delta R^{a}_{\kappa\lambda}}{\delta x^\mu} + R^{\cdot a}_{b\kappa\lambda} A^b_\mu + R^c_{\kappa\lambda}\omega_{\mu b}^{(*)a}\,, \tag{5.8}$$

we have the relations:

$$D_\mu R^{ab}_{\kappa\lambda} + D_\kappa R^{ab}_{\lambda\mu} + D_\lambda R^{ab}_{\mu\kappa} = 0\,, \tag{5.9}$$

$$D_\mu R^{a}_{\kappa\lambda} + D_\kappa R^{a}_{\lambda\mu} + D_\lambda R^{a}_{\mu\kappa} = 0\,. \tag{5.10}$$

In a similar way, from

$$\mathcal{Q}_{(\alpha\beta\gamma)}\Big[D_\alpha^{(*)}, [D_\beta^{(*)}, D_\gamma^{(*)}]\Big] = 0 \tag{5.11}$$

we get for the Q-curvature and torsion the identities below:

$$D_\alpha Q^{ab}_{\beta\gamma} + D_\beta Q^{ab}_{\gamma\alpha} + D_\gamma Q^{ab}_{\alpha\beta} = 0 \tag{5.12}$$

and

$$D_\alpha Q^{a}_{\beta\gamma} + D_\beta Q^{a}_{\gamma\alpha} + D_\gamma Q^{a}_{\alpha\beta} = 0\,, \tag{5.13}$$

where we put

$$D_\alpha Q^{ab}_{\beta\gamma} = \frac{1}{2}\frac{\partial Q^{ab}_{\beta\gamma}}{\partial\bar\xi^\alpha} + \Theta_\alpha^{(*)ac} Q^b_{c\beta\gamma}\,, \tag{5.14}$$

$$D_\alpha Q^{a}_{\beta\gamma} = \frac{\partial Q^{a}_{\beta\gamma}}{\partial\bar\xi^\alpha} + Q^a_{b\beta\gamma}\Psi^b_\alpha + Q^b_{\beta\gamma}\Theta_{\alpha b}^{(*)a}\,. \tag{5.15}$$

## 6. Yang–Mills fields

In this section, we study Yang–Mills fields and we derive the generalized Yang–Mills equations in the framework of our approach. In such a case we consider a vector field $A = (A_\mu, A_\alpha, \bar{A}^\alpha)$ with values in a Lie algebra $\mathcal{G}$ and we define the generalized covariant derivatives by

$$\tilde{D}_\mu = D_\mu + iA_\mu\,, \qquad \tilde{D}_\alpha = D_\alpha + iA_\alpha\,, \qquad \tilde{\bar{D}}^\alpha = \bar{D}^\alpha + i\bar{A}^\alpha\,. \tag{6.1}$$

The commutators $[\tilde{D}_X, \tilde{D}_Y]$, $X, Y = \{\mu, \nu, \alpha, \beta\}$ are defined in the following form

$$[\tilde{D}_\mu, \tilde{D}_\nu] = [D_\mu, D_\nu] + iF_{\mu\nu}\,, \tag{6.2}$$

where,

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu + i[A_\mu, A_\nu]$$

represents the Yang–Mills field, $A_\mu$ is given by

$$A_\mu = A^i_\mu \tau_i\,, \qquad [\tau_i, \tau_j] = C^k_{ij}\tau_k\,, \tag{6.3}$$

the elements $\tau_i$ are the generators which satisfy the commutation relations of the Lie algebra, and $D_\mu$ represent the gauge covariant derivatives (Def. 2.1).

Using (6.3) of the matrices $A_\mu$ we find that

$$F_{\mu\nu} = F^i_{\mu\nu}\tau_i\,, \tag{6.4}$$

where the field strengths are given by

$$F^k_{\mu\nu} = D_\mu A^k_\nu - D_\nu A^k_\mu + iA^i_\mu A^j_\nu C^k_{ij}\,. \tag{6.5}$$

Moreover, the generalized gauge field is defined by the quantities $F_{XY}$, $X, Y = \{\mu, \nu, \alpha, \beta\}$, that is,

$$\begin{aligned} [\tilde{D}_\mu, \tilde{D}_\alpha] &= [D_\mu, D_\alpha] + iF_{\mu\alpha}\,, \\ [\tilde{D}_\mu, \tilde{\bar{D}}^\alpha] = [D_\mu, \bar{D}^\alpha] + i\bar{F}^\alpha_\mu\,, &\qquad [\tilde{D}_\alpha, \tilde{\bar{D}}^\beta] = [D_\alpha, \bar{D}^\beta] + iF^\beta_\alpha\,, \\ [\tilde{D}_\alpha, \tilde{D}_\beta] = [D_\alpha, D_\beta] + iF_{\alpha\beta}\,, &\qquad [\tilde{\bar{D}}^\alpha, \tilde{\bar{D}}^\beta] = [\bar{D}^\alpha, \bar{D}^\beta] + i\bar{F}^{\alpha\beta}\,, \end{aligned} \tag{6.6}$$

with

$$\begin{aligned} F_{\mu\alpha} = D_\mu A_\alpha - D_\alpha A_\mu + i[A_\mu, A_\alpha]\,, &\qquad \bar{F}^\alpha_\mu = D_\mu \bar{A}^\alpha - \bar{D}^\alpha A_\mu + i[A_\mu, \bar{A}^\alpha]\,, \\ F^\beta_\alpha = D_\alpha \bar{A}^\beta - \bar{D}^\beta A_\alpha + i[A_\alpha, \bar{A}^\beta]\,, &\qquad F_{\alpha\beta} = D_\alpha A_\beta - D_\beta A_\alpha + i[A_\alpha, A_\beta]\,, \\ \bar{F}^{\alpha\beta} = \bar{D}^\alpha \bar{A}^\beta &- \bar{D}^\beta \bar{A}^\alpha + i[\bar{A}^\alpha, \bar{A}^\beta]\,. \end{aligned} \tag{6.7}$$

In our space $S^{(2)}(M)$ the Yang–Mills generalized action can be written in the form

$$S_{GF} = \int d^4x\, d^4\xi\, d^4\bar{\xi}\, h(\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \mathrm{tr}\, F_{\mu\alpha}\bar{F}^{\mu\alpha} + \mathrm{tr}\, F_{\alpha\beta}\bar{F}^{\alpha\beta} + \mathrm{tr}\, F^\beta_\alpha F^\alpha_\beta)\,, \tag{6.8}$$

where $F_{\mu\nu}$ represent the internal quantities in the base manifold, $F^{\mu}_{\alpha}$ the field in the spinor bundle and $F_{\alpha\beta}$ the internal quantities in the internal space.

In order to derive the generalized Yang–Mills equations we get the Lagrangian

$$\mathcal{L}_{YM}(A_X, D_X A_Y)\,, \tag{6.9}$$

where $A_X = \{A_\mu, A_\alpha, A^\beta\}$ and $D_X A_Y$ represent

$$D_X A_Y = \{D_\mu A_\nu, D_\alpha A_\nu, \bar{D}^\alpha A_\nu, D_\alpha A_\beta, \bar{D}^\alpha A_\beta, \bar{D}^\alpha \bar{A}^\beta, D_\mu A_\alpha, D_\mu A^\alpha\}\,.$$

By varying the action (6.8) and taking into account the Euler–Lagrange equations

$$D_X\left(\frac{\partial \mathcal{L}_{YM}}{\partial(D_X A_Y)}\right) - \frac{\partial \mathcal{L}_{YM}}{\partial A_Y} = 0\,, \tag{6.10}$$

we obtain the generalized Yang–Mills equations in the following form:

$$\begin{aligned} \tilde{D}^\mu F_{\mu\nu} + \tilde{D}^\alpha F_{\alpha\nu} + \tilde{D}_\alpha \bar{F}^\alpha_\nu &= 0\,,\\ \tilde{D}_\mu F^{\mu\beta} + \tilde{D}_\alpha F^{\alpha\beta} + \tilde{D}^\alpha F^\beta_\alpha &= 0\,,\\ \tilde{D}_\mu F^\mu_\beta + \tilde{D}_\alpha F^\alpha_\beta + \tilde{D}^\alpha F_{\alpha\beta} &= 0\,, \end{aligned} \tag{6.11}$$

where for the derivation of the equations (6.12) we used the trace properties of the generators $\tau_\alpha$ with the normalization condition

$$\mathrm{tr}\,(\tau^\alpha \tau^\beta) = \frac{1}{2}\delta^{\alpha\beta}\,.$$

## 7. Yang–Mills–Higgs field

In this last paragraph we shall give the form of Yang–Mills–Higgs field in a sufficiently generalized form. The usual case has been studied with the appropriate Lagrangian $\mathcal{L}$ and the corresponding Euler–Lagrange equations.

Here, we get a scalar field $\phi$ of mass m which is valuated in the Lie algebra $\mathcal{G}$ of consideration and is defined by

$$\phi : M^{(4)} \times \mathbb{C}^4 \times \mathbb{C}^4 \to \mathcal{G}$$

with

$$\phi(x^\mu . \xi_\alpha, \bar{\xi}^\alpha) \in \mathcal{G}\,.$$

If $\phi_i$ is in the adjoint representation, its covariant derivatives are given by

$$\begin{aligned} \tilde{D}_\mu \phi &= D_\mu \phi + [A_\mu, \phi]\,,\\ \tilde{D}_\alpha \phi &= D_\alpha \phi + [A_\alpha, \phi]\,,\\ \tilde{D}^\alpha \phi &= D^\alpha \phi + [A^\alpha, \phi]\,. \end{aligned} \tag{7.1}$$

The first of the relations (7.2), after taking into account (6.3), becomes

$$\tilde{D}_\mu \phi \;=\; D_\mu \phi + A^\alpha_\mu \phi^b C^c_{\alpha c} \tau_b\,; \tag{7.2}$$

for $\tilde{D}_\alpha \phi$, $\tilde{D}^\alpha \phi$ similar relations are produced.

The generalized Lagrangian is given by the following form:

$$\mathcal{L} \;=\; \mathcal{L}_{YM} - \frac{1}{2}\mathrm{tr}\,(\tilde{D}_\mu \phi)(\tilde{D}^\mu \phi) - \frac{1}{2}\mathrm{tr}\,(\tilde{D}_\alpha \phi)(\tilde{D}^\alpha \phi) + \frac{1}{2} m^2 \mathrm{tr}\,\phi^2\,. \tag{7.3}$$

Using (7.2) and getting (6.10) for this Lagrangian $\mathcal{L}$, the generalized Yang–Mills–Higgs equations are as follows:

$$\begin{aligned}
\tilde{D}^\mu F_{\mu\nu} + \tilde{D}^\alpha F_{\alpha\nu} + \tilde{D}_\alpha F^\alpha_\nu + [\phi, \tilde{D}_\nu \phi] &= 0\,,\\
\tilde{D}_\mu F^{\mu\beta} + \tilde{D}_\alpha F^{\alpha\beta} + \tilde{D}^\alpha F^\beta_\alpha + [\phi, \tilde{D}^\beta \phi] &= 0\,,\\
\tilde{D}_\mu F^\mu_\beta + \tilde{D}_\alpha F^\alpha_\beta + \tilde{D}^\alpha F_{\alpha\beta} + [\phi, \tilde{D}_\beta \phi] &= 0\,.
\end{aligned} \tag{7.4}$$

## 8. Conclusion

In Sections 2 and 3 we presented the basic differential structures of second order spinor bundle $S^{(2)}M$ utilizing the fundamental 1-forms $\rho_\mu$. The introduction and use of the Poincaré group and the $d$-connections in spinor bundle as it has up to now appeared, i.e. [15].

From the gauge consideration perspective, the connection coefficients $\omega^{(*)ab}_\mu$, $\Theta^{(*)ab}_\alpha$, which play the role of the gauge potential for gravity, were introduced in spaces where the metric tensor $g_{\mu\nu}(x,\omega)$ depends on internal independent variables $\omega = (\xi, \bar{\xi})$.

Furthermore, the above mentioned nonlinear connections represent a form of a unified gauge potential. The calculated curvatures correspond to the gauge field strength of the spinor bundle $S^{(2)}(M)$. In general, the introduction of spinor variables in a Riemannian space generalizes and enriches it with torsion.

In the last two sections we derived the Yang–Mills and Yang–Mills–Higgs equations in a generalized form.

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