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FIELD EQUATIONS IN SPACES WITH $g_{\mu\nu}(x,\xi,\overline{\xi})$ METRIC. GENERALIZED CONFORMALLY FLAT SPACES

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The gravitational field equations are derived here in the framework of spaces whose metric tensor depends also on spinor variables ξ and $\overline{\xi}$. The attempt is to describe gravity by a tetrad field and the Lorentz connection coefficients in a more generalized framework than the standard one (cf. eg. [11]). An interesting case with generalized conformally flat spaces GCFS with metric $g_{\mu\nu}(x,\xi,\overline{\xi}) = e^{2\sigma(x,\xi,\overline{\xi})}\eta_{\mu\nu}$ is studied.

1. Introduction

The introduction of a metric $g_{\mu\nu}(x,\omega)$ that depends on the position variables x as well as on the spinor variables ω assings a non-Riemannian structure to the space and provides it with torsion. This procedure enables the construction of a non-localized (bilocal) gravitational field, identical to the one proposed by Yukawa [17] that allows a more general description of elementary particles. Further arguments have been developed by some other authors [5, 10, 15]. In our context each point of the space–time is characterized by the influence of two fields: an external one which is the conventional gravitational field in Einstein's sense, and an internal one due to the introduction of the spinor variables. These fields are expected to play the role of a geometrical unification of the fields. If ω is represented by a vector y, then we work in the Finslerian framework [3, 6, 7]. A more general case of the gauge approach in the framework of Finsler and Lagrange geometry has been studied e.g. in [1, 2, 4, 7, 8, 9].

In the following, we consider a space–time and we denote its metric tensor by

$$g_{\mu\nu}(Z^M),$$

where $Z^M = (x^\mu, \xi_\alpha, \overline{\xi}^\alpha), x^\mu, \xi_\alpha, \overline{\xi}^\alpha$ represent the position and the 4-spinor variables ($\overline{\xi}$ denotes the Dirac conjugate of ξ) [15]. With the Greek letters λ, μ, ν and α, β, γ we denote the space–time indices and the spinor indices, also Latin letters a, b, c are used for the Lorentz (flat) indices. The (*)-differential operators $\partial_M^{(*)}$ are defined as

$$\partial_M^{(*)} = \frac{\partial^{(*)}}{\partial Z^M} = \left(\frac{\partial^{(*)}}{\partial x^\mu}, \frac{\partial^{(*)}}{\partial \xi_\alpha}, \frac{\partial^{(*)}}{\partial \overline{\xi}^\alpha}\right), \tag{1.1}$$

with

$$\frac{\partial^{(*)}}{\partial x^\lambda} = \frac{\partial}{\partial x^\lambda} + N_{\alpha\lambda}\frac{\partial}{\partial \xi_\alpha} + \overline{N}^\alpha_\lambda \frac{\partial}{\partial \overline{\xi}^\alpha}, \qquad \frac{\partial^{(*)}}{\partial \xi_\alpha} = \frac{\partial}{\partial \xi_\alpha} + \widetilde{\eta}^{0\alpha}_\beta \frac{\partial}{\partial \xi_\alpha} + \widetilde{\eta}^{\beta\alpha}_0 \frac{\partial}{\partial \overline{\xi}^\beta},$$
$$\frac{\partial^{(*)}}{\partial \overline{\xi}^\alpha} = \frac{\partial}{\partial \overline{\xi}^\alpha} + \eta^0_{\beta\alpha}\frac{\partial}{\partial \xi_\beta} + \eta^\beta_{0\alpha}\frac{\partial}{\partial \overline{\xi}^\beta},$$

where $N_{\alpha\lambda}, \overline{N}^\alpha_\lambda, \widetilde{\eta}^{0\alpha}_\beta, \widetilde{\eta}^{\beta\alpha}_0, \eta^0_{\beta\alpha}, \eta^\beta_{0\alpha}$ are the nonlinear connections [10].

In our study, field equations are obtained from a Lagrangian density of the form

$$L(\Psi^{(A)}, \partial_M^{(*)}\Psi^{(A)}), \tag{1.2}$$

where $\Psi^{(A)}$ is the set

$$\Psi^{(A)} = \{h^a_\mu(x,\xi,\overline{\xi}), \omega^{(*)ab}_\mu(x,\xi,\overline{\xi}), \theta^{(*)ab}_\alpha(x,\xi,\overline{\xi}), \overline{\theta}^{(*)ab\alpha}(x,\xi,\overline{\xi})\}.$$

Thus L is a function of the tetrad field, of the spin connection coefficients and of their (*)-derivatives. The variables $h, \omega^{(*)}, \theta^{(*)}, \overline{\theta}^{(*)}$ are considered as independent.

It is known that gravity can be described by the tetrad field and the Lorentz connection coefficients [11]. The variation of the Palatini action with respect to h and ω yields a set of two equations:

$$R^a_\mu - \tfrac{1}{2}Rh^a_\mu = 0, \tag{1.3a}$$
$$D_\mu[h(h^\nu_a h^\mu_b - h^\nu_b h^\mu_a)] = 0. \tag{1.3b}$$

h is the determinant of the tetrad h^a_μ and D_μ is the gauge covariant derivative

$$D_\mu = \partial_\mu + \sum \omega_\mu,$$

where the sum is taken over all Lorentz indices.

In spaces whose metric tensor depends on spinor variables, an analogous method can be applied, but instead of one connection we have three connections:

$$\omega^{(*)}_\mu(x,\xi,\overline{\xi}), \qquad \theta^{(*)}_a(x,\xi,\overline{\xi}), \qquad \overline{\theta}^{(*)\alpha}(x,\xi,\overline{\xi}).$$

So we choose a Lagrangian density of the form (1.2) from which four equations are obtained. The analogous gauge covariant derivatives of D_μ appear naturally as

$$D^{(*)}_\mu = \partial^{(*)}_\mu + \sum \omega^{(*)}_\mu, \tag{1.4a}$$
$$D^{(*)}_\alpha = \partial^{(*)}_\alpha + \sum \theta^{(*)}_\alpha, \tag{1.4b}$$
$$D^{(*)\alpha} = \partial^{(*)\alpha} + \sum \overline{\theta}^{(*)\alpha}. \tag{1.4c}$$

Transformation laws of the connection coefficients $\omega^{(*)}_{ab\lambda}(x,\xi,\overline{\xi}), \theta^{(*)}_{ab\alpha}(x,\xi,\overline{\xi})$ and $\overline{\theta}^{(*)\alpha}_{ab}(x,\xi,\overline{\xi})$ under local Lorentz transformations are the expected transformation laws for the gauge potentials [10]

$$\omega^{(*)}_{ab\lambda} = L^c_a L^d_b \omega^{(*)}_{cd\lambda} + \frac{\partial^{(*)} L^c_\alpha}{\partial x^\lambda} L_{bc}, \tag{1.5a}$$

$$\overline{\theta}^{(*)'a}_{ab} = \left[L^c_a L^d_b \overline{\theta}^{(*)\beta}_{cd} + \frac{\partial^{(*)} L^c_\alpha}{\partial \xi_\beta} L_{bc} \right] (\Lambda^{-1})^\alpha_\beta, \tag{1.5b}$$

$$\theta^{(*)'}_{ab\alpha} = \Lambda^\beta_\alpha \left[L^c_a L^d_b \theta^{(*)}_{cd\beta} + \frac{\partial^{(*)} L^c_\alpha}{\partial \overline{\xi}_\beta} L_{bc} \right]. \tag{1.5c}$$

The matrices L and Λ belong to the vector and spinor representations of the Lorentz group, respectively.

2. Derivation of the field equations

The field equations will be the Euler–Lagrange equations for a given Lagrangian. Later we shall postulate the explicit form of the Lagrangian density

$$L(\Psi^{(A)}, \partial^{(*)}_M \psi^{(A)}). \tag{2.1}$$

But first we observe that the metric tensor $g_{\mu\nu}$ and the tetrad h^a_μ are related by (cf. [10])

$$g_{\mu\nu}(x,\xi,\overline{\xi}) = h^a_\mu h^b_\nu \eta_{ab}, \tag{2.2a}$$

$$g^{\mu\nu}(x,\xi,\overline{\xi}) = h^\mu_a h^\nu_b \eta^{ab}, \tag{2.2b}$$

where η_{ab} is the Minkowski metric tensor and it is of the form diag $(+1,-1,-1,-1)$. From the relations [16]:

$$g = -h^2, \tag{2.3a}$$

$$dg = g g^{\mu\nu} dg_{\mu\nu}, \tag{2.3b}$$

and using (2.2), we get

$$\frac{\partial h}{\partial h^\mu_a} = -\frac{1}{2} h h^a_\mu, \tag{2.4}$$

where $g = \det(g_{\mu\nu})$.

Now we postulate the Lagrangian density in the form

$$L = h(R + P + Q + S), \tag{2.5}$$

where R, P, Q, S are the scalar curvatures obtained by contraction of the spin curvature tensors:

$$R = h^\mu_a h^\nu_b R^{ab}_{\mu\nu}, \qquad P = h^\mu_a h^\nu_b P^a_{c\mu\alpha} \overline{P}^{bc\alpha}_\nu, \qquad Q = Q_{ab\beta\alpha} \widetilde{Q}^{ab\beta\alpha}, \qquad S = S^\alpha_{ab\beta} S^{ab\beta}_\alpha. \tag{2.6}$$

The spin curvature tensors are given by [10]

$$
\begin{aligned}
P^{ab}_{\lambda\alpha} &= \frac{\partial^{(*)}\omega^{(*)ab}_{\lambda}}{\partial\overline{\xi}^{\alpha}} - \frac{\partial^{(*)}\theta^{(*)ab}_{\alpha}}{\partial x^{\lambda}} + \omega^{(*)a}_{c\lambda}\theta^{(*)cb}_{\alpha} - \theta^{(*)a}_{c\alpha}\omega^{(*)cb}_{\lambda} - (\overline{\theta}^{ab\beta}E_{\beta\lambda\alpha} + F^{\beta}_{\lambda\alpha}\theta^{ab}_{\beta}),\\
\overline{P}^{ab\alpha}_{\lambda} &= \frac{\partial^{(*)}\omega^{(*)ab}_{\lambda}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\overline{\theta}^{(*)ab\alpha}}{\partial x^{\lambda}} + \omega^{(*)a}_{c\lambda}\overline{\theta}^{(*)a\alpha b} - \overline{\theta}^{(*)a\alpha}_{c}\omega^{(*)cb}_{\lambda} - (\overline{\theta}^{ab\beta}\widetilde{F}^{\alpha}_{\beta\lambda} + \widetilde{E}^{\beta\alpha}_{\lambda}\theta^{ab}_{\beta}),\\
S^{ab\alpha}_{\beta} &= \frac{\partial^{(*)}\omega^{(*)ab}_{\beta}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\overline{\theta}^{(*)ab\alpha}}{\partial\overline{\xi}^{\beta}} + \theta^{(*)a}_{c\beta}\overline{\theta}^{(*)c\alpha b} - \overline{\theta}^{(*)a\alpha}_{c}\theta^{(*)cb}_{\beta} - (\theta^{ab}_{\gamma}G^{\alpha}_{\gamma\beta} + H^{\gamma\alpha}_{\beta}\theta^{ab}_{\gamma}),\\
R^{ab}_{\mu\nu} &= \frac{\partial^{(*)}\omega^{(*)ab}_{\mu}}{\partial x^{\nu}} - \frac{\partial^{(*)}\theta^{(*)ab}_{\nu}}{\partial x^{\mu}} + \omega^{(*)ac}_{\mu}\omega^{(*)b}_{c\nu} - \omega^{(*)ac}_{\nu}\omega^{(*)b}_{c\mu} - (\overline{\theta}^{ab\beta}A_{\beta\mu\nu} + \overline{A}^{\beta}_{\mu\nu}\theta^{ab}_{\beta}),\\
\widetilde{Q}^{\beta\alpha}_{ab} &= \frac{\partial^{(*)}\overline{\theta}^{(*)\beta}_{ab}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\overline{\theta}^{(*)\alpha}_{ab}}{\partial\xi_{\beta}} + \overline{\theta}^{(*)\beta}_{ac}\theta^{(*)ca}_{b} - \overline{\theta}^{(*)\alpha}_{ac}\overline{\theta}^{(*)c\beta}_{b} - (\overline{\theta}^{\gamma}_{ab}\widetilde{K}^{\beta\alpha}_{\gamma} + \widetilde{J}^{\gamma\beta\alpha}\theta_{ab\gamma}),\\
Q^{ab}_{\beta\alpha} &= \frac{\partial^{(*)}\theta^{(*)ab}_{\beta}}{\partial\overline{\xi}^{\alpha}} - \frac{\partial^{(*)}\theta^{(*)ab}_{\alpha}}{\partial x^{\lambda}} + \theta^{(*)a}_{c\beta}\theta^{(*)c}_{b\alpha} - \theta^{(*)a}_{c\alpha}\theta^{(*)cb}_{\beta} - (\theta^{ab\gamma}J_{\gamma\beta\alpha} + K^{\gamma}_{\beta\alpha}\theta^{ab}_{\gamma}),
\end{aligned}
\tag{2.7}
$$

where:

$$
\begin{aligned}
A_{\beta\mu\nu} &= \frac{\partial^{(*)}N_{\beta\mu}}{\partial x^{\nu}} - \frac{\partial^{(*)}N_{\beta\nu}}{\partial x^{\mu}}, & \overline{A}^{\beta}_{\mu\nu} &= \frac{\partial^{(*)}\overline{N}^{\beta}_{\mu}}{\partial x^{\nu}} - \frac{\partial^{(*)}\overline{N}^{\beta}_{\nu}}{\partial x^{\mu}},\\
E_{\beta\lambda\alpha} &= \frac{\partial^{(*)}N_{\beta\lambda}}{\partial\overline{\xi}^{\alpha}} - \frac{\partial^{(*)}\eta^{0}_{\beta\alpha}}{\partial x^{\lambda}}, & F^{\beta}_{\lambda\alpha} &= \frac{\partial^{(*)}\overline{N}^{\beta}_{\lambda}}{\partial\overline{\xi}^{\alpha}} - \frac{\partial^{(*)}\eta^{\beta}_{0\alpha}}{\partial x^{\lambda}},\\
\widetilde{F}^{\alpha}_{\beta\lambda} &= \frac{\partial^{(*)}N_{\beta\lambda}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\widetilde{\eta}^{0\alpha}_{\beta}}{\partial x^{\lambda}}, & \widetilde{E}^{\beta\alpha}_{\lambda} &= \frac{\partial^{(*)}\overline{N}^{\beta}_{\lambda}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\widetilde{\eta}^{\beta\alpha}_{0}}{\partial x^{\lambda}},\\
G^{\alpha}_{\gamma\beta} &= \frac{\partial^{(*)}\eta^{0}_{\gamma\beta}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\widetilde{\eta}^{0\alpha}_{\gamma}}{\partial\overline{\xi}^{\beta}}, & H^{\gamma\alpha}_{\beta} &= \frac{\partial^{(*)}\eta^{\gamma}_{0\beta}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\widetilde{\eta}^{\gamma\alpha}_{0}}{\partial\overline{\xi}^{\beta}},\\
J_{\gamma\beta\alpha} &= \frac{\partial^{(*)}\eta^{0}_{\gamma\beta}}{\partial\overline{\xi}^{\alpha}} - \frac{\partial^{(*)}\eta^{0}_{\gamma\alpha}}{\partial\overline{\xi}^{\beta}}, & K^{\gamma}_{\beta\alpha} &= \frac{\partial^{(*)}\eta^{\gamma}_{0\beta}}{\partial\overline{\xi}^{\alpha}} - \frac{\partial^{(*)}\eta^{\gamma}_{0\alpha}}{\partial\overline{\xi}^{\beta}},\\
\widetilde{K}^{\beta\alpha}_{\gamma} &= \frac{\partial^{(*)}\eta^{0\beta}_{\gamma}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\widetilde{\eta}^{0\alpha}_{\gamma}}{\partial\xi_{\beta}}, & \widetilde{J}^{\gamma\beta\alpha} &= \frac{\partial^{(*)}\widetilde{\eta}^{\gamma\beta}_{0}}{\partial\xi_{\alpha}} - \frac{\partial^{(*)}\widetilde{\eta}^{\gamma\alpha}_{0}}{\partial\xi_{\beta}}.
\end{aligned}
$$

Lagrangian (2.5) is the only possible scalar that can be made from the curvature tensors (2.7) and it must be the sum of the first-order quantity R and the second-order quantities P, Q and S. The mixing of the quantities of different order is not impossible. It is known that the Einstein–Maxwell Lagrangian is the sum of the first-order quantity R and the second-order quantity $F_{\mu\nu}F^{\mu\nu}$. So, our Lagrangian (2.5) is physically acceptable.

The Euler–Lagrange equations for the objects

$$\Psi^{(A)} = \{h^{\mu}, \omega_{\mu}^{(*)}, \theta_{\alpha}^{(*)}, \overline{\theta}^{(*)\alpha}\}$$

are of the form

$$\partial_M^{(*)}\left(\frac{\partial L}{\partial(\partial_M^{(*)}\Psi^{(A)})}\right) - \frac{\partial L}{\partial \Psi^{(A)}} = 0, \tag{2.8}$$

where $\partial_M^{(*)}$ was defined in (1.1). From the variation of L with respect to the tetrad we have

$$\frac{\partial L}{\partial h_b^{\nu}} = 0. \tag{2.9}$$

Taking into account (2.3) and (2.4) we get the equation

$$H_{\nu}^{b} - \tfrac{1}{2}h_{\nu}^{b} = 0, \tag{2.10}$$

where

$$H_{\nu}^{b} = R_{\nu}^{b} + P_{\nu}^{b} = h_a^{\mu}R_{\mu\nu}^{ab} + h_a^{\mu}P_{c\mu\alpha}^{a}\overline{P}_{\nu}^{bc\alpha}, \tag{2.11}$$

and

$$H = R + P. \tag{2.12}$$

From the variation of L with respect to $\omega_{\mu}^{(*)}$ we get

$$\partial_{\mu}^{(*)}\frac{\partial L}{\partial(\partial_{\mu}^{(*)}\omega_{\nu}^{(*)ab})} + \partial^{(*)\alpha}\frac{\partial L}{\partial(\partial^{(*)\alpha}\omega_{\nu}^{(*)ab})} + \partial_{\alpha}^{(*)}\frac{\partial L}{\partial(\partial_{\alpha}^{(*)}\omega_{\nu}^{(*)ab})} - \frac{\partial L}{\partial\omega_{\nu}^{(*)ab}} = 0. \tag{2.13}$$

The spin-connection coefficients $\omega_{\nu}^{(*)ab}$ are contained in R and P:

$$h(R+P) = hh_a^{\mu}h_b^{\nu}(R_{\mu\nu}^{ab} + P_{c\mu\alpha}^{a}\overline{P}_{\nu}^{bc\alpha}).$$

From relation (2.13) we get the following variation of the term hR with respect to $\omega_{\mu}^{(*)}$:

$$\partial_{\mu}^{(*)}\frac{\partial(hR)}{\partial(\partial_{\mu}^{(*)}\omega_{\nu}^{(*)ab})} + \partial^{(*)\alpha}\frac{\partial(hR)}{\partial(\partial^{(*)\alpha}\omega_{\nu}^{(*)ab})} + \partial_{\alpha}^{(*)}\frac{\partial(hR)}{\partial(\partial_{\alpha}^{(*)}\omega_{\nu}^{(*)ab})} - \frac{\partial(hR)}{\partial\omega_{\nu}^{(*)ab}}. \tag{2.14}$$

By a direct calculation, the first term of (2.14) can be written as

$$\partial_{\mu}^{(*)}[h(h_a^{\nu}h_b^{\mu} - h_a^{\mu}h_b^{\nu})].$$

The second and the third terms of (2.14) are equal to zero. The fourth term equals

$$h(h_c^{\nu}h_b^{\mu} - h_b^{\nu}h_c^{\mu})\omega_{a\mu}^{(*)c} + h(h_c^{\nu}h_a^{\mu} - h_a^{\nu}h_c^{\mu})\omega_{b\mu}^{(*)c}. \tag{2.15}$$

Consequently, the first and the fourth terms can be rewritten as

$$D_{\mu}^{(*)}[h(h_a^{\nu}h_b^{\mu} - h_b^{\nu}h_a^{\mu})], \tag{2.16}$$

where we have used the gauge covariant derivative $D_\mu^{(*)}$ from (1.4). Contribution from the P-part is equal to

$$\partial_\mu^{(*)}\frac{\partial(hP)}{\partial(\partial_\mu^{(*)}\omega_\nu^{(*)ab})}+\partial^{(*)\alpha}\frac{\partial(hP)}{\partial(\partial^{(*)\alpha}\omega_\nu^{(*)ab})}+\partial_\alpha^{(*)}\frac{\partial(hP)}{\partial(\partial_\alpha^{(*)}\omega_\nu^{(*)ab})}-\frac{\partial(hR)}{\partial\omega_\nu^{(*)ab}}. \tag{2.17}$$

The first term of (2.17) is equal to zero. The second and the third terms can be written as

$$\partial^{(*)\alpha}(hh_c^\mu h_a^\nu P^c_{b\mu\alpha}), \tag{2.18}$$

$$\partial_\alpha^{(*)}(hh_c^\mu h_a^\nu \overline{P}^{c\alpha}_{b\mu}), \tag{2.19}$$

respectively. The fourth term may be written as

$$hh_a^\nu h_l^\mu \theta^{(*)}_{bc\alpha}\overline{P}^{cl\alpha}_\mu - hh_l^\nu h_k^\mu \theta^{(*)l}_{a\alpha}\overline{P}^{k\alpha}_{b\mu} - hh_l^\mu h_a^\nu P^l_{c\mu\alpha}\overline{\theta}^{(*)c\alpha}_b - hh_k^\nu h_l^\mu P^l_{b\mu\alpha}\overline{\theta}^{(*)k\alpha}_a. \tag{2.20}$$

The sum of (2.18), (2.19) and (2.20) is equal to

$$D_\alpha^{(*)}(hh_a^\nu h_l^\mu \overline{P}^{l\alpha}_{b\mu}) + D^{(*)\alpha}(hh_a^\nu h_l^\mu P^{l\alpha}_{b\mu}). \tag{2.21}$$

So, (2.17) is written in the form

$$D_\mu^{(*)}[h(h_a^\nu h_b^\mu - h_b^\nu h_a^\mu)] + D_\alpha^{(*)}(hh_a^\nu h_l^\mu \overline{P}^{l\alpha}_{b\mu}) + D^{(*)\alpha}(hh_a^\nu h_l^\mu \overline{P}^{l\alpha}_{b\mu}) = 0. \tag{2.22}$$

Taking the variation of L with respect to $\theta_\alpha^{(*)}$ we have contributions from $(P+Q+S)$. The field equation is

$$\partial_\mu^{(*)}\frac{\partial(hL)}{\partial(\partial_\mu^{(*)}\theta_\nu^{(*)ab})}+\partial^{(*)\alpha}\frac{\partial(hL)}{\partial(\partial^{(*)\alpha}\theta_\nu^{(*)ab})}+\partial_\alpha^{(*)}\frac{\partial(hL)}{\partial(\partial_\alpha^{(*)}\theta_\nu^{(*)ab})}-\frac{\partial(hL)}{\partial\theta_\nu^{(*)ab}}=0. \tag{2.23}$$

We proceed in the same way as before. The contribution from the hP term is

$$-D_\mu^{(*)}(hh_a^\mu h_c^\nu \overline{P}^{ca}_{b\nu}). \tag{2.24}$$

The contribution from the hQ term gives

$$D_\beta^{(*)}(2h\widetilde{Q}^{[\alpha\beta]}_{ab}). \tag{2.25}$$

Similarly, the hS term yields

$$2D^{(*)\beta}(hS^\alpha_{ab\beta}). \tag{2.26}$$

So, the third equation is written in the form

$$D_\mu^{(*)}(hh_a^\mu h_c^\nu \overline{P}^{ca}_{b\nu}) - D_\beta^{(*)}(2h\widetilde{Q}^{[\alpha\beta]}_{ab}) - 2D^{(*)\beta}(hS^\alpha_{ab\beta}) = 0. \tag{2.27}$$

Finally, the variation with respect to $\theta^{(*)\alpha}$ yields the equation "conjugate" to (2.27):

$$D_\mu^{(*)}(hh_c^\mu h_a^\nu P^c_{b\nu\alpha}) - D^{(*)\beta}(2hQ_{ab[\alpha\beta]}) - 2D_\beta^{(*)}(hS^\beta_{ab\alpha}) = 0. \tag{2.28}$$

3. Generalized conformally flat spaces of the $g_{\mu\nu}(x,\xi,\overline{\xi}) = e^{2\sigma(x,\xi,\overline{\xi})}\eta_{\mu\nu}$ metric

In this section we study the form of the spin-connection coefficients, spin-curvature tensors, and the field equations for generalized conformally flat spaces (GCFS) $(M, g_{\mu\nu}(x,\xi,\overline{\xi}) = e^{2\sigma(x,\xi,\overline{\xi})}\eta_{\mu\nu})$, where $\eta_{\mu\nu}$ represents the Lorentz metric tensor $\eta_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$, and $\xi,\overline{\xi}$ represent the internal variables. The case of conformally related metrics of the Riemannian and the generalized Lagrange spaces has been extensively studied in [8, 9]. It is remarkable that in the above mentioned GCFS spaces, some spin connection and spin-curvature tensors vanish.

As pointed out in [10], the absolute differential DV^μ of a vector field $V^\mu(x,\xi,\overline{\xi})$ is expressed in terms of the coefficients

$$\{\Gamma^\mu_{\nu\lambda}, \overline{C}^{\mu\alpha}_\nu, C^\mu_{\nu\alpha}\}. \tag{3.1}$$

Considering the absolute differentials of the spinor variables $\xi_\alpha, \overline{\xi}^\alpha$:

$$D\xi_\alpha = d\xi_\alpha - N_{\alpha\lambda}dx^\lambda - \widetilde{\eta}^{0\beta}_\alpha D\xi_\beta - D\overline{\xi}^\beta \eta^0_{\alpha\beta},$$
$$D\overline{\xi}^\alpha = d\overline{\xi}^\alpha - \overline{N}^\alpha_\lambda dx^\lambda - \widetilde{\eta}^{\alpha\beta}_0 D\xi_\beta - D\overline{\xi}^\beta \eta^\alpha_{0\beta},$$

which depend on the nonlinear connections:

$$\{N_{\alpha\lambda}, \overline{N}^\alpha_\lambda, \widetilde{\eta}^{0\beta}_\alpha, \widetilde{\eta}^{\alpha\beta}_0 \eta^0_{\alpha\beta}, \eta^\alpha_{0\beta}\}, \tag{3.2}$$

and expressing DV^μ in terms of $dx^\lambda, D\xi_\alpha, D\overline{\xi}^\alpha$, we obtain the connection coefficients

$$\{\Gamma^{(*)\mu}_{\nu\alpha}, \overline{C}^{(*)\mu\alpha}_\nu, C^{(*)\mu}_{\nu\alpha}\} \tag{3.3}$$

related to the coefficients (3.1) via the non-linear connections (3.2) [10].

By imposing the postulates of the length preservation for the parallel vector fields and symmetry of the derived coefficients

$$\{\Gamma^{(*)}_{\nu\mu\lambda}, \overline{C}^\alpha_{\nu\mu}, C_{\nu\mu\alpha}\} \tag{3.4}$$

in the first two tensor indices, we have the relations:

$$\Gamma^{(*)}_{\nu\mu\lambda} = \frac{1}{2}\left(\frac{\partial^{(*)}g_{\mu\{\nu}}{\partial x^{\lambda\}}} - \frac{\partial^{(*)}g_{\nu\lambda}}{\partial x^\mu}\right),$$
$$\overline{C}^\alpha_{\nu\mu} = \frac{1}{2}\frac{\partial g_{\nu\mu}}{\partial \xi_\alpha}, \qquad C_{\nu\mu\alpha} = \frac{1}{2}\frac{\partial g_{\nu\mu}}{\partial \overline{\xi}^\alpha}, \tag{3.5}$$

where $\tau_{\{\mu\nu\}} = \tau_{\mu\nu} + \tau_{\nu\mu}$.

THEOREM 3.1. *For the GCFS spaces we infer the following*:
(a) *The coefficients* (3.4) *have the explicit form*

$$\Gamma^*_{\nu\mu\lambda} = e^{2\sigma}(\eta_{\mu\{\nu}\sigma^*_{\lambda\}} - \eta_{\nu\lambda}\sigma^*_\mu), \qquad \overline{C}^{\mu\alpha}_\nu = \delta^\mu_\nu\sigma^\alpha, \qquad C^\mu_{\nu\alpha} = \overline{\sigma}_\alpha\delta^\mu_\nu, \tag{3.6}$$

where $\sigma^\alpha = \partial\sigma/\partial\xi_\alpha, \overline{\sigma}_\alpha = \partial\sigma/\partial\overline{\xi}^\alpha, \sigma^*_\lambda = \partial^*\sigma/\partial x^\lambda$ *are the derivation operators of scalar fields involving the coefficients* (3.2).

(b) *The following relations hold:*

$$\begin{aligned}
\Gamma^{\mu}_{\nu\lambda} &= \Gamma^{*\mu}_{\nu\lambda} - \delta^{\mu}_{\nu}\sigma^{\alpha}N_{\alpha\lambda} - \delta^{\mu}_{\nu}\overline{\sigma}_{\alpha}\overline{N}^{\alpha}_{\lambda}, \\
\overline{C}^{*\mu\alpha}_{\nu} &= \overline{C}^{\mu\alpha}_{\nu} + \delta^{\mu}_{\nu}\sigma^{\beta}\widetilde{\eta}^{0\alpha}_{\beta} + \delta^{\mu}_{\nu}\overline{\sigma}_{\beta}\widetilde{\eta}^{\beta\alpha}_{0}, \\
C^{*\mu}_{\nu\alpha} &= C^{\mu}_{\nu\alpha} + \delta^{\mu}_{\nu}\sigma^{\beta}\eta^{0}_{\beta\alpha} + \delta^{\mu}_{\nu}\eta^{\beta}_{0\alpha}.
\end{aligned} \tag{3.7}$$

Proof: Computational, using the consequences (3.5) of the above postulates and identifying the absolute differentials expressed in terms of (3.1) and (3.3). □

Considering the absolute differentials of a Dirac spinor field $\psi(x,\xi,\overline{\xi})$ and of its adjoint $\overline{\psi}(x,\xi,\overline{\xi})$ we have the coefficients

$$\{\Gamma^{\beta}_{\gamma\lambda}, \widetilde{C}^{\beta\alpha}_{\gamma}, C^{\beta}_{\gamma\alpha}\}. \tag{3.8}$$

Expressing $D\psi$ and $D\overline{\psi}$ in terms of $dx^{\lambda}, D\xi_{\alpha}, D\overline{\xi}^{\alpha}$, we are led to the spin-connection coefficients I:

$$\{\Gamma^{*\beta}_{\gamma\lambda}, \widetilde{C}^{*\beta\alpha}_{\gamma}, C^{*\beta}_{\gamma\alpha}\} \tag{3.8$'$}$$

connected to (3.8) [10]. In a similar manner, the absolute differential of a Lorentz vector $V^{a}(x,\xi,\overline{\xi})$ produces the coefficients

$$\{\omega_{ba\lambda}, \overline{\theta}^{\alpha}_{ba}, \theta_{ba\alpha}\}, \tag{3.9}$$

where the raising and lowering of the indices $a, b, ... = 1, ..., 4$ are performed by means of η_{ab}, and also the spin-connection coefficients II:

$$\{\omega^{*}_{ba\lambda}, \overline{\theta}^{*\alpha}_{ba}, \theta^{*}_{ba\alpha}\} \tag{3.10}$$

related to the coefficients (3.9) [(3.13)/10]. Similarly to the previous work of Takano and Ono [10], we shall postulate the invariance of length of the parallel Lorentz vector fields, and the vanishing of the absolute differentials and covariant derivatives of the tetrads h^{μ}_{α}, which involve the connection coefficients (3.3) and (3.10).

In the GCFS, the tetrads are given by $h^{a}_{\mu}(x,\xi,\overline{\xi}) = e^{\sigma(x,\xi,\overline{\xi})}\delta^{a}_{\mu}$ and lead to the dual entities $h^{\mu}_{a}(x,\xi,\overline{\xi}) = e^{-\sigma(x,\xi,\overline{\xi})}\delta^{\mu}_{a}$. In general, the above postulates produce the relations:

$$\begin{aligned}
\omega_{ab\lambda} &= \left(\frac{\partial h^{\mu}_{a}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\nu\lambda}h^{\nu}_{a}\right)h_{\mu b}, \\
\overline{\theta}^{\alpha}_{ab} &= \left(\frac{\partial h^{\mu}_{a}}{\partial \xi_{\alpha}} + \overline{C}^{\mu\alpha}_{\nu}h^{\nu}_{a}\right)h_{\mu b}, \\
\theta_{ab\alpha} &= \left(\frac{\partial h^{\mu}_{a}}{\partial \overline{\xi}^{\alpha}} + C^{\mu}_{\nu\alpha}h^{\nu}_{a}\right)h_{\mu b},
\end{aligned} \tag{3.11}$$

$$\omega^{*}_{ab\lambda} = \left(\frac{\partial^{*} h^{\mu}_{a}}{\partial x^{\lambda}} + \Gamma^{*\mu}_{\nu\lambda}h^{\nu}_{a}\right)h_{\mu b}. \tag{3.11$'$}$$

For the GCFS case we are led to

THEOREM 3.2. *The spin-connection coefficients* (II) *and the coefficients* (3.9) *are subject to*

$$\omega_{ba\lambda} = h_{\mu a}\Gamma^{\mu}_{b\lambda} - \sigma_\lambda \eta_{ba}, \qquad \overline{\theta}^{\alpha}_{ab} = 0, \qquad \theta_{ab\alpha} = 0, \tag{3.12}$$

$$\omega^*_{ba\lambda} = \eta_{\lambda(a}\sigma^*_{b)}, \qquad \overline{\theta}^{*\alpha}_{ab} = 0, \qquad \theta^*_{ab\alpha} = 0, \tag{3.12'}$$

$$\omega^*_{ba\lambda} = \omega_{ba\lambda}, \tag{3.12''}$$

where $h_{\mu a} = e^{\sigma}\eta_{\mu a}$ *and* $T_{(ab)} = T_{ab} - T_{ba}$.

Proof: Relations (3.11) imply (3.12); (3.11′) and $\omega^*_{ba\lambda} = \omega_{ba\lambda} + \theta^{\beta}_{ba}N_{\beta\lambda} + \overline{N}^{\beta}_{\lambda}\theta_{ba\beta}$ produce (3.12″) and

$$\overline{\theta}^{*\alpha}_{ba} = \overline{\theta}^{\alpha}_{ba} + \overline{\theta}^{\beta}_{ba}\widetilde{\eta}^{0\alpha}_{\beta} + \widetilde{\eta}^{\beta\alpha}_{0}\overline{\theta}_{ba\beta}, \qquad \theta^*_{ba\alpha} = \theta_{ba\alpha} + \overline{\theta}^{\beta}_{ba}\eta^{0}_{\beta\alpha} + \eta^{\beta}_{0\alpha}\theta_{ba\beta}.$$

From (3.12(2)) and (3.12(3)) we infer (3.12′(2)) and (3.12′(3)). □

The connections (3.3) and (3.8′) give rise to 8 curvature tensors as described in (5.2) of [10]. But also the spin-connections (II) connected to (3.3) lead to six spin-curvature tensors (2.7)

$$\{R_{ab\lambda\mu}, P_{ab\lambda\alpha}, \overline{P}^{\alpha}_{ab\lambda}, S^{\alpha}_{ab\beta}, Q_{ab\beta\alpha}, \widetilde{Q}^{\beta\alpha}_{ab}\}. \tag{3.13}$$

Taking into account Theorems 3.1 and 3.2 we can express these tensors as follows.

THEOREM 3.3. *In the GCFS spaces the spin-curvature tensors are given by*

$$R_{ab\lambda\mu} = \eta_{\lambda(b}\sigma^*_{\mu a)} + \eta_{\mu(a}\sigma^*_{\lambda b)} + \eta_{\mu(b}\sigma^*_{\lambda}\sigma^*_{a)} + \eta_{\lambda(a}\sigma^*_{\mu}\sigma^*_{b)} + \eta_{(\mu a}\eta_{\lambda)b}\eta^{cd}\sigma^*_{c}\sigma^*_{d}, \tag{3.14}$$

$$P_{ab\lambda\alpha} = \eta_{\lambda(b}\sigma^*_{\alpha a)}, \qquad \overline{P}^{\alpha}_{ab\lambda} = \eta_{\lambda(b}\sigma^{*\alpha}_{a)},$$
$$S^{\alpha}_{ab\beta} = 0, \qquad Q_{ab\beta\alpha} = 0, \qquad \widetilde{Q}^{\beta\alpha}_{ab} = 0, \tag{3.14'}$$

where $\sigma^*_{xy} = \partial^{*2}\sigma/\partial x^x \partial x^y$; $x, y = \{\lambda, \alpha, a\}$ and $\eta_{\lambda(b}\sigma^*_{\mu a)} = \eta_{\lambda b}\sigma^{*\alpha}_{\mu a} - \eta_{\lambda a}\sigma^*_{\mu b}$.

Proof: Relations (3.14′) are directly implied by (3.12′(2)) and (3.12′(3)). (3.6(1)) leads to (3.14(1)) after a straightforward calculation. Also, using Theorem 3.2, we infer that

$$P_{ab\lambda\alpha} = \omega^*_{ab\lambda,\alpha}, \qquad \overline{P}_{ab\lambda\alpha} = \omega^*_{ab\lambda,\alpha}, \tag{3.15}$$

where

$$\omega^*_{ab\lambda,\alpha} = \frac{\partial^*\omega_{ab\lambda}}{\partial\overline{\xi}^{\alpha}}, \qquad \omega^*_{ab\lambda,\alpha} = \frac{\partial^*\omega_{ab\lambda}}{\partial\xi_{\alpha}}.$$

Then (3.12′(1)) leads to (3.14(2)) and (3.14(3)). □

As a consequence of this theorem we state the following

COROLLARY 3.4. *In the GCFS space* $(M, g_{\mu\nu})$ *the Ricci tensor fields have the form*

$$R^d_{\mu} = e^{-\sigma}(2\eta^{bd}\sigma^*_{\mu}\sigma^*_{b} - 2\eta^{bd}\sigma^*_{\mu b} - \delta^d_{\mu}\eta^{a\lambda}\sigma^*_{\lambda a} - 2\delta^d_{\mu}\eta^{ef}\sigma^*_{c}\sigma^*_{f}),$$
$$P^b_{\nu} = -3e^{-\sigma}(\eta^{bc}\sigma^*_{\alpha c}\sigma^{*\alpha}_{\nu} - \sigma^*_{\alpha\nu}\sigma^{*\alpha}_{e}\eta^{eb}). \tag{3.16}$$

Proof: Using Theorem 3.3 and $R^d_\mu = h^\lambda_c R^{cd}_{\lambda\mu}$, $P^b_\nu = h^\mu_a P^a_{c\mu\alpha}\overline{P}^{bc\alpha}_{\ \ \ \nu}$ we obtain relations (3.16). □

Remark 3.5: It follows that the scalar curvature takes the form

$$R = R^d_\mu h^\mu_d = -6e^{-2\sigma}(\eta^{bd}\sigma^*_{db} + \eta^{ef}\sigma^*_e\sigma^*_f). \tag{3.17}$$

Furthermore, it can be easily seen that

$$P = P^b_\nu h^\nu_b \equiv 0.$$

As we have previously remarked, the scalar curvature fields $Q = Q_{ab\beta\alpha}\widetilde{Q}^{ab\beta\alpha}$ and $S = S_{ab\alpha\beta}S^{ab\alpha\beta}$ vanish identically. Then the employed Lagrangian density (2.5)

$$L = h(R + P + Q + S), \qquad \det(g_{\mu\nu}) = -h^2,$$

reduces to $L = e^\sigma(R + P)$ and depends on the fields $\varphi \in \{h^b_\nu, \omega^*_{ab\lambda}, \theta^*_{ab\alpha}, \overline{\theta}^{*\alpha}_{ab}\}$. The Euler–Lagrange equations

$$\partial^*_M\left(\frac{\partial L}{\partial(\partial^*_M\varphi)}\right) - \frac{\partial L}{\partial\varphi} = 0 \tag{3.18}$$

for these fields produce the field equations (2.10), (2.22), (2.27) and (2.28).

We shall obtain their form for the GCFS as follows.

THEOREM 3.6. *The field equations for the GCFS are*

$$\delta^d_\mu\eta^{ef}(2\sigma^*_{ef} - \sigma^*_e\sigma^*_f) + 2\eta^{bd}(\sigma^*_{\mu b} - \sigma^*_\mu\sigma^*_b) + 3\eta^{ed}\sigma^*_{\alpha\mu}\sigma^{*\alpha}_e - 3\eta^{dc}\sigma^*_{\alpha c}\sigma^{*\alpha}_\mu = 0, \tag{F1}$$

$$\sigma^*_{(b}\delta^\nu_{a)} - 3\sigma^*_\alpha\sigma^{*\alpha}_{(b}\delta^\nu_{a)} - 3\delta^\nu_a\sigma^{*\alpha}_{\alpha b} = 0, \tag{F2}$$

$$2\sigma^*_a\sigma^*_{\alpha b} + \sigma^*_{a\alpha b} = 0, \tag{F3}$$

$$2\sigma^*_\mu\eta^{\mu d}\eta_{\alpha b}\sigma^*_{\alpha d} - 2\sigma^*_a\sigma^*_{\alpha b} + \eta^{\mu d}\eta_{ab}\sigma^*_{\mu\alpha d} - \sigma^*_{a\alpha b} = 0, \tag{F4}$$

where we have put $\sigma^{*\beta}_{\alpha b} = \partial^{*3}\sigma/\partial\xi_\beta\partial\overline{\xi}^\alpha\partial x^b$.

Proof: By virtue of relations (2.10) and (2.11), and using Corollary 3.4 and Remark 3.5, we get (F1).

Considering Theorem 3.2 we infer that $D^*_\alpha = \partial^*_\alpha$ and $D^{*\alpha} = \partial^{*\alpha}$. Also from $\omega^{(*)}_{\alpha b\lambda} = \omega_{\alpha b\lambda} = -\omega_{ba\lambda}$, we derive $D^*_\mu = \partial^*_\mu$. Taking into account (2.22), we obtain relation (F2) by a straightforward computation. Also, by means of Theorem 3.3 and noticing that $\overline{P}^{\mu\alpha}_{b\mu} = -3\sigma^{*\alpha}_b$, after substituting to (2.27), we infer (F3). Finally, from (2.28) we derive (F4).

4. Geodesics and geodesic deviation

We shall now give the form of geodesics in spaces with the $g_{\mu\nu}(x,\xi,\overline{\xi})$ metric.

A curve c in a space $(M, g_{\mu\nu}(x,\xi,\overline{\xi}))$ is defined as a smooth mapping c: $I \to U \subset M$: $t \to (x(t), \xi(t), \overline{\xi}(t))$, where U is an open set of M and t is an arbitrary parameter.

DEFINITION 4.1. A curve c is a *geodesic* if it satisfies the set of equations:

$$\frac{D\dot{x}^{\mu}}{ds} \equiv \frac{d^2x^{\mu}}{ds^2} + \dot{x}^{\nu}(\Gamma^{\mu}_{\nu\lambda x}\dot{x}^{\lambda} + \overline{C}^{\mu\alpha}_{\nu}\dot{\xi}_{\alpha} + C^{\mu}_{\nu\alpha}\dot{\overline{\xi}}^{\alpha}) = 0, \tag{4.1a}$$

$$\frac{D^2\xi_{\alpha}}{ds^2} \equiv \frac{D}{ds}[\dot{\xi}_{\alpha} - \xi_{\gamma}(\Gamma^{\gamma}_{\alpha\lambda}\dot{x}^{\lambda} + \widetilde{C}^{\gamma\beta}_{\alpha}\dot{\xi}_{\beta} + C^{\gamma}_{\alpha\beta}\dot{\overline{\xi}}^{\beta})] = 0, \tag{4.1b}$$

$$\frac{D^2\overline{\xi}^{\alpha}}{ds^2} \equiv \frac{D}{ds}[\dot{\overline{\xi}}^{\alpha} + \overline{\xi}^{\gamma}(\Gamma^{\alpha}_{\gamma\lambda}\dot{x}^{\lambda} + \widetilde{C}^{\alpha\beta}_{\gamma}\dot{\xi}_{\beta} + C^{\alpha}_{\gamma\beta}\dot{\overline{\xi}}^{\beta})] = 0, \tag{4.1c}$$

where $\dot{x}^{\mu} = dx^{\mu}/ds, \dot{\xi}_{\alpha} = d\xi_{\alpha}/ds, \dot{\overline{\xi}}^{\alpha} = d\overline{\xi}^{\alpha}/ds$, and the coefficients $\Gamma^{\mu}_{\nu\lambda}, \Gamma^{\gamma}_{\alpha\lambda}, \overline{C}^{\mu\alpha}_{\nu}$, $\widetilde{C}^{\alpha\beta}_{\gamma}, C^{\mu}_{\nu\alpha}, C^{\gamma}_{\alpha\beta}$ satisfy the postulates imposed by Y. Takano and T. Ono [10].

PROPOSITION 4.2. (a) *If* $\overline{C}^{\mu\alpha}_{\nu} = 0$ *and* $C^{\mu}_{\nu\alpha} = 0$, *then* $\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\lambda\nu}$ *and relation* (4.1a) *becomes*

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\nu\lambda}(x,\xi(x),\overline{\xi}(x))\frac{dx^{\nu}}{ds}\frac{dx^{\lambda}}{ds} = 0. \tag{4.2}$$

(b) *For the GCFS, equation* (4.1) *has the form*

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\nu\lambda}\dot{x}^{\nu}\dot{x}^{\lambda} + \dot{x}^{\mu}(\sigma^{\alpha}\dot{\xi}_{\alpha} + \overline{\sigma}_{\alpha}\dot{\overline{\xi}}^{\alpha}) = 0. \tag{4.3}$$

In this case $\overline{C}^{\mu\alpha}_{\nu} = 0, C^{\mu}_{\nu\alpha} = 0$ *hold true iff* $\sigma^{\alpha} = \overline{\sigma}_{\alpha} = 0$, *i.e., for* σ *depending only on* x.

Proof: Equations (4.2) and (4.3) are consequences of Definition (4.1a) and relations (3.6). □

Remark 4.3: The spinor parts of equations (4.2) and (4.3) also write as

$$\begin{aligned} &\ddot{\xi}_{\alpha} - \dot{\xi}_{\gamma}T^{\gamma}_{\alpha} - \xi_{\gamma}\dot{T}^{\gamma}_{\alpha} - (\dot{\xi}_{\gamma} - \xi_{\delta}T^{\delta}_{\gamma})T^{\gamma}_{\alpha} = 0, \\ &\ddot{\overline{\xi}}^{\alpha} + \dot{\overline{\xi}}^{\alpha}\overline{T}^{\gamma}_{\alpha} + \overline{\xi}^{\gamma}\dot{\overline{T}}^{\alpha}_{\gamma} + (\dot{\overline{\xi}}^{\gamma} + \overline{\xi}^{\delta}\overline{T}^{\gamma}_{\delta})T^{\alpha}_{\gamma} = 0, \end{aligned} \tag{4.4}$$

where

$$T^{\gamma}_{\alpha} \equiv \Gamma^{\gamma}_{\alpha\lambda}\dot{x}^{\lambda} + \widetilde{C}^{\gamma\beta}_{\alpha}\dot{\xi}_{\beta} + C^{\gamma}_{\alpha\beta}\dot{\overline{\xi}}^{\beta} = \overline{T}^{\gamma}_{\alpha}.$$

Having the equations of geodesics, it remains to derive the equations of geodesic deviation of our spaces. This geodesic deviation can be given a physical meaning if we consider two very close geodesic curves and the curvature tensor is Riemannian.

In the general case of GCFS, the spinor variables are independent of the position, so it is difficult to convey a physical meaning to the equations of geodesic deviation. For this reason it is convenient to study the deviation of the geodesics in the case where the spinor field $\xi_{\alpha} = \xi_{\alpha}(x^{\mu})$ (and $\overline{\xi}^{\alpha} = \overline{\xi}^{\alpha}(x^{\mu})$) is defined on the manifold. This spinor field associates a spinor—and its conjugate—to every point of the space–time.

In this case, from Proposition 4.2 and relation (4.2) the Christoffel symbols $\Gamma^{\mu}_{\nu\lambda}$ are symmetric in the lower indices and the equation of geodesics is similar to the Riemannian one, except that the connection coefficients have the additional dependence on the spinors $\xi_\alpha(x^\mu), \overline{\xi}^\alpha(x^\mu)$. Thus our approach is more general. The equation of geodesic deviation in our case is given by

$$\frac{D^2\zeta^\lambda}{ds^2} + R^\lambda_{\mu\nu\varrho}\frac{dx^\mu}{ds}\frac{d\zeta^\nu}{ds}\frac{dx^\varrho}{ds} = 0. \tag{4.5}$$

The above curvature tensor $R^\lambda_{\mu\nu\varrho}(x, \xi(x), \overline{\xi}(x))$ has a modified Riemannian form. This equation has additional contributions from the spinor parts which enter the curvature tensor R and the covariant derivative. In (4.5) ζ^μ denotes the deviation vector, and s the arc length.

For the GCFS, the deviation equation has the above form, where the curvature tensor depends on the function $\sigma(x, \xi(x), \overline{\xi}(x))$ and its derivatives, as we have proved in Theorem 3.3, relation (3.14). After a direct calculation from (4.5) and

$$R^\lambda_{\mu\nu\varrho} = R_{ab\mu\varrho}h^{b\lambda}h^a_\nu, \tag{4.6}$$

where $h^{b\lambda} = e^{-\sigma}\eta^{b\lambda}, h^a_\nu = e^\sigma\delta^a_\nu$, we get the equation of geodesic deviation for the GCFS, with $\overline{C}^{\mu\alpha}_\nu = 0, C^\mu_{\nu\alpha} = 0$, in the form

$$\begin{aligned}\frac{D^2\zeta^2}{ds^2} &+ (\delta^\lambda_{(\mu}\sigma^*_{\varrho)\nu} + \eta_{\nu(\varrho}\sigma^*_{\mu)b}\eta^{b\lambda} + \delta^\lambda_{(\varrho}\sigma^*_{\mu)}\sigma^*_\nu + \\ &+ \eta_{\nu(\mu}\sigma^*_{\varrho)}\sigma^*_b h^{b\lambda} + \eta_{\nu(\varrho}\delta^\lambda_{\mu)}\eta^{cd}\sigma^*_c\sigma^*_d)\frac{dx^\mu}{ds}\frac{d\zeta^\nu}{ds}\frac{dx^\varrho}{ds} = 0.\end{aligned} \tag{4.7}$$

5. Conclusions

(a) In Section 2 we derived the gravitational field equations in spaces whose metric tensor depends on spinor variables. Equations (2.10) and (2.22) are generalizations of the conventional equations (1.3a) and (1.3b). They are reduced to equations (1.3a) and (1.3b) when the coefficients

$$(\omega^{(*)}_\mu, \theta^{(*)}_\alpha, \overline{\theta}^{(*)\alpha}) \to (\omega_\mu).$$

Relations (2.27) and (2.28) give rise to new results.

(b) Equations (F1)–(F4) represent the field equations on the GCFS $(M, g_{\mu\nu}(x, \xi, \overline{\xi}))$. The solutions of these equations are the subject of further concern. They represent an application of the gauge approach, for spaces with the metric $g(x, \xi, \overline{\xi})$, studied by two of the authors in [12, 13].

(c) The vanishing of the curvatures $S^\alpha_{ab\beta}, Q_{ab\beta\alpha}, \widetilde{Q}^{\beta\alpha}_{ab}$ (Theorem 3.3), reduces the 6 spin curvatures of the theory of Y. Takano and T. Ono to the three ones $R_{ab\lambda\mu}$, $P_{ab\lambda\alpha}, \overline{P}^\alpha_{ab\lambda}$. This simplifies considerably the study of the generalized conformally flat spaces.

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