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# Modified Einstein and Finsler like theories on tangent Lorentz bundles

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In this paper, we study modifications of general relativity, GR, with nonlinear dispersion relations which can be geometrized on tangent Lorentz bundles. Such modified gravity theories, MGTs, can be modeled by gravitational Lagrange density functionals $f(\mathbf{R}, \mathbf{T}, F)$ with generalized/modified scalar curvature $\mathbf{R}$, trace of matter field tensors $\mathbf{T}$ and modified Finsler like generating function $F$. In particular, there are defined extensions of GR with extra dimensional "velocity/momentum" coordinates. For four-dimensional models, we prove that it is possible to decouple and integrate in very general forms the gravitational fields for $f(\mathbf{R}, \mathbf{T}, F)$-modified gravity using nonholonomic $2 + 2$ splitting and nonholonomic Finsler like variables $F$. We study the modified motion and Newtonian limits of massive test particles on nonlinear geodesics approximated with effective extra forces orthogonal to the four-velocity. We compute the constraints on the magnitude of extra-accelerations and analyze perihelion effects and possible cosmological implications of such theories. We also derive the extended Raychaudhuri equation in the framework of a tangent Lorentz bundle. Finally, we speculate on effective modeling of modified theories by generic off-diagonal configurations in Einstein and/or MGTs and Finsler gravity. We provide some examples for modified stationary (black) ellipsoid configurations and locally anisotropic solitonic backgrounds.

*Keywords*: Modified theories of gravity; Einstein spaces; tangent Lorentz bundle; Finsler geometry; exact solutions.

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## 1. Introduction

The late-time cosmic accelerating discovered and confirmed in 1998–1999[1,2] opened new directions of research in cosmology and seems to change paradigms in modern gravity and standard particle physics. In spite of various efforts, the source of acceleration of universe, dark energy and dark matter effects etc are far from being understood. For reviews of theoretical works and observational data, see Refs. 3–8 and references therein. Different theoretical models in which the Einstein–Hilbert action is replaced by functions $f(R, q)$, (where $R$ is the Ricci scalar and a function $q$ is be used, for instance, for the trace of energy–momentum of matter, torsion fields, etc.), have been investigated in a number of papers. These have stated the conditions of existence of viable cosmological models, analyzed the constraints obtained from the classical tests of general relativity and quantum gravity models, studied the galactic dynamics and test particle propagations with and without dark matter, etc. They have also explored possible connections with modified Newtonian dynamics (MOND) and the Pioneer anomaly and considered the astrophysical and cosmological implications of nonminimal coupling matter–geometry models,[9–13] Finsler like generalizations,[14–19] etc.

The gravitational field equations in general relativity, GR, and extra-dimensional extensions (including models on (co) tangent bundles with commutative and noncommutative variables) have been found to possess a decoupling property with respect to certain nonholonomic frames of reference. This allows us to integrate such systems of partial differential equations (PDE) in general off-diagonal forms.[a,20,21] Such methods of constructing off-diagonal solutions in various gravity theories were elaborated by introducing Finsler like variables in Einstein gravity and various modifications.

Finsler like variables can be naturally introduced on a (co) tangent bundle, $T\mathbf{V}$, to a Lorentz manifold, $\mathbf{V}$, with possible noncommutative extensions, for various classical and quantum gravity models with modified dispersion relations.[21,22] Considering nonintegrable (equivalently nonholonomic/anholonomic) $2+2$ splitting on (pseudo) Riemannian spacetimes, we can mimic certain locally anisotropic configurations with prescribed fibred local structures. A Finsler nonlinear quadratic

[a]The metrics for these classes of solutions cannot be diagonalized via coordinate transform and the geometric/physical objects may depend on all coordinates via generating and integration functions and various parameters.

element $F(x,y) = F(x^k, y^a = dx^a/d\tau)$ is present in such theories as a nonlinear generating function/metric for three fundamental geometric objects. These are the nonlinear connection (N-connection), $\mathbf{N} = \{N_i^a(x,y)\}$, a lift to total metric, $\mathbf{g} = \{g_{\alpha\beta}\}$, and distinguished connection, $\mathbf{D} = \{\Gamma^{\alpha}_{\beta\gamma}\}$ (d-connection, which is different from the Levi–Civita connection, $\nabla = \{\Gamma^{\alpha}_{\beta\gamma}\}$).[b] There are various models of (generalized) Finsler geometry and gravity theories depending on explicit assumptions on data $(F : \mathbf{g}, \mathbf{N}, \mathbf{D})$ and how corresponding curvature, torsion and other tensors are derived. Such theories can be metric compatible, or noncompatible, see critical remarks in Refs. 19 and 23. In all cases, minimal extensions of GR with a well defined axiomatic can be encoded into certain extended principles of general covariance and relativity. Considering sections $y^a(x^k)$ on a basic Lorentz manifold, we generate certain effective "osculating" (pseudo) Riemannian metrics $\tilde{g}_{ij}(x) := \mathbf{g}_{ij}(x, y(x))$ derived in nonlinear forms from $F(x, y(x))$ (such approaches were considered in Finsler modifications of gravity and cosmology in Refs. 14 and 15). This class of theories is with $f(R, T, \ldots)$ modifications of GR and (via nonholonomic frame and off-diagonal deformations of metrics and distortions of fundamental geometric structures) can be related to various anisotropic modifications of Hořava–Lifshitz, and/or covariant anisotropic models of gravity, etc.[5,7,8,13,22] Conventionally, we denote such modified gravity theories with gravitational Lagrangians $f(R, T, F)$.

It is the purpose of the present paper to study extensions of standard GR to certain forms with Lagrange density $f(R, T, F)$ when the constructions can be modeled by generic off-diagonal and nonholonomic effects in an effective Einstein or Finsler like gravity theory. We shall consider standard models of matter with respect to certain N-adapted frames of reference which via nonholonomic constraints and off-diagonal interactions may mimic exotic fluids and states of matter, quantum effects. Such theories are with modified dispersion relations and/or anomalies, anisotropies and "noncompactified" extra dimensions with (co) velocity-like variables, etc. The field equations and the covariant divergence of the stress–energy tensor can be derived in two equivalent geometric and variational forms working with N-adapted geometric objects. Nonholonomic constraints and off-diagonal gravitational interactions can model nontrivial matter configurations, for instance, nonlinear scalar field interactions. So, there are possible alternative explanations for

[b]We write boldface symbols for spaces (and geometric objects on such spaces) enabled with so-called horizontal (h) and vertical (v) splitting, $\mathbf{N} : T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V}$, when coordinates and indices split in the form $u^\alpha = (x^i, y^a)$ [in brief, $u = (x,y)$] and $\beta = (j, b)$. Such $h$–$v$-splitting exist naturally of vector/tangent bundles, but can be introduced formally, for instance, as a $2+2$ splitting to diads on (pseudo) Riemannian manifolds (when $i, j \ldots, = 1, 2$ and $a, b, \ldots = 3, 4$). For (pseudo) Finsler models on tangent bundles, the indices run values of type $i, j \ldots, = 1, 2, 3, 4$ and $a, b, \ldots = 6, 7, 8, 9$. The term "pseudo" will be used for any necessary local signature of metrics $(\pm, \pm, \pm, \pm)$. Readers may consider details on such constructions in Refs. 23 and 20.

inflation scenarios, late time accelerations and dark energy/matter effects. We shall demonstrate the possibility of reconstruction of various types Friedman–Lemaître–Robertson–Worker (FLRW) cosmology and anisotropic modifications by appropriate choices of the above mentioned functionals and/or generating/integration functions for generic off-diagonal solutions.

We shall speculate on possible theories of reduction tangent bundle models to standard ones with effective Einstein equations in GR, for $F(x, y) \to F(x, y(x))$. In general, such constructions may result in nontrivial torsion, nonzero covariant divergence of the stress–energy tensor, etc. We argue that following certain general principles on metric compatible constructions completely determined by a fundamental metric tensor we can provide an equivalent encoding of off-diagonal coefficients of metrics into nonholonomic frames. We will state the conditions so that $f(R, T, F)$ with general data $(F : \mathbf{g}, \mathbf{N}, \mathbf{D})$ can be effectively described by certain $(\tilde{g}_{ij}(x), \nabla(x))$ and/or equivalent $(\mathbf{g}, \mathbf{D} = \nabla + \mathbf{Q})$. All geometric objects being determined by the same metric structure. The motion of massive test particles in such modified theories is modeled by nonlinear geodesic configurations (with effective extra acceleration). This is due to the off-diagonal/nonholonomic interactions and nonlinear coupling between matter and geometry. We shall investigate the Newtonian limit of such models and compute certain expressions for the extra acceleration. The observational data for the perihelion of the Mercury can be used to impose a general constraint on magnitude of such extra-acceleration and local anisotropy effects.

The present paper is structured as follows. Section 2 is devoted to a brief introduction into the Finsler osculating gravity and its relation to GR and modifications. There are derived field equations of $f(R, T, F)$ gravity. Some particular cases and the conditions of effective modeling via generic off-diagonal solutions in GR and Finsler–Cartan gravity are analyzed in Sec. 3. We briefly discuss the procedure of reconstructing gravity theories with scalar field and off-diagonal interactions. The equations of constrained motion in modified backgrounds of massive test particles, and the corresponding Newtonian limits of effective locally anisotropic models, are analyzed. In Sec. 4, we develop a geometric method of decoupling and integrating the field equations in modified gravity. We show how such equations can be solved in generic off-diagonal form as nonholonomic deformations of de Sitter black holes to certain rotoid and/or locally anisotropic solitonic configurations. Finally, we discuss and conclude our results in Sec. 5.

## 2. Modified Einstein and Finsler Osculating Gravity

In the present section, we provide an introduction into modified theories with local anisotropies which can be modeled as effective GR theories for certain nonholonomic constraints resulting in zero torsion structure but generic off-diagonal terms in metrics.

### 2.1. *Motivations for the $f(R, T, F)$ gravity*

We analyze two approaches to modifications of the GR theory.

#### 2.1.1. *Action for $f(R, T)$ theories*

We can consider models on a four-dimensional (4-d) pseudo-Riemannian manifold enabled with metric structure $g_{ij}(x^k)$ defining a quadratic linear element

$$ds^2 = g_{ij}(x)dx^i dx^j, \tag{1}$$

for $x = \{x^k\}$, a gravity theory corresponding to action

$$S = \int \sqrt{|g|} d^4x \{(16\pi)^{-1} f(R, T) + {}^m L\},$$

where $R = g^{ij} R_{ij}$ is the scalar corresponding to contraction of the inverse metric $g^{ij}$ with the Ricci tensor $R_{ij}$ constructed for the Levi–Civita connection, ${}^m L$ is the matter Lagrangian density which via corresponding variational calculus results in the stress–energy tensor

$$T_{ij} = \frac{-2(\sqrt{|g|})^{-1} \delta(\sqrt{|g|}{}^m L)}{\delta g^{ij}},$$

and its trace, $T = g^{ij} T_{ij}$.[c] We obtain the Hilbert–Einstein action if $f(R, T) = R$. Such constructions are reviewed in Refs. 5, 7, 8 and 13.

#### 2.1.2. *The Finsler–Cartan gravity*

In the second class of theories, we consider instead of (1) a nonlinear quadratic element

$$\begin{aligned} ds^2 &= F^2(x^i, y^j) \\ &\approx -(cdt)^2 + g_{\widehat{i}\widehat{j}}(x^k) y^{\widehat{i}} y^{\widehat{j}} \left[1 + \frac{1}{r} \frac{\rho_{\widehat{i}_1 \widehat{i}_2 \cdots \widehat{i}_{2r}}(x^k) y^{\widehat{i}_1} \cdots y^{\widehat{i}_{2r}}}{(g_{\widehat{i}\widehat{j}}(x^k) y^{\widehat{i}} y^{\widehat{j}})^r}\right] + O(\rho^2), \end{aligned} \tag{2}$$

for $y^i = dx^i/d\tau$ with a real parameter $\tau$ in $x^i(\tau)$, where values $\rho_{\widehat{i}_1 \widehat{i}_2 \cdots \widehat{i}_{2r}}(x)$ are parametrized by 3-d spacelike "hat" indices running values $\widehat{i} = 1, 2, 3$ have to be computed using certain experimental/observational data and/or theoretical models. To spacetime geometry and/or geometric mechanics, locally anisotropic field theories with effective nonlinear metrics of type (2), models of quantum gravity, etc. we can naturally associate[22,24] certain local modified dispersion relations for propagation of light. For a corresponding frequency, $\omega$ and wave vector $k_i$, one

[c]We use the natural system of units when the Newton constant, $G$, and light speed, $c$, are subjected to the conditions $G = c = 1$ and the gravitational constant is $\kappa^2 := 8\pi$.

computes locally

$$\omega^2 = c^2[g_{\widehat{i}\widehat{j}}k^{\widehat{i}}k^{\widehat{j}}]^2 \left( \frac{1 - \frac{1}{r}\rho_{\widehat{i}_1\widehat{i}_2\ldots\widehat{i}_{2r}} y^{\widehat{i}_1} \cdots y^{\widehat{i}_{2r}}}{[g_{\widehat{i}\widehat{j}}y^{\widehat{i}}y^{\widehat{j}}]^{2r}} \right), \quad (3)$$

when the local wave vectors $k_i \rightarrow p_i \sim y^a$ are related to momentum type variables $p_i$ which are dual to "fiber" coordinates $y^a$.

Nonlinear metric elements (2) are usually considered in Finsler geometry when certain homogeneity conditions are imposed, $F(x^i, \beta y^j) = \beta F(x^i, y^j)$, for any $\beta > 0$). The value $F$ is considered to be a fundamental (generating) Finsler function usually satisfying the condition that the Hessian

$$\tilde{g}_{ij}(x^i, y^j) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \quad (4)$$

is not degenerate. For physical applications related to "small" deformations of GR, we can consider that $g_{ij} = (-1, g_{\widehat{i}\widehat{j}}(x^k))$ in the limit $\rho \rightarrow 0$ correspond to a metric on a (pseudo) Riemannian manifold with local coordinates $(x^i)$ and signature of metric of type $(- + ++)$. In such cases, we elaborate (pseudo) Finsler models on tangent bundles to Lorentz manifolds.

There are substantial differences between geometric and physical theories constructed for (pseudo) Riemannian quadratic elements (1) and those with nonlinear (Finsler type) ones (2). In the first case, the data $(g_{ij}, \nabla)$ provide a complete geometric model for which gravity theories are derived for corresponding Lagrange densities. We need more assumptions in order to construct some self-consistent geometries from a generating function $F(x, y)$. A metric compatible model of (pseudo) Finsler–Cartan geometry completely determined by $F$ and $\tilde{g}_{ij}$, up to necessary classes of frame/coordinate transform $e^{\alpha'} = e^{\alpha'}_{\ \alpha}(x, y)e^{\alpha}$, can be constructed from a triple $(F : \mathbf{N}, \mathbf{g}, \mathbf{D})$ of fundamental geometric objects:

(i) The nonlinear connection (N-connection) structure.

$$\mathbf{N} : \ T\mathbf{TV} = h\mathbf{TV} \oplus v\mathbf{TV}, \quad (5)$$

i.e. a nonholonomic (equivalently, nonintegrable/anholonomic) distribution with horizontal (h) and vertical (v) splitting. This value can be introduced in coefficient form, $\mathbf{N} = \{\mathbf{N}^{a'}_{i'} = e^{a'}_{\ a} e_{i'}^{\ i} \tilde{N}^a_i\}$, where

$$\tilde{N}^a_j := \frac{\partial \tilde{G}^a(x, y)}{\partial y^j}, \quad \text{for } \tilde{G}^k = \frac{1}{4}\tilde{g}^{kj}\left(y^i \frac{\partial^2 L}{\partial y^j \partial x^i} - \frac{\partial L}{\partial x^j}\right).$$

An N-adapted frame structure is defined naturally as $\tilde{\mathbf{e}}_\nu = (\tilde{\mathbf{e}}_i, e_a)$, where

$$\tilde{\mathbf{e}}_i = \frac{\partial}{\partial x^i} - \tilde{N}^a_i(u)\frac{\partial}{\partial y^a} \quad \text{and} \quad e_a = \frac{\partial}{\partial y^a}, \quad (6)$$

and the dual frame (coframe) structure is $\tilde{\mathbf{e}}^\mu = (e^i, \tilde{\mathbf{e}}^a)$, where

$$e^i = dx^i \quad \text{and} \quad \mathbf{e}^a = dy^a + \tilde{N}^a_i(u)dx^i. \quad (7)$$

The following nonholonomy relations are satisfied:

$$[\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\beta] = \tilde{\mathbf{e}}_\alpha \tilde{\mathbf{e}}_\beta - \tilde{\mathbf{e}}_\beta \tilde{\mathbf{e}}_\alpha = \tilde{W}^\gamma_{\alpha\beta} \tilde{\mathbf{e}}_\gamma, \tag{8}$$

with anholonomy coefficients $\tilde{W}^b_{ia} = \partial_a \tilde{N}^b_i$ and $\tilde{W}^a_{ji} = \tilde{\Omega}^a_{ij}$.[d]

(ii) Using data $(\tilde{g}_{ij}, \tilde{\mathbf{e}}_\alpha)$, we can define a canonical (Sasaki type) metric structure

$$\tilde{\mathbf{g}} = \tilde{g}_{ij}(x,y) e^i \otimes e^j + \tilde{g}_{ij}(x,y) \tilde{\mathbf{e}}^i \otimes \tilde{\mathbf{e}}^j \tag{9}$$

$$= g_{ij}(x,y) e^i \otimes e^j + h_{ab}(x,y) \mathbf{e}^a \otimes \mathbf{e}^b, \tag{10}$$

which can be related to an "arbitrary" metric structure $\mathbf{g} = \{\mathbf{g}_{\alpha'\beta'}\}$ via frame transforms, $\mathbf{g}_{\alpha'\beta'} = e^\alpha_{\ \alpha'} e^\beta_{\ \beta'} \tilde{\mathbf{g}}_{\alpha\beta}$.

(iii) For any metric $\mathbf{g} = \tilde{\mathbf{g}}$, we can construct in standard form the Levi–Civita connection $\nabla = \{\Gamma^\alpha_{\beta\gamma}\}$, which does not preserve under parallelism the N-connection splitting (5). In Finsler theories, one introduces distinguished connections (d-connections) $\mathbf{D} = \{\Gamma^\alpha_{\beta\gamma}\}$ which is adapted to the N-connection structure, i.e. preserves the nonholonomic $h$–$v$-splitting. It is possible to construct Einstein–Finsler type theories for d-connections with are compatible with the metric structure, $\mathbf{Dg} = \mathbf{0}$. For instance, this is the case of the well known Cartan d-connection, which is metric compatible, but the Chern and/or Berwald d-connections are not metric compatible which is less related to standard models of physics, see discussions and critical remarks in Refs. 19 and 23.

Using N-adapted differential forms and the d-connection 1-form is $\Gamma^\alpha_{\ \beta} = \Gamma^\alpha_{\ \beta\gamma} \mathbf{e}^\gamma$, we can define and compute the torsion and curvature 2-forms

$$\mathcal{T}^\alpha := \mathbf{D}\mathbf{e}^\alpha = d\mathbf{e}^\alpha + \Gamma^\alpha_{\ \beta} \wedge \mathbf{e}^\beta, \quad \text{and}$$
$$\mathcal{R}^\alpha_{\ \beta} := \mathbf{D}\Gamma^\alpha_{\ \beta} = d\Gamma^\alpha_{\ \beta} - \Gamma^\gamma_{\ \beta} \wedge \Gamma^\alpha_{\ \gamma} = \mathbf{R}^\alpha_{\ \beta\gamma\delta} \mathbf{e}^\gamma \wedge \mathbf{e}^\delta.$$

For instance, the $h$–$v$-coefficients $\mathbf{T}^\alpha_{\ \beta\gamma} = \{T^i_{\ jk}, T^i_{\ ja}, T^a_{\ ji}, T^a_{\ bi}, T^a_{\ bc}\}$ of $\mathcal{T}^\alpha$ are computed using formulas

$$T^i_{\ jk} = L^i_{\ jk} - L^i_{\ kj}, \quad T^i_{\ ja} = -T^i_{\ aj} = C^i_{\ ja}, \quad T^a_{\ ji} = \Omega^a_{\ ji},$$
$$T^a_{\ bi} = \frac{\partial N^a_i}{\partial y^b} - L^a_{\ bi}, \quad T^a_{\ bc} = C^a_{\ bc} - C^a_{\ cb}. \tag{11}$$

See, for instance, Ref. 23 for N-adapted coefficients of curvature, $\mathbf{R}^\alpha_{\ \beta\gamma\delta}$, and

$$\Omega^a_{ij} = \frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b}.$$

[d] An N-connection can be canonically determined by $F$ following a geometric/variational principle for an effective regular Lagrangian $L = F^2$ and action integral $S(\tau) = \int_0^1 L(x(\tau), y(\tau)) d\tau$, for $y^k(\tau) = dx^k(\tau)/d\tau$. The Euler–Lagrange equations $\frac{d}{d\tau}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = 0$ are equivalent to the "nonlinear geodesic" (equivalently, semi-spray) equations $\frac{d^2 x^k}{d\tau^2} + 2\tilde{G}^k(x,y) = 0$, where $\tilde{g}^{kj}$ is inverse to ${}^v\tilde{g}_{ij} \equiv \tilde{g}_{ij}$ (4).

For our purposes (in order to decouple and integrate in very general forms the gravitational field equations), it is convenient to work with the so-called canonical d-connection $\mathbf{D}$ completely defined by a metric $\mathbf{g} = \tilde{\mathbf{g}}$ in metric compatible form, $\mathbf{D}\mathbf{g} = 0$, and with zero $h$- and $v$-torsions, $T^i_{\ jk} = 0$ and $T^a_{\ bc} = 0$. In general, there are nonzero values $T^i_{\ ja}, T^a_{\ ji}$ and $T^a_{\ bi}$, see (11).[e] Here we note that there is a canonical distortion relation

$$\mathbf{D} = \nabla + \mathbf{Z}, \tag{12}$$

where both connections $\mathbf{D}$ and $\nabla$ and the distortion tensor $\mathbf{Z}$ (it is an algebraic combination of nontrivial torsion coefficients, see explicit formulas in Ref. 23) are uniquely defined by the same metric structure $\mathbf{g}$.

We can construct a variant of Einstein–Finsler theory $\mathbf{D}$ following standard geometric rules as in general relativity but reconsidering the constructions on tangent bundles/manifolds. The scalar curvature is by definition

$$ {}_s^F R := \mathbf{g}^{\beta\gamma}\ \mathbf{R}_{\beta\gamma} = g^{ij} R_{ij} + h^{ab} R_{ab} = {}^h R + {}^v R. \tag{13}$$

This scalar curvature is similar to that for the Levi–Civita connection in the Einstein gravity. In both cases of a (pseudo) Riemannian geometry and/or a Finsler space, such a value is uniquely defined on the corresponding total tangent bundle by contracting the total metric tensor and the respective Riemannian tensor. Formulas are similar but with that difference that in the second case we consider a Finsler like connection.

It should be noted that Finsler like variables can be introduced in standard GR considering a generating function $F = \mathcal{F}(x, y)$ determining a $2 + 2$ splitting. For such models, indices $i, j, \ldots = 1, 2$ and $a, b, \ldots = 3, 4$ which is adapted to a fibred structure on a Lorentz manifold. We can use similar geometric constructions with $4+4$ splitting when Finsler models are on tangent bundles, and distinguish this via conventional $i, j, \ldots = 1, 2, 3, 4$ and $a, b, \ldots = 5, 6, 7, 8$.

The gravitational field equations for $\mathbf{D}$ can be postulated in standard geometric form and/or derived via N-adapted variational calculus

$$\mathbf{R}_{\beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta}\ {}_s^F R = \Upsilon_{\beta\delta}, \tag{14}$$

$$L^c_{aj} = e_a(N^c_j), \quad C^i_{jb} = 0, \quad \Omega^a_{\ ji} = 0, \tag{15}$$

for $\Upsilon_{\beta\delta} \to T_{\beta\delta}$ if $\mathbf{D} \to \nabla$. We have to consider the constraints (15) in order to get zero torsion (11) and distortion tensors, $\mathbf{Z} = 0$, which constraints $\widehat{\mathbf{D}} = \nabla$

[e] In our former works, we used the symbol $\widehat{\mathbf{D}}$ for the canonical d-connection. Here, we write the N-adapted coefficients of $\mathbf{D}$ are $\Gamma^\gamma_{\ \alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})$

$$L^i_{jk} = \frac{1}{2} g^{ir}(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}), \quad \widehat{C}^a_{bc} = \frac{1}{2} h^{ad}(e_c h_{bd} + e_c h_{cd} - e_d h_{bc}),$$

$$L^a_{bk} = e_b(N^a_k) + \frac{1}{2} h^{ac}(e_k h_{bc} - h_{dc} e_b N^d_k - h_{db} e_c N^d_k), \quad C^i_{jc} = \frac{1}{2} g^{ik} e_c g_{jk}.$$

in N-adapted frames, (12). It is convenient to work with equations of type (14) and (15) if we want to study in an unified form both the Einstein gravity and Finsler generalized theories. Such N-adapted Finsler like variables result into a very important property of decoupling respective PDE which allows to construct solutions in very general forms.

2.1.3. *The osculating approximation and $f(R,T,F)$ gravity*

Considering arbitrary frame/coordinate transforms on $\mathbf{V}$ and $T\mathbf{V}$, we mix the variables and do not "see" explicit dependencies on $F(x,y)$. Fixing a system of reference, we can introduce an osculating (pseudo) Riemannian metric on the $h$-subspace

$$\mathbf{g}_{ij} = \tilde{g}_{ij}(x, y(x)), \tag{16}$$

where $\tilde{g}_{ij}$ is defined by (4). In general, we can construct exact solutions of (14) for 8-d metrics (9) and/or (10). Nevertheless, the observable spacetime is 4-d and we can verify possible physical implications, directly, only for the $h$-components. Any modifications via $F$ and $\mathbf{g}_{ij}$ (16), and related nonholonomic deformations of ${}^F_sR$ (13) can be parametrized as ${}^F_sR \to f({}^hR, T, F)$, where ${}^hR$ is computed for $\tilde{g}_{ij}$.

In explicit form, we can determine experimentally $F$ for theories with modified dispersions (3) and, for instance, restricted local Lorentz invariance, see reviews of results in Refs. 16–18. There are experimental restrictions for such configurations.[24] Nevertheless, only local considerations are not enough to conclude if a Finsler like theory is physically important, or not. For instance, any data $(F : \mathbf{g}, \mathbf{N}, \mathbf{D})$ can be redefined equivalently via frame transforms into $({}^0F : {}^0\mathbf{g}, {}^0\mathbf{N}, {}^0\mathbf{D})$, where ${}^0F$ is a trivial "Finsler" function resulting in quadratic element (1) but the data $({}^0\mathbf{g}, {}^0\mathbf{N}, {}^0\mathbf{D})$ are constructed as solutions of (14) with possible, or not, Levi–Civita constraints (15). Experimentally, we shall obtain quadratic dispersions in (2) and (3), but the information on locally anisotropic types (Finsler, or other types) is encoded into N-adapted frames ${}^0\mathbf{N}$ and generic off-diagonal terms of ${}^0\mathbf{g}$. One could see observational effects for Finsler brane and black hole/ellipsoid solutions with (non) commutative variables and anisotropic modified dispersions.[21,22]

The principles of generalized covariance can be extended from $\mathbf{V}$ to $T\mathbf{V}$. Various classes of exact solution of gravitational field equations $T\mathbf{V}$ directly or indirectly contain physical information on $F$. Such data can be encoded into N- and d-connections and total metrics. Possible phenomenology and experimental/observational effects can analyzed for metrics of type (16) with certain nonholonomic projections of theories on fundamental $h$-spacetime. Geometrically, we can transform a diagonal configuration on $T\mathbf{V}$, determined with respect to N-adapted bases (6) and (7), into a generic off-diagonal $\mathbf{g}_{ij}[F, \tilde{g}_{ij}]$ when the functional dependence can be stated for well-defined boundary/assymptotic conditions, Cauchy problem, etc. This class of theories is of type $f(R,T,F) \to {}^hR[F, \tilde{g}_{ij}]$, where $F$ can be defined up to certain classes of symmetries under coordinate/frame transforms and, in special cases, reduced local symmetries.

Working with $\mathbf{g}_{ij}[F, \tilde{g}_{ij}]$, we use an effective $\mathcal{F}(x^1, x^2; y^3, y^4)$ generating conventional $2+2$ splitting which for corresponding N-adapted bases we can construct off-diagonal solutions for 4-d gravity theories. In a particular case, we can constrain the integral varieties of solutions in order to extract Levi–Civita configurations. Such generating functions $\mathcal{F}$ can be also determined up to certain classes of frame transforms. In all cases, we can conventionally write $f(R, T, F)$ where $F$ emphasizes possible locally anisotropic/nonholonomic/generic off-diagonal contributions from certain Finsler like models on $\mathbf{V}$, or $T\mathbf{V}$.

### 2.2. *Field equations for the* $f(R, T, F)$ *gravity*

Hereafter, we assume that small Greek indices split in the form $\alpha, \beta, \ldots = (i, a), (j, b) \ldots$, where $i, j, \ldots = 1, 2$ and $a, b = 3, 4$ for a osculating (pseudo) Riemannian metric of type (16)

$$\mathbf{g}_{\alpha\beta}(u^\gamma) = \tilde{\mathbf{g}}_{\alpha\beta}(u^\gamma, y^\alpha(u^\mu)), \tag{17}$$

on a nonholonomic Lorentz manifold $\mathbf{V}$ with local coordinates $u^\alpha = (x^i, y^a)$ with functional dependence on sections $y^\alpha(u^\mu)$ of $T\mathbf{V}$. We shall not state theoretical/experimental constraints for $F(u^\gamma, y^\alpha)$ on open regions of $T\mathbf{V}$ but analyze general and physical important implications of such nontrivial structures on effective (pseudo) Riemannian spacetime $\mathbf{V}$. The metric $\mathbf{g}_{\alpha\beta}(u^\gamma)$ (17) can be parametrized in N-adapted from as (9) and/or (10), for 4-d configurations with nonholonomic splitting into 2-d $h$-components and 2-d $v$-components. It is possible to introduce Finsler like variables on $\mathbf{V}$ if we prescribe an effective nonholonomic distribution $F \to \mathcal{F}(x^i, y^a)$.

For simplicity, we assume that the Lagrangian density of matter $\mathcal{L}(u^\gamma)$ depends only the metric tensor components $\mathbf{g}_{\alpha\beta}$, when

$$\mathbf{T}_{\alpha\beta} = \frac{-2(\sqrt{|\mathbf{g}|})^{-1}\delta(\sqrt{|\mathbf{g}|}\mathcal{L})}{\delta \mathbf{g}^{\alpha\beta}} = \frac{\mathbf{g}_{\alpha\beta}\mathcal{L} - 2\partial\mathcal{L}}{\partial \mathbf{g}^{\alpha\beta}}, \tag{18}$$

for $|\mathbf{g}|$ being the determinant of (17). We denote by $\mathbf{T} := \mathbf{T}^\beta_\beta$. We note that N-adapted variations are obtained with respect to N-elongated frames (6) and (7) assuming that we work on a spacetime with nonholonomic $2+2$ splitting.

Our modified gravity theory is modeled on $\mathbf{V}$ by a functional $f(R, \mathbf{T}, F) \simeq {}^hR$, for $R \simeq {}^F_sR$ as in (13). The action is considered in the forms

$$S = \int \sqrt{|\mathbf{g}|} d^2x \delta^2 y \{(16\pi)^{-1} f(R, \mathbf{T}, F) + \mathcal{L}\} \tag{19}$$

$$\simeq \int \sqrt{|\tilde{\mathbf{g}}|} d^2x \delta^2 y \{{}^F_sR + \mathcal{L}\}, \tag{20}$$

where the term (20) with ${}^F_sR$ is a part of theory on $T\mathbf{V}$, which we do not state in explicit form. We shall formulated certain physically important conditions for a theory for (19) which will be described by exact solutions of (14) related to (20).

For the N-adapted variation of $S$ (computations are similar to those for derivation of formulas (11) in Ref. 8, but performed for the canonical d-connection $\mathbf{D}$ and metrics of type $\mathbf{g}$ (10), for $\partial f/\partial R = \partial_R f \neq 0$), we obtain the locally anisotropic gravitational field equations

$$\mathbf{R}_{\beta\gamma} - \frac{1}{2}\frac{f}{(\partial_R f)}\mathbf{g}_{\beta\gamma}$$
$$= \frac{1}{\partial_R f}[8\pi \mathbf{T}_{\beta\gamma} + (\mathbf{D}_\beta \mathbf{D}_\gamma - \mathbf{g}_{\beta\gamma}\mathbf{D}_\alpha \mathbf{D}^\alpha)(\partial_R f) - (\partial_T f)(\mathbf{T}_{\beta\gamma} + \Theta_{\beta\gamma})], \quad (21)$$

where

$$\Theta_{\beta\gamma} := \frac{\mathbf{g}^{\mu\nu}\delta(\mathbf{T}_{\mu\nu})}{\delta \mathbf{g}^{\beta\gamma}} \quad \text{and} \quad \Theta := \Theta_\mu^{\ \mu}. \quad (22)$$

It should be noted that the divergence of $\mathbf{T}_{\beta\gamma}$ is not zero

$$(8\pi(\partial_T f)^{-1} - 1)\mathbf{D}_\mu \mathbf{T}^{\mu\nu} = (\mathbf{T}^{\mu\nu} + \Theta^{\mu\nu})\mathbf{D}_\mu \ln|\partial_T f| + \mathbf{D}_\mu \Theta^{\mu\nu}. \quad (23)$$

Such properties with $\mathbf{D}_\mu \mathbf{T}^{\mu\nu} \neq 0$ are known in Finsler gravity theories and GR in nonholonomic variables, see Ref. 23, when $\mathbf{D}_\mu$ is of type (12) with all components determined by $\mathbf{g}$, and if $\mathbf{D}_\mu \to \nabla_\mu, \nabla_\mu \mathbf{T}^{\mu\nu} = 0$.

Finally, we note that Eq. (21) is similar to (14) with effective sources $\Upsilon_{\beta\delta}$ depending on the physical nature of matter fields determined by $\Theta_{\beta\gamma}$.

## 3. Particular Cases and Effective Gravity Models

In this section, we consider several classes of modified gravity theories with explicit parametrization for sources and functional $f$. We shall analyze the possibility to reconstruct gravity with scalar field and off-diagonal interactions. We will study the effective locally anisotropic motion and Newtonian limits. We also will provide the Raychaudhuri equation on the tangent Lorentz bundle $TV$.

### 3.1. *Effective Finsler like and Einstein configurations*

#### 3.1.1. *Assumptions on stress–energy tensors*

The calculation of $\Theta_{\beta\gamma}$ is possible if the matter Lagrangian is postulated. Using formulas (18) and (22), we find

$$\Theta_{\beta\gamma} = \frac{\mathbf{g}_{\beta\gamma}\mathcal{L} - 2\mathbf{g}^{\alpha\tau}\partial^2\mathcal{L}}{\partial \mathbf{g}^{\alpha\tau}\partial \mathbf{g}^{\beta\gamma} - 2\mathbf{T}_{\beta\gamma}}. \quad (24)$$

There are three such important models of matter fields subjected to nonholonomic constraints:

(i) For perfect fluids, we assume that with respect to N-adapted frames, the four-velocity field satisfy the conditions $\mathbf{v}_\alpha \mathbf{v}^\alpha = 1$ and $\mathbf{v}^\alpha \mathbf{D}_\mu \mathbf{v}_\alpha = 0$. There is not a unique definition but we shall take the Lagrangian density $L = -p$. For conventional energy density $\rho$ and pressure $p$, we can parametrize

$$\mathbf{T}_{\beta\gamma} = (\rho + p)\mathbf{v}_\beta \mathbf{v}_\gamma - p\mathbf{g}_{\beta\gamma}, \tag{25}$$

when (24) is computed $\Theta_{\beta\gamma} = -2\mathbf{T}_{\beta\gamma} - p\mathbf{g}_{\beta\gamma}$.

(ii) We can consider a scalar field $\varphi(x,y)$ with zero mass, with Lagrange density ${}^{\varphi}\mathcal{L} = \mathbf{g}^{\alpha\tau}(\mathbf{D}_\alpha\varphi)(\mathbf{D}_\tau\varphi)$, when $\Theta_{\beta\gamma} = -{}^{\varphi}\mathbf{T}_{\beta\gamma} + (1/2){}^{\varphi}\mathbf{T}\mathbf{g}_{\beta\gamma}$.

(iii) A different relation, $\Theta_{\beta\gamma} = -{}^{F}\mathbf{T}_{\beta\gamma}$, is computed for the electromagnetic (antisymmetric) tensor field $\mathbf{F}_{\alpha\beta}$ with the Lagrangian density, ${}^{F}\mathcal{L} = -(16\pi)^{-1}\mathbf{g}^{\alpha\tau}\mathbf{g}^{\beta\gamma}\mathbf{F}_{\alpha\beta}\mathbf{F}_{\tau\gamma}$.

The assumptions above on stress–energy fields are based on the principle of general covariance in GR when the formulas for gravity–matter field interactions are the same with respect to arbitrary frames of reference. Possible contributions from modified $f$-terms, extra $v$-dimensions and local anisotropies are encoded in N-adapted frames of reference and d-connection $\mathbf{D}$.



#### 3.1.2. *Models with* $f(R,T,F) = {}^{h}R + 2f(T)$

The scalar curvature ${}^{h}R$ is the first term in (13), in 8-d, taken for $\mathbf{g}_{\alpha\beta}(u^\gamma)$ (17), reduced to 4-d, and $f(T)$ is an arbitrary function taken for the traces of the stress–energy tensor of matter. We identify ${}^{h}R$ with the scalar curvature ${}^{F}_{s}R$ of $\mathbf{D}$ adapted to a $2+2$ N-connection splitting via a prescribed generating function $\mathcal{F}(u)$. The gravitational field equations (21) transform into a variant for Finsler gravity, see (14), with source

$$\Upsilon_{\beta\delta}[f(T), \mathbf{T}_{\alpha\beta}, \Theta_{\alpha\beta}] = f(T)\mathbf{g}_{\beta\delta} + [8\pi - 2\partial_T f(T)]\mathbf{T}_{\beta\delta} - 2\partial_T f(T)\Theta_{\beta\delta}.$$

Such a 4-d effective gravity model transforms into a theory for $\nabla$ if the conditions (15) for zero torsion are imposed. In both cases of connections $\mathbf{D}$ and/or $\nabla$, the functional dependence $\Upsilon_{\beta\delta}[f(T), \mathbf{T}_{\alpha\beta}, \Theta_{\alpha\beta}]$ does not allow to obtain standard Einstein manifolds even we neglect terms with $\mathbf{T}_{\beta\delta}$ and $\Theta_{\beta\delta}$. The term $f(T)\mathbf{g}_{\beta\delta}$ mimic a locally anisotropic polarization resulting from $T(u)$ of a gravitational constant $\lambda$, when, for instance, $f(T) = \lambda T(u)$. Such generic off-diagonal solutions were studied in a series of our works, see Ref. 23.

For trivial N-connection structure, the constructions presented in this subsection reduce to those analyzed in Sec. III.A. of Ref. 13. Nevertheless, there are known locally anisotropic cosmological configurations with nonzero N-connection coefficients, i.e. off-diagonal generalizations of FRWL universes studied in Ref. 25. In all such cases, we can consider perfect fluid or dust universe approximations and construct cosmologies with effective cosmological constant and locally anisotropic polarizations. The generic off-diagonal solutions can be constructed with generalized group symmetries which may contain information on symmetries for the Finsler generating function $F$ in some 8-d models.

3.1.3. *Modified theories with* $f(R,T,F) = {}^1f({}^F_sR) + {}^2f(T)$

This is an example when the effective 4-d gravity is geometrically more "sensitive" to Finsler contributions. For simplicity, assuming a matter content for a perfect fluid, the field equations (21) for $f = {}^1f({}^F_sR) + {}^2f(T)$ are reformulated in the form (14) with modified gravitational constant and effective source

$$\Upsilon_{\beta\delta} = 8\pi {}^{ef}G\mathbf{T}_{\beta\gamma} + {}^{ef}\mathbf{T}_{\beta\gamma},$$

where the effective values are computed

$$ {}^{ef}G = \frac{[1 + (8\pi)^{-1}\partial_T({}^2f)]}{\partial_R({}^1f)},$$

$$2\partial_R({}^1f){}^{ef}\mathbf{T}_{\beta\gamma} = [(1 - 2\partial_R)({}^1f) + (1 + 2\partial_T)({}^2f)]\mathbf{g}_{\beta\gamma} + 2(\mathbf{D}_\beta\mathbf{D}_\gamma - \mathbf{g}_{\beta\gamma}\mathbf{D}^\alpha\mathbf{D}_\alpha)\partial_R({}^1f).$$

We can compute variations of ${}^{ef}G$ by a fundamental Finsler function $F$ in tangent Lorentz bundle even we follow an osculating approximation to 4-d. If we impose the conditions (15), we get a Levi–Civita configuration, but with effective (matter and time) dependent coupling. In general, locally anisotropic Finsler like contributions are "inverse" ones comparing to matter modifications.

In the scenario with ${}^{ef}G$ and ${}^{ef}\mathbf{T}_{\beta\gamma}$, the cosmic acceleration may result in three possible forms: With locally anisotropic Finsler like contributions, depending on matter content of the universe (for the matter and geometry coupling, etc. and modification of the Hilbert–Einstein terms in Lagrange density) and via an effective source term in the right part of effective Einstein equations. It should be noted here that such nonholonomic deformations of theories are with generic N-connection structure.

Of course, we can consider another types of parametrization of $f(R,T,F)$ resulting in different classes of effective sources and off-diagonal deformations. If a generalized principle of relativity is considered for such classes of theories, we can model such theories as certain branches of nonholonomic manifolds/bundles geometries. We can use this for defining such transforms when the effective gravitational equations decouple and can be integrated in general forms.

### 3.2. *Equivalence of models with* $f(R,T,F)$ *to effective GR or Einstein–Finsler gravity*

Frame and conformal transforms change the geometric and matter field components of theories when a theory of type (19) can be modeled as (20) and/or inversely. The gravitational field equations are also modified (21) both via functionals $f(\ldots)$ and $F(\ldots)$. For simplicity, we consider the first action for $f = {}^F_sR + {}^2f(T)$, where ${}^F_sR$

is of type (13), $q_A$ label matter fields with energy–momentum tensor

$$\mathbf{T}_{\alpha\beta} = -2(\sqrt{|g|})^{-1}\delta \int \frac{d^4x\sqrt{|\mathbf{g}|}\ \mathcal{L}(\mathbf{g}_{\alpha\beta}, q_A)}{\delta \mathbf{g}_{\alpha\beta}}, \tag{26}$$

computed on the 4-d ($h$-part) base spacetime. The $v$-components of the Ricci tensor in 8-d are stated equal to an effective polarized cosmological constant ${}^{v}R = \Lambda(x)$, we considered such models in Ref. 21. The field equations are of type (14), which in 4-d result in

$$\mathbf{R}_{\beta\gamma} - \frac{1}{2}\mathbf{g}_{\beta\gamma}{}_{s}^{F}R = [8\pi + \partial_T({}^2f)]\mathbf{T}_{\beta\gamma} + {}^{ef}\mathbf{T}_{\beta\gamma},$$

with effective gravitational constant

$$ {}^{ef}G = [8\pi + \partial_T({}^2f)], \tag{27}$$

and effective energy–momentum tensor, ${}^{ef}\mathbf{T}_{\beta\gamma} = [{}^2f + \partial_T({}^2f)]\mathbf{g}_{\beta\gamma}$. Putting all terms together, we get

$$\mathbf{R}_{\beta\gamma} - \frac{1}{2}[{}^{h}R + \Lambda(x) +\ 2({}^2f + 2\partial_T({}^2f))]\mathbf{g}_{\beta\gamma} = [8\pi + \partial_T({}^2f)]\mathbf{T}_{\beta\gamma}, \tag{28}$$

for the canonical d-connection $\mathbf{D}$, when the term ${}^2f(T)$ modifies both the gravitational constant and may compensate a cosmological constant $\Lambda = \Lambda_0$, or polarizations to $\Lambda(x)$ and possible contributions by Finsler modifications in ${}^{h}R$.

The theory described by (28) becomes an effective Einstein like theory if we impose the Levi–Civita conditions (15), $\mathbf{D} \to \nabla$, and fix such a parametrizations where ${}^{h}R + \Lambda(x) + 2({}^2f + 2\partial_T({}^2f)) = {}^{\nabla}R$, where ${}^{\nabla}R$ is the scalar curvature of $\nabla$. Nevertheless, variations of effective gravitational constant (27) are still possible. The gravitational field equations can be integrated in very general off-diagonal forms following methods elaborated in Refs. 20 and 21.



### 3.3. *Extracting scalar fields from modified/Finsler gravity*

#### 3.3.1. *Conformal transforms and effective scalar fields*

We show how the Finsler generating function $F(x, y(x)) := \chi(x)$ can be used to mimic scalar interactions in modified gravity with conformal transforms $\mathbf{g}_{\alpha\beta} \to {}^{\chi}\mathbf{g}_{\alpha\beta} = e^{\chi(x)}\mathbf{g}_{\alpha\beta}$. Let us consider a functional $f({}_{s}^{F}R, \chi)$ being an algebraic function on ${}_{s}^{F}R$ and $\chi$. We introduce the action

$$S = \int d^4x\sqrt{|\mathbf{g}|}\left\{\frac{1}{2\kappa^2}f({}_{s}^{F}R, \chi) + \mathcal{L}(e^{\chi(x)}\mathbf{g}_{\alpha\beta}, q_A)\right\}. \tag{29}$$

Varying on $\chi$, we get the relation, $\partial_\chi f({}_{s}^{F}R, \chi) = -\kappa^2 {}^{\chi}\mathbf{T}$, where the trace of energy–momentum tensor for the effective scalar field is computed ${}^{\chi}\mathbf{T} := {}^{\chi}\mathbf{g}^{\alpha\beta}{}^{\chi}\mathbf{T}_{\alpha\beta}$, see formula (26) for re-scaled metric $e^{\chi(x)}\mathbf{g}_{\alpha\beta}$, which results in ${}^{\chi}\mathbf{T}_{\alpha\beta}$.

There are parametrization where we can invert and find $\chi = \chi({}_{s}^{F}R, {}^{\chi}\mathbf{T})$ and then re-define $\widehat{f}({}_{s}^{F}R, {}^{\chi}\mathbf{T}) \equiv f({}_{s}^{F}R, \chi({}_{s}^{F}R, {}^{\chi}\mathbf{T}))$. Under conformal N-adapted transform

$\mathbf{g}_{\alpha\beta} \to e^{-\chi(x)}\mathbf{g}_{\alpha\beta}$, we induce from a modified Finsler action (29) an action for the canonical d-connection $\mathbf{D}$ modified gravity with effective scalar fields $\chi$

$$S = \int d^4 x e^{-2\chi(x)} \sqrt{|\mathbf{g}|} \left\{ \frac{1}{2\kappa^2} f({}^{\chi}_{s}R, \chi) + \mathcal{L}(\mathbf{g}_{\alpha\beta}, q_A) \right\},$$
$$\text{where } {}^{\chi}_{s}R = e^{\chi(x)} \left[ {}^{F}_{s}R + 3\mathbf{g}^{\alpha\beta} \left( (\mathbf{D}_{\alpha}\chi)(\mathbf{D}_{\beta}\chi) - \frac{1}{2}(\mathbf{e}_{\alpha}\chi)(\mathbf{e}_{\beta}\chi) \right) \right], \qquad (30)$$

when the N-elongated operators $\mathbf{e}_{\alpha}$ are certain frame transforms of (6) adapted to a nonholonomic $2+2$ splitting. Such bases are also physical frames if the matter fields $q_A$ couple minimally with $\mathbf{g}_{\alpha\beta}$ but not with $\chi$ and the action is re-written in the form

$$S = \int d^4 x e^{-2\chi(x)} \sqrt{|\mathbf{g}|} \left\{ \frac{1}{2\kappa^2} \widehat{f}({}^{\chi}_{s}R\ , \mathbf{T}) + \mathcal{L}(\mathbf{g}_{\alpha\beta}, q_A) \right\}, \qquad (31)$$

where $\mathbf{T} = {}^{\chi}\mathbf{T} + {}^{q}\mathbf{T}$ is computed for sets of fields $(\chi, q_A)$.

#### 3.3.2. *FLRW-geometries induced by Finsler modifications*

We briefly analyze possible cosmological implications of the models (30) and/or (31) when $\mathcal{L} = 0$ and $f({}^{\chi}_{s}R, {}^{\chi}\mathbf{T}) = {}^{F}_{s}R + f({}^{\chi}\mathbf{T})$ with a corresponding re-definition of $F(x, y(x)) := \chi(x)$ to get an effective[f] ${}^{\chi}\mathbf{T} = -\omega(\chi)\mathbf{g}^{\alpha\beta}(\mathbf{D}_{\alpha}\chi)(\mathbf{D}_{\beta}\chi) - 4V(\chi)$. Such a Finsler modified gravity theory

$$S = \int d^4 x \sqrt{|\mathbf{g}|} \left\{ \frac{1}{2\kappa^2} {}^{F}_{s}R + f[-\omega(\chi)\mathbf{g}^{\alpha\beta}(\mathbf{D}_{\alpha}\chi)(\mathbf{D}_{\beta}\chi)] - \frac{1}{2}\omega(\chi)\mathbf{g}^{\alpha\beta}(\mathbf{D}_{\alpha}\chi)(\mathbf{D}_{\beta}\chi) \right\}$$

contains (in our case, induced) $k$-essence cosmology models studied in Refs. 26–29. This follows from the Friedman equations

$$\frac{3}{2\kappa^2} H^2 = -f[\Phi] + \frac{1}{2}\Phi - (2\partial_{\Phi} f[\Phi] - \Phi)\Phi,$$
$$\frac{1}{\kappa^2}(3H^2 + 2\dot{H}) = f[\Phi] - \frac{1}{2}\Phi, \qquad (32)$$

for signature $(+,+,+,-)$, where $\Phi = \omega(\chi)\dot{\chi}^2$ is computed using the derivative on time-like variable $t$, $\dot{\chi} = \partial\chi/\partial t$.

We can construct a simple solution for the model

$$A(\chi) = f[-\omega(\chi)\mathbf{g}^{\alpha\beta}(\mathbf{D}_{\alpha}\chi)(\mathbf{D}_{\beta}\chi)] - \frac{1}{2}\omega(\chi)\mathbf{g}^{\alpha\beta}(\mathbf{D}_{\alpha}\chi)(\mathbf{D}_{\beta}\chi)$$
$$= A_0 \exp[-2\ln(\chi/\chi_0)\mathbf{g}^{\alpha\beta}(\mathbf{D}_{\alpha}\chi)(\mathbf{D}_{\beta}\chi)],$$

with some constants $A_0$ and $\chi_0$. The solution of (32) is very similar to that presented in Ref. 13, $H = H_0/t, \chi = t$, when $3H_0^2 - 2H_0 + (\kappa\chi_0)^2 A_0 = 0$. In general, $k$-essence

[f]We can introduce, for instance, a term $-4\mathcal{V}(\chi)$ for nonlinear interactions, as in various cosmological models; for simplicity, we restrict our considerations to models with $\mathcal{V}(\chi) = 0$.

models, there are no de Sitter type solutions if (in our denotations) $\chi = $ const. and $A(0) > 0$.

It is also interesting to note that physical implications of Finsler modifications in gravity via functionals $f(\ldots, F)$ may be very different than those in "standard" $k$-essence cosmology. For instance, the constraint $F(t, y(t)) = \chi(t) = $ const defines a nonholonomic distribution in spacetime which may be "fiber like" with formal extra dimensional velocity coordinates $y^i(t)$, for a $2+2$ distribution as we explained in footnote b. The evolution in time depends on such parametrization. Pseudo–Riemannian configurations can be obtained if $\mathbf{D} \to \nabla$ (15).

### 3.4. *Locally anisotropic motion and the Newtonian limit*

In $f(R, T, F)$ models, the divergence of energy–momentum tensor of matter (23) is not zero. For theories which can be modeled as an effective Einstein gravity, this is a consequence of distortion $\mathbf{D} = \nabla + \mathbf{Z}$ (12). Nontrivial distributions $F$ result in transferring solutions of $f(R, T)$ theories into Finsler like models and off-diagonal Einstein configurations. The coupling between metric, geometry and constraints on nonlinear dynamics induces supplementary accelerations acting on test particles. The goal of this section, is to study the equation of motion of test particles in dependence of both $f$- and $F$-functionals and distortions $\mathbf{Z}$. We shall derive the equations of motion, compute the Newtonian limits and investigate constraints on such theories which can be derived from the observational data.

#### 3.4.1. *Modified equations for anisotropic motion of test particles*

We compute the divergence (23) for the case of perfect fluid model energy–momentum tensor (25). Introducing the projector operator ${}^{\perp}\mathbf{g}_{\mu\lambda} = \mathbf{g}_{\mu\lambda} - \mathbf{v}_{\mu}\mathbf{v}_{\lambda}$, for which ${}^{\perp}\mathbf{g}_{\mu\lambda}\mathbf{v}^{\mu} = 0$ and ${}^{\perp}\mathbf{g}_{\mu\lambda}\mathbf{T}^{\lambda\nu} = -{}^{\perp}\mathbf{g}^{\nu}_{\lambda}p$, and following a calculus with $\mathbf{D}_{\alpha}$ and decompositions with respect to N-adapted frames $\mathbf{e}_{\nu}$ (6) and $\mathbf{e}^{\nu}$ (7) (see similar details for $\nabla$ in Sec. V of Ref. 13), we obtain that the equations of motion of a particle in background $(\mathbf{g}, \mathbf{D} = \{\Gamma^{\mu}{}_{\nu\lambda}\})$ can be expressed

$$\frac{d^2 u^{\mu}}{ds^2} + \Gamma^{\mu}{}_{\nu\lambda}\mathbf{v}^{\nu}\mathbf{v}^{\lambda} = (\mathbf{g}^{\mu\nu} - \mathbf{v}^{\nu}\mathbf{v}^{\mu})\mathbf{e}_{\nu}q. \tag{33}$$

The term $\mathbf{e}_{\nu}q$ can be found from divergence (23)

$$\mathbf{e}_{\nu}q = 8\pi(\mathbf{e}_{\nu}p)(\rho + p)^{-1}[8\pi + \partial_T f({}_{s}^{F}R, \mathbf{T}, F)]^{-1}. \tag{34}$$

Equation (34) can be integrated using approximative methods and additional assumptions on the matter fluid model. For instance, we can chose a linear barotropic equation of state, $p = w\rho$, with a constant $w \ll 1$, when $\rho + p \approx \rho$ and $\mathbf{T} = \rho - 3p$. The value $\partial_T f$ depends only on $\rho$ and $F$ and the deviations from geodesic motion are determined by a term $\partial_T f = {}^{1}\xi(\rho) + {}^{2}\xi(\rho, F)$. Such contributions split into two terms: ${}^{1}\xi$ derived from modifications of type $f(R, T)$ and ${}^{2}\xi$

derived from an anisotropic $F$. Fixing a value $\rho = \rho_0$, we can expend

$$\partial_T f = {}^1\xi(\rho_0) + {}^2\xi\,(\rho_0, F) + (\rho - \rho_0)\left[\frac{\partial^1\xi}{\partial\rho}_{|\rho_0} d\rho + \frac{\partial^2\xi}{\partial\rho}_{|\rho_0} d\rho\right]$$
$$= 8\pi[{}^1a + {}^2a(F) + ({}^1b + {}^2b(F))(\rho - \rho_0)],$$

where ${}^1a = {}^1\xi/8\pi, {}^1b = \frac{\partial^1\xi}{\partial\rho}_{|\,\rho_0} d\rho$ and the anisotropic (Finsler generating depending generating function coefficients) are ${}^2a(F) = {}^2\xi\,/8\pi, {}^2b = \frac{\partial^2\xi}{\partial\rho}_{|\rho_0} d\rho$. We can write (34) in the form

$$[{}^1a + {}^2a(F) - ({}^1b + {}^2b(F))\rho_0]\mathbf{e}_\nu q$$
$$= w\mathbf{e}_\nu \ln\left\{\frac{\rho}{[{}^1a + {}^2a(F) + ({}^1b + {}^2b(F))(\rho - \rho_0)]}\right\},$$

and get the approximate solution

$$q = \ln\left\{\left[\frac{C\rho}{{}^1a + {}^2a(F) + ({}^1b + {}^2b(F))(\rho - \rho_0)}\right]^{w/[{}^1a + {}^2a(F) - ({}^1b + {}^2b(F))\rho_0]}\right\}, \qquad (35)$$

where $C$ is an integration constant.

The solution (35) depends parametrically on generating Finsler function via ${}^2a(F)$ and ${}^2b(F)$ and a logarithmic anisotropic variations on $\rho$. Such solutions cannot be expressed in exact form and there are necessary certain approximate series decompositions.

We extended the Raychaudhuri equations by using $\mathbf{D}$ connections in the framework of a tangent Lorentz bundle $T\mathbf{V}$. We consider a nonlinear congruence of geodesics (33) and we use an analogous method with Ref. 30 for deriving of Raychaudhuri equations, for a velocity field $\mathbf{v}^\mu$ on $T\mathbf{V}$ the commutation relations of $\mathbf{v}^\mu$ gives us

$$\mathbf{D}_\alpha\mathbf{D}_\beta\mathbf{v}^\gamma - \mathbf{D}_\beta\mathbf{D}_\alpha\mathbf{v}^\gamma = \mathbf{R}^\gamma{}_{\epsilon\alpha\beta}\mathbf{v}^\epsilon - \mathbf{T}^\delta_{\alpha\beta}\mathbf{D}_\delta\mathbf{v}^\gamma - \mathbf{T}^\epsilon_{\alpha\beta}\mathbf{D}_\epsilon\mathbf{v}^\gamma.$$

Imposing the relations $\mathbf{g}^\perp_{\mu\nu}\mathbf{v}^\mu = 0$ and $\mathbf{v}^\alpha\mathbf{D}_\beta\mathbf{v}_\alpha = 0$, the previous equation is written

$$\mathbf{v}^\beta\mathbf{D}_\alpha\mathbf{D}_\beta\mathbf{v}^\gamma = -\mathbf{v}^\beta\mathbf{v}_\alpha\mathbf{D}_\beta\mathbf{v}^\gamma + \mathbf{R}^\gamma{}_{\epsilon\alpha\delta}\mathbf{v}^\epsilon\mathbf{v}^\delta - \mathbf{T}^\delta_{\beta\alpha}\mathbf{D}_\delta\mathbf{v}^\beta\mathbf{v}^\gamma - \mathbf{T}^\epsilon_{\alpha\beta}\mathbf{v}^\beta\mathbf{D}_\epsilon\mathbf{v}^\gamma.$$

We decompose the term $\mathbf{D}_\alpha\mathbf{v}^\beta$ with respect to the kinematics terms of expansion, shear and vorticity $\theta, \sigma, \omega$ as

$$\mathbf{D}_\alpha\mathbf{v}^\beta = \frac{1}{7}\theta\mathbf{h}^\beta_\alpha + \sigma^\beta_\alpha + \omega^\beta_\alpha,$$

where correspondingly, $\theta = \mathbf{D}_\alpha\mathbf{v}^\beta\mathbf{h}^\alpha_\beta,\ \sigma_{\alpha\beta} = \mathbf{D}_\alpha\mathbf{v}_\beta + \mathbf{D}_\beta\mathbf{v}_\alpha - \frac{1}{7}\theta\mathbf{h}_{\alpha\beta}, \omega_{\alpha\beta} = \mathbf{D}_\alpha\mathbf{v}_\beta - \mathbf{D}_\beta\mathbf{v}_\alpha,\ \mathbf{h}^\alpha_\beta = \mathbf{g}^{\alpha\gamma}\mathbf{h}_{\beta\gamma}$. We obtain

$$\mathbf{v}^\alpha\mathbf{D}_\alpha\theta = \mathbf{R}_{\alpha\beta}\mathbf{v}^\alpha\mathbf{v}^\beta - \mathbf{T}^\delta_{\alpha\gamma}\mathbf{v}^\alpha\mathbf{D}_\delta\mathbf{v}^\gamma - \mathbf{D}_\alpha\mathbf{v}^\beta\ \mathbf{D}_\beta\mathbf{v}^\alpha$$
$$= \mathbf{R}_{\alpha\beta}\mathbf{v}^\alpha\mathbf{v}^\beta - \mathbf{T}^\delta_{\alpha\beta}\left(\frac{1}{3}\theta\mathbf{h}^\beta_\delta + \sigma^\beta_\delta + \omega^\beta_\delta\right)\mathbf{v}^\alpha - \frac{1}{3}\theta - \sigma^\alpha_\beta\sigma^\beta_\alpha - \omega^\alpha_\beta\omega^\beta_\alpha.$$

These equations are the (modified) Raychaudhuri equations on a Lorentz tangent bundle. For vanishing nonholonomically induced torsion structures, they transform into the well known Raychaudhuri equations for the Levi–Civita connection, but on the tangent bundle to a pseudo-Riemannian manifold.

Using two test particles, we can study spacetime structure in Finsler like gravity theories using (modified) Raychaudhuri equations. There are possible also observable locally anisotropic effects on a single test particle which do not follow standard geodesic equations as in general relativity but certain nonlinear geodesic ones, see footnote d and Eq. (33) for modified equations of motion of a test particle in a Finsler like background.

### 3.4.2. *Corrections to the Newton law and perihelion effects*

We use pressureless dust and associated variational principle to study modifications and anisotropies in the Newtonian limit of theories. The equations for nonlinear geodesics (33) of test particles can be derived from $\delta^{p} S = 0$, where

$$ {}^{p}S = \int {}^{p}L ds, \quad {}^{p}L = e^{q}\sqrt{|\mathbf{g}_{\alpha\beta}v^{\alpha}v^{\beta}|}, $$

and limit of weak gravitational fields of $ds$ is characterized by $ds \approx (1+2\varphi - \mathbf{v}^2)^{1/2} \approx (1+\varphi - \mathbf{v}^2/2)dt$. In the above formula, $\varphi$ is the Newton potential and $\mathbf{v}$ is the velocity of the fluid in the 3-d space.

The solution (35) can approximated $e^{q} \approx 1 + U(\rho, F)$, for

$$ U(\rho,F) = w[{}^{1}a + {}^{2}a(F) - ({}^{1}b + {}^{2}b(F))\rho_0]^{-1} \ln \times \left\{ \frac{C\rho}{[{}^{1}a + {}^{2}a(F) + ({}^{1}b + {}^{2}b(F))(\rho - \rho_0)]} \right\}, $$

and the variation of respective action is $\delta^{p} S = \delta[1 + U(\rho, F) + \varphi - \mathbf{v}^2/2]dt = 0$. This allows us to compute the 3-d acceleration of the particle using the 3-d gradient grad (for nontrivial $F$, we should perform a N-adapted calculus of this gradient),

$$ {}^{\mathrm{tot}}\mathbf{a} = -\mathrm{grad}\,[\varphi + U(\rho, F)] = {}^{s}\mathbf{a} + {}^{p}\mathbf{a} + {}^{N}\mathbf{a} + {}^{E}\mathbf{a}, $$

where ${}^{s}\mathbf{a} = -\mathrm{grad}\,\varphi$

$$ \begin{aligned} {}^{s}\mathbf{a} &= -\mathrm{grad}\,\varphi, \text{the Newtonian gravitational acceleration}, \\ {}^{p}\mathbf{a}(\rho,p,F) &= -\frac{C}{{}^{1}a + {}^{2}a(F) - ({}^{1}b + {}^{2}b(F))\rho_0}\frac{1}{\rho}\,\mathrm{grad}\,p, \\ &\quad \text{the hydrodinamical acceleration}, \\ {}^{N}\mathbf{a}(\dots,F) &\approx \text{N-connection terms}, \\ {}^{E}\mathbf{a}(\rho,p,F) &= \frac{{}^{1}b + {}^{2}b(F)}{1 + {}^{1}a + {}^{2}a(F) - ({}^{1}b + {}^{2}b(F))\rho_0} \\ &\quad \times \frac{\mathrm{grad}\,p}{1 + {}^{1}a + {}^{2}a(F) + ({}^{1}b + {}^{2}b(F))(\rho_0 - \rho)}, \end{aligned} \tag{36} $$

where the last term ${}^{E}\mathbf{a}$ is a supplementary acceleration induced from modifications of the action and (Finsler) anisotropies of gravitational field. Such a term, ${}^{E}\mathbf{a} \simeq {}^{E}a(\rho, p)$, was used in Sec.V(c) of Ref. 13 for computing possible modification of perihelion procession of Mercury

$$\triangle\varphi = \frac{6\pi G M_{\odot}}{a(1-\epsilon^2)} + \frac{2\pi a^2\sqrt{1-\epsilon^2}}{G M_{\odot}}\ ({}^{E}a),$$

where $\epsilon$ is the eccentricity of orbit, $a$ is the distance between Mercury and Sun, $M_{\odot}$ is the Sun's mass. The observational data constrain for modifications resulting from a $f(R,\rho)$ theory where computed ${}^{E}a \leq 1.28 \times 10^{-9}$ cm/s$^2$. The terms ${}^{p}\mathbf{a}, {}^{N}\mathbf{a}, {}^{E}\mathbf{a}$ in (36) depend anisotropically on $F$. So, we have to take into consideration such terms when the anisotropic effects and constraints for the $f(R,\rho,F)$ models are computed.



## 4. Decoupling and Integrability of $f(R,T,F)$ Gravity

The field equations in modified gravity theories are very "sophisticated" systems of nonlinear PDE. Surprisingly, it is possible to decouple and integrate such PDE in general forms using the anholonomic deformation method[20,25] (see also references therein). In this section, we reformulate the method for constructing generic off-diagonal solutions for the $f(R,T,F)$ gravity, with trivial or nontrivial contributions from a Finsler generating function $F$, when $\mathbf{g}_{\alpha\beta}(u^{\gamma}) = \tilde{\mathbf{g}}_{\alpha\beta}(u^{\gamma}, y^{\alpha}(u^{\mu}))$ (17). Finally, we shall provide examples of solutions for ellipsoid and solitonic configurations.

### 4.1. *The anholonomic deformation method for modified gravity*

For simplicity, we shall prove integrability of the system (28) for $\mathbf{T}_{\beta\gamma} = 0$,

$$\mathbf{R}^{\beta}_{\ \gamma} \ = \ \frac{1}{2}\Upsilon(x^i)\delta^{\beta}_{\ \gamma}, \qquad (37)$$

$$\text{where } \frac{1}{2}\Upsilon(x^i) := \tilde{\Lambda}(x^i) + \ [{}^{2}f + 2\partial_T({}^{2}f)], \qquad (38)$$

for $[{}^{2}f + 2\partial_T({}^{2}f)]_{|T=0}$ being a nontrivial function on $x^i$ and an effective anisotropically polarized cosmological "constant" $\tilde{\Lambda}(x^i) = \frac{1}{2}[{}^{h}R(x^i) + \Lambda(x^i)]$. The source $\Upsilon(x^i)$ contains information on possible contributions from a Finsler generating function $F$ and modifications by ${}^{2}f(\mathbf{T})$. The solutions of this system of nonlinear PDE define nonholonomic Einstein manifolds.

#### 4.1.1. *Decoupling of field equations*

The decoupling property can be proven for metrics with one Killing symmetry on $\partial/\partial y^4$, with local coordinates $u^{\alpha} = (x^1, x^2, y^3, y^4)$, when

$$\mathbf{g} = \epsilon_i e^{\psi(x^k)} dx^i \otimes dx^j + h_3(x^k, y^3)\mathbf{e}^3 \otimes \mathbf{e}^3 + h_4(x^k, y^3)\mathbf{e}^4 \otimes \mathbf{e}^4, \qquad (39)$$

for $\mathbf{e}^3 = dy^3 + w_i(x^k, y^3)dx^i,\ \mathbf{e}^4 = dy^4 + n_i(x^k, y^3)dx^i$. Such metrics are of type (7) up to frame transforms, $\epsilon_i = \pm 1$ depending on a chosen signature for the spacetime metric. We shall use brief denotations of partial derivatives, for instance, $s^\bullet = \partial/\partial x^1, s' = \partial/\partial x^2$ and $s^* = \partial/\partial y^3$ and construct exact solutions in such N-adapted frames when $h_4^* \neq 0$ and $\Upsilon(x^i) \neq 0$.

**Gravitational field equations for D**: For ansatz (39), the nonlinear PDEs (37) are equivalent to

$$\epsilon_1 \psi^{\bullet\bullet} + \epsilon_2 \psi'' = \Upsilon, \tag{40}$$

$$\phi^* (\ln|h_4|)^* = \Upsilon h_3, \tag{41}$$

$$\beta w_i + \alpha_i = 0, \tag{42}$$

$$n_i^{**} + \gamma n_i^* = 0, \tag{43}$$

where the coefficients

$$\gamma = (\ln|h_4|^{3/2} - \ln|h_3|)^*, \quad \alpha_i = h_4^* \partial_i \phi \quad \text{and} \quad \beta = h_4^* \phi^*, \tag{44}$$

are determined by $h_3$ and $h_4$ via

$$\phi = \ln|2(\ln\sqrt{|h_4|})^*| - \ln\sqrt{|h_3|}. \tag{45}$$

See detailed computations of the N-adapted coefficients for the Ricci and Einstein tensors in Refs. 20 and 25.

The above system of equations (40)–(43) reflects a very important decoupling property of the Einstein equations for certain classes of metric compatible linear connections and with respect to N-adapted frame (in this section, we consider for simplicity only metrics with one Killing symmetry):

(i) Depending on signature, Eq. (40) is a 2-d D'Alambert, or Laplace, equation which can be integrated for arbitrary source $\Upsilon(x^k)$.
(ii) The system of two equations (41) and (45) is for three unknown functions $h_3(x^k, y^3), h_4(x^k, y^3)$ and $\phi(x^k, y^3)$ if a source $\Upsilon(x^k)$ is prescribed. It contains only partial derivatives $* = \partial/\partial y^3$. We can integrate in general form and define certain functionals $h_3[\phi]$ and $h_4[\phi]$ for any prescribed generating function $\phi, \phi^* \neq 0$, and integration functions and parameters, see below formulas (50).
(iii) Equations (42) and (43) are respectively algebraic ones (for $w_i$) and contains only first and second derivatives on $\partial/\partial y^3$ of $n_i$. For any defined $h_3$ and $h_4$, we can compute the coefficients $\alpha_i, \beta$ and $\gamma$ following formulas (44) and integrate all equations for $N$-coefficients in general form.

We conclude that with respect to N-adapted frames (6) and (7) determined by $N_i^a = (w_i, n_i)$ the modified Einstein equations (37) decouple into PDE with derivatives of 2-d and 1st-order, and algebraic equations for corresponding coefficients of metric.[g] This property can be proven in explicit form for **D** and contain additional

[g]Via conformal and frame transforms and introducing additional multiples, we can prove decoupling properties for various classes of metrics depending on $y^4$, i.e. on all coordinated on a manifold **V** of finite dimension. For simplicity, we omit such constructions in this work.

information on $f(R,T,F)$ modifications of gravity via source $\Upsilon(x^k)$ and induced torsion of $\mathbf{D}$.

In this work, we construct solutions with nonzero $\phi^*$ and $\Upsilon(x^k)$ because we are interested to investigate possible nontrivial contributions from modified gravity via nontrivial sources $\Upsilon(x^k)$. Vacuum configurations $\phi^* = 0$ and $\Upsilon(x^k) = 0$ can be studies by similar methods, see Refs. 20 and 25.

**Constraints for the Levi–Civita connection** $\nabla$: For ansatz of type (39), the zero-torsion constraints (15) can be satisfied if

$$w_i^* = \mathbf{e}_i \ln|h_4|, \quad \partial_i w_j = \partial_j w_i, n_i^* = 0. \tag{46}$$

The first condition "brock" the decoupling property of the system (40)–(43). The constraints of vanishing the torsion for $\nabla$ relate additionally the N-coefficients with $h_4$. Nevertheless, we can solve such conditions in explicit form via additional frame and coordinate transforms and/or re-parametrization of generating functions etc. For instance, we may fix any convenient value for $\ln|h_4|$ and use it as a generating function in the system (41) and (45) in order to define $h_3$ and $\phi$. Next step will consist in determining $w_i$ from algebraic equations (42). Equation (43) became trivial for $n_i^* = 0$ which allows us to introduce any $n_i(x^k), \partial_k n_i = \partial_i n_k$, in the off-diagonal metric ansatz.

In general, we can consider (46) as a class of nonholonomic constraints on integral varieties of solutions for $\mathbf{D}$ which results in subvarieties with torsionless configurations for $\nabla$.

#### 4.1.2. *General solutions for modified field equations*

The $h$-metric is given by $\epsilon_i e^{\psi(x^k)} dx^i \otimes dx^j$, where $\psi(x^k)$ is a solution of (40) considered as a 2-d d'Alambert/Laplace equation (40). It depends on $f(R,T,F)$ via source $\Upsilon(x^k)$ (38).

As a second step, we integrate in general form the system (41) and (45), for $\phi^* \neq 0$. Defining $A := (\ln|h_4|)^*$ and $B = \sqrt{|h_3|}$, we re-write such equations in the form

$$\phi^* A = \Upsilon B^2, \quad B e^{\phi} = 2A. \tag{47}$$

Considering $B \neq 0$, we obtain $B = (e^{\phi})^*/2\Upsilon$ as a solution of a system of quadratic algebraic equations. This formula can integrated on $dy^3$ which allows us to find $\sqrt{|h_3(x^k, y^3)|} = \sqrt{|{}^0h_3(x^k)|} + \partial_3 e^{\phi(x^k,y^3)}/2\Upsilon(x^k)$. We can write

$$h_3 = \frac{{}^0h_3(1 + (e^{\phi})^*}{2\Upsilon\sqrt{|{}^0h_3|})^2}, \tag{48}$$

if the local signature of the term ${}^0h_3$ is the same as $h_3$. Introducing the value $h_3$ in (47) and integrating on $y^3$, we find

$$h_4 = {}^0h_4 \exp[(8\Upsilon)^{-1}\ e^{2\phi}],$$

where ${}^0h_4 = {}^0h_4(x^k)$ is an integration function.

The N-connection coefficients can be found from (42), $w_i = -\partial_i\phi/\phi^*$, and integrating two times on $y^3$ in (43),

$$n_k = {}^1n_k + {}^2n_k \int \frac{dy^3 h_3}{(\sqrt{|h_4|})^3}, \tag{49}$$

where ${}^1n_k(x^i)$ and ${}^2n_k(x^i)$ are integration functions.

Putting all terms together in (39), we obtain a formal general solution of the gravitational field equations (37) in $f(R,T,F)$ gravity via quadratic element

$$ds^2 = \epsilon_i e^{\psi[\Upsilon]}(dx^i)^2 + {}^0h_3\left(1+\frac{(e^\phi)^*}{2\Upsilon\sqrt{|{}^0h_3|}}\right)^2 \left[dy^3 - \frac{\partial_i\phi}{\phi^*}dx^i\right]^2 + {}^0h_4 \exp\left[(8\Upsilon)^{-1}e^{2\phi}\right]\left[dy^4 + \left({}^1n_k + {}^2n_k\int dy^3\frac{h_3}{(\sqrt{|h_4|})^3}\right)dx^i\right]^2. \tag{50}$$

Such solutions depend on generating functions $\phi(x^i, y^3)$ and $\psi[\Upsilon(x^k)]$ and on integration functions ${}^0h_3(x^k), {}^0h_4(x^k)$, ${}^1n_k(x^k), {}^2n_k(x^k)$ as we described in above formulas. This class of modified spacetimes are characterized by nontrivial torsion with coefficients computed following formulas (11) using only the coefficients of metric (and respective N-connection). In general, we can introduce additional parameters and derive new symmetries because of existing Killing symmetry, see Refs. 20, 23 and 25. We can also consider that the class of solutions (50) is for the $f(R,T)$ gravity when certain Finsler like variables were introduced in order to be able to decouple the field equations and get very general classes of solutions as effective 4-d nonholonomic Einstein equations.

Constraining additionally the class of generating and integration functions in (50), we construct exact solutions for the Levi–Civita connection $\nabla$. We have to consider solutions with ${}^2n_k = 0, \partial_i({}^1n_k) = \partial_k({}^1n_i)$ and $w_i = -\partial_i\phi/\phi^*$ and $h_4$ are subjected to conditions (46). Even for solutions with $\nabla$, we get only effective Einstein spaces with locally anisotropic polarizations. This is because the source $\Upsilon(x^k)$ (38) determines the diagonal coefficients of (50), with respect to N-adapted frames. This results, in general, in effective polarization of the gravitational constant as in (27). We can generate solutions for Einstein manifolds if we fix $\Upsilon(x^k) = \text{const}$.

Finally, we note that metrics of type (50) are generic off-diagonal, i.e. we cannot diagonalize such solutions via coordinate transform. This follows from the fact that the anholonomy coefficients, see formulas (8), are not zero for arbitrary generating and integration functions.

### 4.2. *Examples of exact solutions*

Quadratic elements (50) parametrize formal integrals of a system of nonlinear PDE for modified gravity. It may describe certain physical real situations if the generating and integration functions and parameters are subjected to realistic boundary/asymptotic conditions with associated symmetries and conservation laws.

In spherical coordinates $u^\alpha = (x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t)$, a diagonal metric

$$^\circ\mathbf{g} = \underline{q}^{-1}(r)dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta d\varphi \otimes d\varphi - \underline{q}(r)dt \otimes dt, \tag{51}$$

defines an empty de Sitter space if $\underline{q}(r) = 1 - 2\frac{m(r)}{r} - \Lambda\frac{r^2}{3}$, where $\Lambda$ is a cosmological constant. The total mass–energy within the radius $r$ is defined by a function $m(r)$. For $m(r) = 0$ we obtain an empty space with a cosmological horizon at $r = r_c = \sqrt{3/\Lambda}$. If $m(r) = m_0 = \text{const}$ and $\Lambda = 0$, we get the Schwarzschild solution. The metric (51) is an example of diagonal solution of (37) in GR, when $f(R,T,F) = R$, $N_i^a = 0, \mathbf{D} = \nabla$ and $\frac{1}{2}\Upsilon(x^i) = \Lambda = \text{const}$, see source (38).

In this section, we analyze two classes of solutions related to possible $f(R,T,F)$ modifications of GR. In the first case, we construct metrics for possible off-diagonal deformations and polarizations of coefficients of de Sitter black holes resulting in ellipsoidal configurations. In the second case, the de Sitter black holes are embedded self-consistently into certain solitonic background configurations.



#### 4.2.1. *Ellipsoid configurations in $f(R,T,F)$ gravity*

The generic off-diagonal ansatz is chosen

$$\begin{aligned} ds^2 &= e^{\psi(\xi,\vartheta)}(d\xi^2 + \ d\vartheta^2) + h_3(\xi,\vartheta,\varphi)(\mathbf{e}_\varphi)^2 + h_4(\xi,\vartheta,\varphi)(\mathbf{e}_t)^2, \\ \mathbf{e}_\varphi &= d\varphi + w_1(\xi,\vartheta,\varphi)d\xi + w_2(\xi,\vartheta,\varphi)d\vartheta, \\ \mathbf{e}_t &= dt + n_1(\xi,\vartheta,\varphi)d\xi + n_2(\xi,\vartheta,\varphi)d\vartheta, \end{aligned} \tag{52}$$

for $h_3 = \eta_3(\xi,\theta,\varphi)r^2(\xi)\sin^2\theta, h_4 = \eta_4(\xi,\theta,\varphi)\varpi^2(\xi)$, local coordinates $x^1 = \xi, x^2 = \vartheta = r(\xi)\theta, y^3 = \varphi, y^4 = t$, with $\xi = \int dr/|\underline{q}(r)|^{\frac{1}{2}}$. We get a diagonal configuration if $w_i = 0, n_i = 0, \eta_3 = 1, \eta_4 = 1$ and $\psi = 0$,

$$^\circ\mathbf{g} = d\xi \otimes d\xi + r^2(\xi)d\theta \otimes d\theta + r^2(\xi)\sin^2\theta d\varphi \otimes d\varphi - \underline{q}(\xi)dt \otimes dt, \tag{53}$$

with coefficients $\check{g}_1 = 1, \check{g}_2 = r^2(\xi), \check{h}_3 = r^2(\xi)\sin^2\theta, \check{h}_4 = -q(\xi)$. In variables $(r,\theta,\varphi)$, the metric (53) is equivalent to (51).

The ansatz (52) is an example of solutions of type (50) if the coefficients are generated following similar methods taking ${}^0h_3 = \check{h}_3$ and ${}^0h_4 = \check{h}_4$.

The coefficients of this metric determine exact solutions if

$$\begin{aligned} &\psi^{\bullet\bullet}(\xi,\vartheta) + \psi''(\xi,\vartheta) = \Upsilon(\xi,\vartheta), \\ &h_3 = \frac{\check{h}_3(1 + \partial_\varphi(e^\phi)}{2\Upsilon\sqrt{|\check{h}_3|})^2}, \quad h_4 = \check{h}_4 \exp[(8\Upsilon)^{-1}e^{2\phi}], \\ &w_i = -\partial_i\phi/\phi^*; \quad n_i = {}^1n_i(\xi,\vartheta) + {}^2n_i(\xi,\vartheta)\int \frac{d\varphi h_3}{(\sqrt{|h_4|})^3}, \end{aligned} \tag{54}$$

for any nonzero $h_a$ and $h_a^*$, and (integrating) functions ${}^1n_i(\xi,\vartheta), {}^2n_i(\xi,\vartheta)$ and generating function $\phi(\xi,\vartheta,\varphi)$.

For nonholonomic ellipsoid de Sitter configurations (for simplicity, we consider rotoid configurations with small eccentricity $\varepsilon$), we parametrize

$$\begin{aligned} {}^{\rm rot}_{\lambda}\mathbf{g} &= e^{\psi(\xi,\vartheta)}(d\xi^2 + d\vartheta^2) + r^2(\xi)\sin^2\theta\left(\frac{1+\partial_\varphi e^{\phi}}{2\Upsilon\sqrt{|\underline{q}|}}\right)^2 \\ &\quad \times \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi - [\underline{q}(\xi) + \varepsilon\zeta(\xi,\vartheta,\varphi)]\mathbf{e}_t \otimes \mathbf{e}_t, \\ \mathbf{e}_\varphi &= d\varphi - \frac{\partial_\xi \phi}{\partial_\varphi \phi}d\xi - \frac{\partial_\vartheta \phi}{\partial_\varphi \phi}d\vartheta, \quad \mathbf{e}_t = dt + n_1(\xi,\vartheta,\varphi)d\xi + n_2(\xi,\vartheta,\varphi)d\vartheta, \end{aligned} \tag{55}$$

where $n_i$ (49) are computed in some forms $n_i \sim \varepsilon \ldots$ for corresponding coordinates and values $h_3$ and $h_4$. The function

$$\zeta = \underline{\zeta}(\xi,\vartheta)\sin(\omega_0\varphi + \varphi_0), \tag{56}$$

for some constant parameters $\omega_0$ and $\varphi_0$, we can state $\underline{\zeta}(\xi,\vartheta) \simeq \underline{\zeta} = \text{const}$, is chosen to generate an anisotropicrotoid configuration for the smaller "horizon" (when the term before $\mathbf{e}_t \otimes \mathbf{e}_t$ became $h_4 = 0$),

$$\frac{r_+ \simeq 2\ m_0}{(1 + \varepsilon\underline{\zeta}\sin(\omega_0\varphi + \varphi_0))},$$

where $\varepsilon$ is the eccentricity. The generating function ${}_{|}\phi(\xi,\tilde{\vartheta},\varphi)$ contained in (54) is related to $\zeta(\xi,\tilde{\vartheta},\varphi)$ via formula $e^{2\phi} = 8\Upsilon \ln|1 - \varepsilon\zeta/\underline{q}(\xi)|$, for which a rotoid configuration (56) can be fixed. We construct rotoid deformations of the de Sitter black hole metric (51) if the function $\zeta$ (56) is introduced into the last formula and define a generating function/functional $\phi = \phi(\underline{q},\Upsilon,\zeta,F)$. In general, such off-diagonal deformations do not result in other classes of black hole solutions. If we consider small deformations on parameter $\varepsilon$ for which

$$h_3 = \check{h}_3(1+\varepsilon\chi_3), \quad h_4 = \check{h}_4(1+\varepsilon\chi_4), \quad w_i \sim \varepsilon\check{w}_i, \quad n_i \sim \varepsilon\check{n},$$

metrics of type (55) describe stationary black ellipsoid solutions with coefficients computed with respect to N-adapted frames, see discussion and references in Refs. 23 and 20. If we restrict the integral variety of such solutions to satisfy the conditions (46), we generate exact off-diagonal solutions for the Levi–Civita connection $\nabla$.

#### 4.2.2. *Black holes and locally anisotropic solitonic backgrounds*

Another example of off-diagonal solutions with local anisotropies in modified gravity can be constructed as a nonlinear superposition of the de Sitter black hole solution and solitonic waves. We consider a nonstationary ansatz

$$ds^2 = e^{\psi(\xi,\vartheta)}[d\xi^2 + d\vartheta^2] - \underline{q}(\xi)\left(1 + \frac{\partial_t e^{\phi(\xi,\vartheta,t)}}{2\Upsilon(\xi,\vartheta)\sqrt{|\underline{q}(\xi)|}}\right)^2 \left[dt - \frac{\partial_\xi \phi}{\partial_t \phi}d\xi - \frac{\partial_\vartheta \phi}{\partial_t \phi}d\vartheta\right]^2$$
$$+ r^2(\xi)\sin^2\vartheta \exp[(8\Upsilon(\xi,\vartheta))^{-1}e^{2\phi(\xi,\vartheta,t)}]$$
$$\times\left[d\varphi + \left({}^1n_1(\xi,\vartheta) + {}^2n_1(\xi,\vartheta)\int dt \frac{h_3(\xi,\vartheta,t)}{(\sqrt{|h_4(\xi,\vartheta,t)|})^3}\right)d\xi\right.$$
$$\left. + \left({}^1n_2(\xi,\vartheta) + {}^2n_2(\xi,\vartheta)\int dt \frac{h_3(\xi,\vartheta,t)}{(\sqrt{|h_4(\xi,\vartheta,t)|})^3}\right)d\vartheta\right]^2, \qquad (57)$$

for local coordinates $x^1 = \xi, x^2 = \vartheta, y^3 = t, y^4 = \varphi$ and $\underline{q}(\xi) = \underline{q}(r(\xi))$ computed as in (51).

**Solitonic backgrounds with radial Burgers equation:** We take $\phi(\xi,\vartheta,t) = \eta(\xi,\vartheta,t)$, when $y^3 = t$ is a time like coordinate, as a solution of KdP equation,[31,32]

$$\pm\eta'' + (\partial_t \eta + \eta\eta^{\bullet} + \epsilon\eta^{\bullet\bullet\bullet})^{\bullet} = 0, \qquad (58)$$

with dispersion $\epsilon$ and possible dependencies on a set of parameters $\theta$. It is supposed that in the dispersionless limit $\epsilon \to 0$ the solutions are independent on $x^2$ and determined by Burgers' equation $\partial_t \eta + \eta\eta^{\bullet} = 0$. Introducing generating functions $\phi$ determined by solutions of such 3-d solitonic equations in (57), we generate solitonic nonholonomic deformations of the de Sitter black hole solutions. In general, the new off-diagonal solutions do not have black hole properties. We can consider other types of solitonic solutions. Such configurations always define exact solutions of gravitational field equations (37) for $\mathbf{D}$. Constraining the solitonic integral varieties via conditions (46), we generate solutions for the Levi–Civita connection $\nabla$.

**Solitonic backgrounds with angular Burgers equation:** In this case $\phi = \widehat{\eta}(\xi,\vartheta,t)$ is a solution of KdP equation

$$\pm\widehat{\eta}^{\bullet\bullet} + (\partial_t \widehat{\eta} + \widehat{\eta}\widehat{\eta}' + \epsilon\widehat{\eta}''')' = 0. \qquad (59)$$

In the dispersionless limit $\epsilon \to 0$, the solutions are independent on $x^1 = \xi$ and determined by Burgers' equation $\partial_t \widehat{\eta} + \widehat{\eta}\ \widehat{\eta}' = 0$. Introducing $\phi = \widehat{\eta}$ in (57), we generate solutions of (37) with angular anisotropy. For small values of $\widehat{\eta}$, we can model 3-d solitonic polarizations of the de Sitter black holes.

Finally, we note that nonholonomic constraints and off-diagonal interactions with terms induced from modified and/or locally anisotropic gravity (for instance, via solitonic waves) may preserve the black hole character of certain classes or "disperse" them into effective nonlinear vacuum configurations with polarized cosmological constants.

## 5. Discussion and Conclusions

In this work, we have merely presented a flavor of a geometric formalism in modified $f(R,T)$ theories starting from principles of general covariance and relativity for nonholonomic deformations of fundamental geometric/physical objects and field equations on Lorentz manifolds and their tangent bundles. The goal was to prove a general decoupling property and further integrability in very general forms of gravitational field equations. Surprisingly, such systems of nonlinear PDE can be solved in general forms with respect to certain classes of nonholonomic frames for (auxiliary) Finsler type connections. Imposing nonholonomic constraints on certain classes of generic off-diagonal integral varieties, the solutions can be transformed into configurations for the torsionless Levi–Civita connection.

There are two very different approaches to connect Finsler like geometries to Einstein gravity and modifications:

(i) The first one is to introduce on (pseudo) Riemannian/Lorentz manifolds a nonintegrable (nonholonomic) $2+2$ splitting with a conventional fibered structure. Such constructions are very similar to those in Finsler–Cartan geometry with metric compatible connections. A reason to introduce Finsler like variables in GR is that we can elaborate a geometric method of integrating the gravitational field equations. Here we note that it is possible to elaborate also certain new geometric methods of quantization of gravity theories using almost Kähler–Finsler variables (following respective gauge like, A-brane and deformation quantization formalisms). In such cases, a Finsler geometry is modeled via nonholonomic distributions as an auxiliary tool and does not change the paradigm of an originally considered Einstein or modified gravity theory.

(ii) The second approach is related to more fundamental locally anisotropic modifications of the concept of spacetime and gravitational interactions. They can be motivated by certain theoretical arguments, for instance, in quantum gravity, modified dispersion relations, and theories with possible violations of local Lorentz symmetry, etc. Roughly speaking, a very general class of modified theories of gravity have to be elaborated on tangent bundles of Lorentz manifolds when certain geometric/physical principles are used for extensions of GR to Einstein–Finsler type gravity theories. Such models may play a physically important role because they accept an axiomatic very similar to that for the Einstein gravity, such theories can be integrated in very general form and there are known well defined methods of quantization of such models. Certain anisotropic solutions from Finsler gravity seem to play an important role in explaining locally anisotropic effects in modern cosmology. On tangent Lorentz manifolds, the gravity models are for an extra dimensional spacetimes (eight dimensions) which can be reduced to “effective” 4-d metrics via osculating procedure.

In another turn, the late-time acceleration of the universe and dark energy/matter effects are intensively studied in the framework of theories with modifications of Lagrange density, $R \rightarrow f(R, T, \ldots)$. Various such models with exotic anisotropic states of matter, modified gravitational and matter field interactions, torsion etc. were elaborated. The corresponding field equations are very sophisticated nonlinear PDE which request advanced analytic and numerical methods for constructing solutions and analysis of possible physical implications. One of the main goals of this work is to extend the anholonomic deformation method of constructing exact solutions in Einstein and/or Finsler gravity theories in such a form which would allow to integrate in very general forms certain classes of $f(R, T, \ldots)$ gravity models. From a "modest" pragmatic point of view, a Finsler generating function $F$ is just a formally prescribed nonholonomic distribution on various spacetime models which allows us to decouple and solve physically important field equations. But the method may be also formulated to encode possible modifications from locally anisotropic gravity models on tangent bundles. Conventionally, various geometric and physical assumptions for modified gravity theories are denoted as $f(R, T, F)$. We proved that via nonholonomic frame transforms and deformations we can model a $f(R, T, F)$ theory as a $f(R, T)$ one, or as an effective Einstein, or Finsler, theory.

In the present work, we investigated generalized gravity theories with arbitrary coupling (including anisotropies, parametric dependencies and nonholonomic constraints) between matter and geometry. Using geometric and variational methods, all adapted to possible nonlinear connection structures, we derived the gravitational field equations. We considered several important particular cases that may present interest in modern cosmology and astrophysics. We concluded that off-diagonal terms of metrics, modified matter and time dependent terms in generalized Einstein equations play the role of effective cosmological constant, exotic matter, effective torsion fields, anholonomic frame effects, etc. For well defined geometric and physical conditions, such effects can be modified by nonholonomic distributions on Einstein manifolds and their tangent bundles.

We studied the modified equations of motion of test particles in modified $f(R, T, F)$ theories and evaluated possible contributions of effective extra-forces with locally anisotropic terms. There were formulated the nonholonomically modified Raychaudhury equations on tangent Lorentz bundle. Certain Newton limits with corresponding corrections were computed. Using perihelion procession, an upper limit to extra-acceleration and anisotropies was obtained. As an explicit example, we took the de Sitter black hole metric and deformed it into new classes of exact solutions with rotoid symmetry. The effective source (an anisotropically polarized cosmological constant) contains contributions from possible modifications of gravitational actions and/or from Finsler like anisotropies. We provided some examples when black hole solutions are modified ("dissipated") into off-diagonal vacuum configurations with complex locally anisotropic structure and effective cosmological constant.

We conclude that the predictions of $f(R,T,F)$ theories could be very different from those in $f(R,T)$, $f(R)$ and/or GR theories. Generalized principles of covariance and relativity may state certain conditions of equivalence and mutual transforms of various models. The study of these theoretical issues and related phenomena may provide some specific effects which may help to distinguish different gravitational models. In our forthcoming work, we shall explore in more detail such theories by elaborating certain models of modified/anisotropic cosmological evolution and possible dark energy/matter effects.

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