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SOME CONNECTIONS AND VARIATIONAL PRINCIPLE TO THE FINSLERIAN SCALAR-TENSOR THEORY OF GRAVITATION

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A generalized scalar-tensor theory of the Finslerian gravitational field is constructed on the basis of the fibered bundle of a base manifold and consists of two G-numbers (Grassmannian noncommutative) y^+, y^-, playing the role of fibres. In the framework of our approach, we study: the connection structures, torsions, curvatures and the gravitational field equations derived from variational principles of the appropriate Lagrangian densities.

1. Introduction

Scalar-tensor theories were first introduced by Jordan [6]. In these theories, the gravitational "constant" is not a constant, but a scalar component of the gravitational field. Such a consideration constituted a subject of a great deal of interest some years ago. Brans-Dicke's scalar-tensor theory has a dominant position amongst them [3]. This theory is identified with the conventional general relativity based on the Riemannian geometry in the physical interpretation of the metric $g_{\mu\nu}$ and differs only in that a new scalar field φ enters the gravitational field equations, which is coupled with the mass density of the universe [12].

The field equations corresponding to scalar-tensor field theories are usually obtained from a variational principle, in which the Lagrangian density is a function of $g_{\kappa\lambda}$ and its first two derivatives as well as a scalar field φ and its first derivative $\varphi_{,\kappa}$.

In addition, recently a theory of noncommutative geometry has been introduced and developed by Connes [4] in the theoretical framework of a unified description of space-time geometry with quantum theory.

In the framework of the Finslerian approach, G. S. Asanov and H. Rund [1, 2, 9] have studied the problem of the variational principle on general relativity in a sufficiently generalized form. In the present paper, our basic idea is to give the connection structures, torsions, curvatures and the gravitational field equations derived from a variational principle. The noncommutative structure on the fibered bundle, which is dominated by some components of the torsion (cf. (3.3d)) of the Finslerian gravitational field, is being studied without using the noncommutative geometry.

This consideration, as we shall see in Sections 2 and 3, leads us to a form of a scalar-tensor theory of gravitational field, more generalized than Brans-Dicke's because of the contributions of two independent noncommutative G-numbers y^+, y^-, which represent the fibres. In this case, the elements of our space are expressed in a form of independent variables $Z^M \equiv (x^\mu, y^+, y^-)$, $\mu = 1, 2, 3, 4$, and the scalar field φ of Brans-Dicke's theory intrinsically enters into the metrical structure of the space.

Therefore, it is legitimate to ask for the connection structure, the metric form, the scalar curvature and field equations in the framework of a form which is analogous with the one regarded by the authors in previous papers about the spinor bundle $S^{(2)}(M) = M \times \mathbb{C}^{2\cdot 4}$ with $(x^\kappa, \xi_\alpha, \bar{\xi}^\alpha) \in S^{(2)}(M)$ [10] and the second order deformed bundle $T^{(2)}(DF)$ constructed locally by $X^A = (x^\kappa, y^i, y^0 = \lambda)$ [11]. By utilizing the vector bundle consideration [5, 7], the Finslerian gravitational field can be regarded as the unified field over the total space of the vector bundle whose base manifold is the (x)-field and fibre at each point x is the (y)-field.

Consequently, we can study the Finslerian gravitational field based on the geometry of the total space of this vector bundle.

2. Connections

In the following, we shall first set the concept of connections of the total space of the tangent bundle with a local system of coordinates $(x^i, y^i) \in TM$ in the form in which it was developed in [8]. Next, we shall define the connection structures for the case where the bundle has independent variables $Z^M \equiv (x^\alpha, y^+, y^-)$, instead of the vectorial variable y. Here, y^+ and y^-, as we mentioned in Section 1, represent two noncommutative G-numbers.

A local adapted frame on the tangent bundle TM is composed of $(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^i})$ and its dual basis $(dx^\alpha, \delta y^i)$,

$$\frac{\partial}{\partial Z^M} \equiv \left(\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - N^i_\alpha \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i}\right), \qquad dZ^M \equiv (dx^\alpha, \delta y^i = dy^i + N^i_\alpha dx^\alpha), \quad (2.1)$$

where $Z^M \equiv (x^\alpha, y^i)$ with $M \equiv (\alpha, i) = 1, 2, ..., 8$, and N^i_α denote the nonlinear connection representing physically the interaction between the (x)- and (y)-fields.

The metrical structure in this case is given by

$$G \equiv G_{MN} dZ^M dZ^N \equiv g_{\alpha\beta} dx^\alpha \otimes dx^\beta + g_{ij} \delta y^i \otimes \delta y^j, \qquad (2.2)$$

where $G_{MN} \equiv (g_{\alpha\beta}, g_{ij})$ is the metric tensor.

The connection structure and the covariant derivative for an arbitrary vector $V^M = (V^\alpha, V^i)$ have the form

$$\nabla_{\partial/\partial Z^N} \frac{\partial}{\partial Z^M} = \Gamma^L_{MN} \frac{\partial}{\partial Z^L}, \tag{2.3}$$

with $\Gamma^L_{MN} \equiv (L^\alpha_{\beta\gamma}, L^i_{j\gamma}, C^\alpha_{\beta k}, C^i_{jk})$, and

$$\begin{aligned} V^\alpha_{|\gamma} &= \frac{\delta V^\alpha}{\delta x^\gamma} + L^\alpha_{\beta\gamma} V^\beta, \\ V^\alpha|_k &= \frac{\partial V^\alpha}{\partial y^k} + C^\alpha_{\beta k} V^\beta, \\ V^i_{|\gamma} &= \frac{\delta V^i}{\delta x^\gamma} + L^i_{j\gamma} V^j, \\ V^i|_k &= \frac{\partial V^i}{\partial y^k} + C^i_{jk} V^j. \end{aligned}$$

We assume that the connection is metrical, i.e. $g_{\alpha\beta|\gamma} = 0$, $g_{\alpha\beta}|_k = 0$ and $g_{ij|\gamma} = 0$, $g_{ij}|_k = 0$.

To make some use of the above case of connections in our paper, we can specialize the connection structure (2.3) by taking $Z^M \equiv (x^\alpha, y^0)$ as the independent variables, where y^0 is an independent scalar. Then the connection structure is reduced to

$$\Gamma^L_{MN} \equiv (L^\alpha_{\beta\gamma}, L^0_{0\gamma}, C^\alpha_{\beta 0}, C^0_{00}),$$

namely $i, j, k, \ldots = 0$ in (2.3), and the metrical structure takes the form

$$G \equiv g_{\alpha\beta} dx^\alpha \otimes dx^\beta + g_{00} \delta y^0 \otimes \delta y^0, \tag{2.4}$$

where $\delta y^0 = dy^0 + N^0_\alpha dx^\alpha$.

Now, by generalizing the previous description of the bundle with two noncommutative G-numbers y^+, y^- (internal variables) playing the role of fibres, we develop the geometry of this bundle without making use of noncommutative geometry [4], as we mentioned in Section 1.

The adapted frame is set as follows:

$$\begin{aligned} \frac{\partial}{\partial Z^M} &\equiv \left(\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - N^+_\alpha \frac{\partial}{\partial y^+} - N^-_\alpha \frac{\partial}{\partial y^-}, \frac{\partial}{\partial y^+}, \frac{\partial}{\partial y^-} \right), \\ dZ^M &\equiv (dx^\alpha, \delta y^+ = dy^+ + N^+_\alpha dx^\alpha, \delta y^- = dy^- + N^-_\alpha dx^\alpha). \end{aligned} \tag{2.5}$$

Here, we have introduced two kinds of nonlinear connections, N^+_α and N^-_α.

Concerning the adapted basis (2.5), the metrical structure is given by

$$G \equiv G_{MN} dZ^M dZ^N = g_{\alpha\beta} dx^\alpha \otimes dx^\beta + g_{++} \delta y^+ \otimes \delta y^+ + g_{--} \delta y^- \otimes \delta y^-, \tag{2.6}$$

where we have put $g_{\alpha+} = g_{+\alpha} = 0$, $g_{\alpha-} = g_{-\alpha} = 0$ and $g_{+-} = -g_{-+} = 0$.

The connection structure is given by

$$\nabla_{\partial/\partial Z^K}\frac{\partial}{\partial Z^M}=\Gamma^N_{MK}\frac{\partial}{\partial Z^N}\,,\qquad K,M=\{\alpha,\beta,\ldots,+,-\},$$

with $\Gamma^N_{MK}=\{L^N_{M\alpha},C^N_{M+},E^N_{M-}\}$.

$$\nabla_{\delta/\delta x^\alpha}\frac{\delta}{\delta x^\beta}=L^\gamma_{\beta\alpha}\frac{\delta}{\delta x^\gamma}\,,\quad \nabla_{\delta/\delta x^\alpha}\frac{\partial}{\partial y^+}=L^+_{+\alpha}\frac{\partial}{\partial y^+}\,,\quad \nabla_{\delta/\delta x^\alpha}\frac{\partial}{\partial y^-}=L^-_{-\alpha}\frac{\partial}{\partial y^-}\,,\tag{2.7a}$$

$$\nabla_{\partial/\partial y^+}\frac{\delta}{\delta x^\alpha}=C^\beta_{\alpha+}\frac{\delta}{\delta x^\beta}\,,\quad \nabla_{\partial/\partial y^+}\frac{\partial}{\partial y^+}=C^+_{++}\frac{\partial}{\partial y^+}\,,\quad \nabla_{\partial/\partial y^+}\frac{\partial}{\partial y^-}=C^-_{-+}\frac{\partial}{\partial y^-}\,,\tag{2.7b}$$

$$\nabla_{\partial/\partial y^-}\frac{\delta}{\delta x^\alpha}=E^\beta_{\alpha-}\frac{\delta}{\delta x^\beta}\,,\quad \nabla_{\partial/\partial y^-}\frac{\partial}{\partial y^+}=E^+_{+-}\frac{\partial}{\partial y^+}\,,\quad \nabla_{\partial/\partial y^-}\frac{\partial}{\partial y^-}=E^-_{--}\frac{\partial}{\partial y^-}\,.\tag{2.7c}$$

For an arbitrary vector $V^M\equiv(V^\alpha,V^+,V^-)$, the covariant derivatives can be defined as follows:

$$\begin{aligned}V^\alpha_{|\gamma}&=\frac{\delta V^\alpha}{\delta x^\gamma}+L^\alpha_{\beta\gamma}V^\beta,\\ V^+|_+&=\frac{\partial V^+}{\partial y^+}+C^+_{++}V^+,\\ V^-\|_-&=\frac{\partial V^-}{\partial y^-}+E^-_{--}V^-,\end{aligned}\tag{2.8}$$

etc. Here, the metrical conditions $g_{\alpha\beta|\gamma}=0$, $g_{++}|_+=0$ and $g_{--}\|_-=0$ are assumed.

3. Torsions and curvatures

We assign the torsion tensors and curvature tensors for the case of our vector bundle. The torsion tensor field is given by

$$T(X,Y)=\nabla_XY-\nabla_YX-[X,Y],$$

for every differentiable vector fields X and Y defined on the bundle $(X,Y\in\mathcal{X}(TM))$. Consequently, we have the relations,

$$\begin{aligned}T\left(\frac{\delta}{\delta x^\beta},\frac{\delta}{\delta x^\alpha}\right)&=T^\gamma_{\alpha\beta}\frac{\delta}{\delta x^\gamma}+T^+_{\alpha\beta}\frac{\partial}{\partial y^+}+T^-_{\alpha\beta}\frac{\partial}{\partial y^-}\,,\\ T\left(\frac{\partial}{\partial y^+},\frac{\delta}{\delta x^\alpha}\right)&=T^\gamma_{\alpha+}\frac{\delta}{\delta x^\gamma}+T^+_{\alpha+}\frac{\partial}{\partial y^+}+T^-_{\alpha+}\frac{\partial}{\partial y^-}\,,\\ T\left(\frac{\partial}{\partial y^-},\frac{\delta}{\delta x^\alpha}\right)&=T^\gamma_{\alpha-}\frac{\delta}{\delta x^\gamma}+T^+_{\alpha-}\frac{\partial}{\partial y^+}+T^-_{\alpha-}\frac{\partial}{\partial y^-}\,,\\ T\left(\frac{\partial}{\partial y^-},\frac{\partial}{\partial y^+}\right)&=T^\gamma_{+-}\frac{\delta}{\delta x^\gamma}+T^+_{+-}\frac{\partial}{\partial y^+}+T^-_{+-}\frac{\partial}{\partial y^-}\,,\\ T\left(\frac{\partial}{\partial y^+},\frac{\partial}{\partial y^+}\right)&=T^\gamma_{++}\frac{\delta}{\delta x^\gamma}+T^+_{++}\frac{\partial}{\partial y^+}+T^-_{++}\frac{\partial}{\partial y^-}\,,\\ T\left(\frac{\partial}{\partial y^-},\frac{\partial}{\partial y^-}\right)&=T^\gamma_{--}\frac{\delta}{\delta x^\gamma}+T^+_{--}\frac{\partial}{\partial y^+}+T^-_{--}\frac{\partial}{\partial y^-}\,.\end{aligned}\tag{3.1}$$

The relation (3.1) can be written in the equivalent form

$$\nabla_{\partial/\partial Z^M}\frac{\partial}{\partial Z^N} - \nabla_{\partial/\partial Z^N}\frac{\partial}{\partial Z^M} - \left[\frac{\partial}{\partial Z^M}, \frac{\partial}{\partial Z^N}\right], \tag{3.2}$$

where we have put $\frac{\partial}{\partial Z^M} = \left\{\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^+}, \frac{\partial}{\partial y^-}\right\}$

The comparison between (3.1) and (3.2), after straightforward calculations, gives us the following relations for the torsion tensor components:

$$T^\gamma_{\alpha\beta} = L^\gamma_{\alpha\beta} - L^\gamma_{\beta\alpha}, \qquad T^+_{\alpha\beta} = -R^+_{\beta\alpha}, \qquad T^-_{\alpha\beta} = -V^-_{\beta\alpha}, \tag{3.3a}$$

$$T^\gamma_{\alpha+} = C^\gamma_{\alpha+}, \qquad T^+_{\alpha+} = \frac{\partial N^+_\alpha}{\partial y^+} - L^+_{\alpha+}, \qquad T^-_{\alpha+} = \frac{\partial N^-_\alpha}{\partial y^+}, \tag{3.3b}$$

$$T^\gamma_{\alpha-} = -E^\gamma_{-\alpha}, \qquad T^+_{\alpha-} = \frac{\partial N^+_\alpha}{\partial y^-}, \qquad T^-_{\alpha-} = -L^-_{-\alpha} + \frac{\partial N^-_\alpha}{\partial y^-}, \tag{3.3c}$$

$$T^\gamma_{+-} = 0, \qquad T^+_{+-} = -E^+_{-+}, \qquad T^-_{+-} = -C^-_{-+}, \tag{3.3d}$$

$$T^\gamma_{++} = 0, \qquad T^+_{++} = C^+_{++} = 0, \qquad T^-_{++} = 0, \tag{3.3e}$$

$$T^\gamma_{--} = 0, \qquad T^+_{--} = 0, \qquad T^-_{--} = 0. \tag{3.3f}$$

For the derivation of the above relations, we have taken into account the brackets:

$$\left[\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right] = R^+_{\alpha\beta}\frac{\partial}{\partial y^+} + V^-_{\alpha\beta}\frac{\partial}{\partial y^-}, \tag{3.4}$$

$$\left[\frac{\partial}{\partial y^+}, \frac{\delta}{\delta x^\alpha}\right] = \left[\frac{\partial}{\partial y^-}, \frac{\delta}{\delta x^\alpha}\right] = \left[\frac{\partial}{\partial y^+}, \frac{\partial}{\partial y^-}\right] = \left[\frac{\partial}{\partial y^+}, \frac{\partial}{\partial y^+}\right] = \left[\frac{\partial}{\partial y^-}, \frac{\partial}{\partial y^-}\right] = 0,$$

where we have put $R^+_{\alpha\beta} = \frac{\delta N^+_\beta}{\delta x^\alpha} - \frac{\delta N^+_\alpha}{\delta x^\beta}$, $V^-_{\alpha\beta} = \frac{\delta N^-_\beta}{\delta x^\alpha} - \frac{\delta N^-_\alpha}{\delta x^\beta}$.

In the relation (3.3d), the two last terms, T^+_{-+} and T^-_{-+} embody the noncommutative structure. The connection has no torsion, if and only if all the coefficients (3.3) are equal to zero.

With the above mentioned connection coefficients, we can associate the curvature tensors with the adapted basis $(\partial/\partial Z^M) \equiv X_M = \{X_\alpha, X_+, X_-\}$, $X_\alpha = \frac{\delta}{\delta x^\alpha}$, $X_+ = \frac{\partial}{\partial y^+}$, $X_- = \frac{\partial}{\partial y^-}$.

$$\begin{aligned} &R(X_M, X_N)X_L = \mathcal{R}^K_{LNM}X_K, \\ &\mathcal{R}^K_{LNM} = X_M\Gamma^K_{LN} - X_N\Gamma^K_{LM} + \Gamma^Z_{LN}\Gamma^K_{ZM} - \Gamma^Z_{LM}\Gamma^K_{ZN} + \Gamma^K_{LZ}W^Z_{NM}, \end{aligned} \tag{3.5}$$

where W^Z_{NM} are the nonholonomy coefficients in the local basis X_M,

$$[X_M, X_N] = W^Z_{NM}X_Z.$$

Twelve kinds of curvature tensors appear,

$$\mathcal{R}^K_{LMN} \equiv (R^K_{L\alpha\beta}, R^K_{L\alpha+}, R^K_{L\alpha-}, R^K_{L+-}), \tag{3.6}$$

where $R^K_{L\alpha\beta} = (R^\gamma_{\delta\alpha\beta}, R^+_{+\alpha\beta}, R^-_{-\alpha\beta})$, etc. The explicit form of these curvatures can be derived in a similar way to that developed in [10].

We shall adopt the case where all the coefficients of torsion tensors are equal to zero and $g_{\alpha+} = g_{+\alpha} = g_{\alpha-} = g_{-\alpha} = g_{+-} = -g_{-+} = 0$ (cf. (2.6)).

From (3.6), the Ricci tensors are given by

$$\mathcal{R}_{MN} \equiv R^L_{MLN} \equiv (\widetilde{R}_{\alpha\beta}, \widetilde{R}_{\alpha+}, \widetilde{R}_{\alpha-}, \ldots, \widetilde{R}_{+-}, -\widetilde{R}_{-+}, -\widetilde{R}_{+\alpha}, -\widetilde{R}_{-\alpha}), \tag{3.7}$$

or in the form

$$\mathcal{R}_{MN} = \{R^\alpha_{MN\alpha}, R^+_{MN+}, R^-_{MN-}\},$$

where we have put

$$\begin{aligned} R^\alpha_{MN\alpha} &= \frac{\delta L^\alpha_{MN}}{\delta x^\alpha} - \frac{\delta L^\alpha_{M\alpha}}{\delta x^N} + L^\alpha_{K\alpha} L^K_{MN} - L^\alpha_{KN} L^K_{M\alpha}\,, \\ R^+_{MN+} &= \frac{\partial C^+_{MN}}{\partial y^+} - \frac{\partial C^+_{M+}}{\partial y^N} + C^+_{K+} C^K_{MN} - C^+_{KN} C^K_{M+}\,, \\ R^-_{MN-} &= \frac{\partial E^-_{MN}}{\partial y^-} - \frac{\partial E^-_{M-}}{\partial y^N} + E^-_{K-} E^K_{MN} - E^-_{KN} E^K_{M-}\,. \end{aligned}$$

The scalar curvature has the form

$$\mathcal{R} = R_{MN} G^{MN} = \widetilde{R}_{\alpha\beta} g^{\alpha\beta} + \widetilde{R}_{++} g^{++} + \widetilde{R}_{--} g^{--} = R + R^{(+)} + R^{(-)}, \tag{3.8}$$

where in (3.8) we have assumed $G_{MN} = (g_{\beta\gamma}(x^\alpha),\ g_{++}(x^\alpha),\ g_{--}(x^\alpha))$, with $g_{++}(x^\alpha) = g_{--}(x^\alpha) = \lambda^2(x^\alpha)$ ($\lambda(x^\alpha) \neq 0$ is a scalar function).

It should be noted that even if R is the scalar curvature of a Riemannian manifold constructed from $g_{\alpha\beta}$, the scalar curvature $\mathcal{R}$ constructed from G_{MN} differs from the Riemannian one because of the contributions of the internal variables g_{++}, g_{--}. So, as we shall see in Section 4, the above-mentioned scalar curvature $\mathcal{R}$ becomes more general than the one in Brans-Dicke theory.

4. Gravitational field equations

It is usually assumed that the field equations which govern the behaviour of the field are identical with Euler–Lagrange equations of a given problem in the calculus of variations. In our approach, we shall construct the Lagrangian scalar density, which is supposed to be invariant under the coordinate transformations.

In the following, we shall consider the Lagrangian of the type

$$L = L(\varphi\,;\ \varphi_{,\alpha}\,;\ \varphi_{,\alpha\beta}\,;\ g_{\alpha\beta}\,;\ g_{\alpha\beta,\gamma}\,;\ g_{\alpha\beta,\gamma\delta}) \tag{4.1}$$

with $\varphi(x^\alpha) = \lambda^2(x^\alpha) = g_{++}(x^\alpha) = g_{--}(x^\alpha)$.

We shall derive the Euler–Lagrange equations corresponding to (4.1). These will be given by the variation of the action integral,

$$\delta I = \delta \int_D L dx^{(N)} = 0\,, \tag{4.2}$$

where D is the region of integration and $dx^{(N)} = dx^1 \dots dy^+ dy^-$. In our case, the complete system of Euler–Lagrange equations consists of the two sets of equations:

$$
\begin{aligned}
\frac{\delta L}{\delta \varphi} &= \frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial \varphi_{,\alpha}}\right) + \frac{\partial^2}{\partial x^\alpha \partial x^\beta}\left(\frac{\partial L}{\partial \varphi_{,\alpha\beta}}\right) = 0\,, \\
\frac{\delta L}{\delta g_{\alpha\beta}} &= \frac{\partial L}{\partial g_{\alpha\beta}} - \frac{\partial}{\partial x^\gamma}\left(\frac{\partial L}{\partial g_{\alpha\beta,\gamma}}\right) + \frac{\partial^2}{\partial x^\gamma \partial x^\delta}\left(\frac{\partial L}{\partial g_{\alpha\beta,\gamma\delta}}\right) = 0\,,
\end{aligned}
\tag{4.3}
$$

where

$$
\varphi_{,\alpha} = \frac{\partial \varphi}{\partial x^\alpha}\,, \qquad \varphi_{,\alpha\beta} = \frac{\partial^2 \varphi}{\partial x^\alpha \partial x^\beta}\,,
$$

$$
g_{\alpha\beta,\gamma} = \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\,, \qquad g_{\alpha\beta,\gamma\delta} = \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta}\,.
$$

In order to associate the explicit form of the Euler–Lagrange equations (4.3), we assume the Lagrangian

$$
L = \sqrt{|G|}\,\mathcal{R}\,, \tag{4.4}
$$

where $G = |\det(G_{MN})|$, $\sqrt{|G|} = \varphi\sqrt{|g|}$, $g = |\det(g_{\alpha\beta})|$ and $\mathcal{R}$ represents the generalized form of the scalar curvature and is given by (3.8) and $g^{\alpha\beta}$, $g^{++} = g^{--} = \varphi^{-1}$ are the inverse of $g_{\alpha\beta}$, $g_{++} = g_{--} = \varphi$.

By direct calculations, the Ricci tensors yield

$$
\begin{aligned}
\widetilde{R}_{\alpha\beta} &= R_{\alpha\beta} + L^\mu_{\alpha\beta}\varphi_{,\mu}\varphi^{-1} + \frac{1}{2}\varphi_{,\alpha}\varphi_{,\beta}\varphi^{-1} - \varphi_{,\alpha\beta}\varphi^{-1}, \\
\widetilde{R}_{++} = \widetilde{R}_{--} &= \frac{1}{2}g^{\alpha\mu}g^{\beta\nu}g_{\alpha\beta,\mu}\varphi_{,\nu} - \frac{1}{2}g^{\mu\nu}\left(\varphi_{,\mu\nu} + \frac{1}{2}g_{,\mu}\varphi_{,\nu}\right),
\end{aligned}
\tag{4.5}
$$

with $g_{,\mu} = \frac{\partial g}{\partial x^\mu}$, and $R_{\alpha\beta}$ is the Ricci tensor constructed from (symmetric) $g_{\alpha\beta}$. Consequently, the Lagrangian density (4.4), because of (4.5), can be written in the form

$$
L = \sqrt{|g|}\,(\varphi R - 2g^{\alpha\beta}\varphi_{|\alpha\beta} + \frac{1}{4}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}/\varphi)\,, \tag{4.6}
$$

provided that the identity $g_{,\mu} = g g^{\alpha\beta} g_{\alpha\beta,\mu}$ is taken into account. In the above relation, $\varphi_{|\alpha\beta}$ means the second covariant derivative of φ. A comparison of (4.6) with Brans–Dicke's Lagrangian reveals some similarities. The first and third term of (4.6) are like those of B–D theory. By virtue of (4.6), we get the form of derivatives:

$$
\begin{aligned}
\Phi &= L_0 - \frac{1}{2}\sqrt{|g|}\,g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}\varphi^{-2}, \\
\Phi^\kappa &= \sqrt{|g|}\,g^{\alpha\beta}\left(2L^\kappa_{\alpha\beta} + \delta^\kappa_\alpha\varphi_{,\beta}\varphi^{-1}\right), \\
\Phi^{\kappa\lambda} &= -2\sqrt{|g|}\,g^{\kappa\lambda}, \\
\Pi^{\kappa\lambda} &= \varphi\frac{\partial L_0}{\partial g_{\kappa\lambda}} + \left(g^{\alpha\beta}g^{\kappa\lambda} - 2g^{\alpha\kappa}g^{\beta\lambda}\right)\left(2L^\gamma_{\alpha\beta}\varphi_{,\gamma} + \frac{1}{2}\varphi_{,\alpha}\varphi_{,\beta}\varphi^{-1} - 2\varphi_{,\alpha\beta}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\sqrt{|g|}\, g^{\alpha\beta}\left(g^{\gamma\kappa}L^{\lambda}_{\alpha\beta}+g^{\gamma\lambda}L^{\kappa}_{\alpha\beta}\right)\varphi_{,\gamma}, \qquad (4.7)\\
\Pi^{\kappa\lambda,\mu} &= \varphi\frac{\partial L_0}{\partial g_{\kappa\lambda,\mu}}+\sqrt{|g|}\,\varphi_{,\gamma}\left(g^{\kappa\mu}g^{\gamma\lambda}+g^{\lambda\mu}g^{\gamma\kappa}-g^{\kappa\lambda}g^{\gamma\mu}\right),\\
\Pi^{\kappa\lambda,\mu\nu} &= \varphi\frac{\partial L_0}{\partial g_{\kappa\lambda,\mu\nu}},
\end{aligned}
$$

where we have put

$$
\Phi=\frac{\partial L}{\partial\varphi},\qquad \Phi^{\kappa}=\frac{\partial L}{\partial\varphi_{,\kappa}},\qquad \Phi^{\kappa\lambda}=\frac{\partial L}{\partial\varphi_{,\kappa\lambda}},
$$

$$
\Pi^{\kappa\lambda}=\frac{\partial L}{\partial g_{\kappa\lambda}},\qquad \Pi^{\kappa\lambda,\mu}=\frac{\partial L}{\partial g_{\kappa\lambda,\mu}},\qquad \Pi^{\kappa\lambda,\mu\nu}=\frac{\partial L}{\partial g_{\kappa\lambda,\mu\nu}},
$$

and $L_0=\sqrt{|g|}\,R$.

Substituting (4.7) into (4.3), we obtain, after straightforward calculations, the gravitational field equations that govern the behaviour of the fields $\varphi(x^\mu)=g_{++}=g_{--}$, $g_{\alpha\beta}(x^\mu)$. So, we get the equations

$$
\frac{\delta L}{\delta\varphi} \equiv \sqrt{|g|}\left(\varphi R+\frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}\varphi^{-1}-g^{\alpha\beta}\varphi_{|\alpha\beta}\right)=0, \qquad (4.8)
$$

$$
\begin{aligned}
\frac{\delta L}{\delta g_{\kappa\lambda}} \equiv{}& \sqrt{|g|}\,E^{\kappa\lambda}+\frac{1}{4}\sqrt{|g|}\left(g^{\alpha\beta}g^{\kappa\lambda}\varphi^{-1}-g^{\alpha\kappa}g^{\beta\lambda}\right)\varphi_{,\alpha}\varphi_{,\beta}\\
&+\sqrt{|g|}\,\varphi_{|\alpha\beta}\left(g^{\alpha\kappa}g^{\beta\lambda}-g^{\alpha\beta}g^{\kappa\lambda}\right)=0, \qquad (4.9)
\end{aligned}
$$

where $E^{\kappa\lambda}=R^{\kappa\lambda}-\frac{1}{2}g^{\kappa\lambda}R$.

In the more general case, if we use the relations (3.5) for the curvatures, we shall get the Ricci tensors $K_{\alpha\beta}$, K_{++}, K_{--} in the following form:

$$
\begin{aligned}
K_{\alpha\beta} &= \widetilde{R}_{\alpha\beta}+\frac{1}{2}\varphi_{,\alpha}\varphi^{-1}\left(\frac{\partial N^{+}_{\beta}}{\partial y^{+}}+\frac{\partial N^{-}_{\beta}}{\partial y^{-}}\right),\\
K_{++} &= \widetilde{R}_{++}+\frac{1}{2}g^{\mu\nu}\varphi_{,\nu}\frac{\partial N^{+}_{\mu}}{\partial y^{+}}, \qquad (4.10)\\
K_{--} &= \widetilde{R}_{--}+\frac{1}{2}g^{\mu\nu}\varphi_{,\nu}\frac{\partial N^{-}_{\mu}}{\partial y^{-}},
\end{aligned}
$$

where $\widetilde{R}_{\alpha\beta}$, $\widetilde{R}_{++}$, $\widetilde{R}_{--}$ are given by (4.5).

Consequently, the generalized Lagrangian $\widetilde{L}$ will be given as follows:

$$
\begin{aligned}
\widetilde{L}\equiv\sqrt{|G|}G^{AB}K_{AB}={}&\varphi\sqrt{|g|}g^{\alpha\beta}(R_{\alpha\beta}-2\varphi^{-1}\varphi_{,\alpha\beta}+\frac{1}{2}\varphi^{-2}\varphi_{|\alpha}\varphi_{|\beta})\\
&+\sqrt{|g|}g^{\alpha\beta}\varphi_{,\alpha}\left(\frac{\partial N^{+}_{\beta}}{\partial y^{+}}+\frac{\partial N^{-}_{\beta}}{\partial y^{-}}\right) \qquad (4.11)
\end{aligned}
$$

or, taking into account (4.6), the relation (4.11) is written in the equivalent form:

$$\widetilde{L} = L + \sqrt{|g|} g^{\alpha\beta} \varphi_{,\alpha} \left(\frac{\partial N_\beta^+}{\partial y^+} + \frac{\partial N_\beta^-}{\partial y^-} \right). \tag{4.12}$$

The field equations produced by the variational principle of the Lagrangian $\widetilde{L}$ entail for the first set of equations

$$\sqrt{|g|} g^{\alpha\beta} (R_{\alpha\beta} - \frac{\varphi_{;\alpha\beta}}{\varphi} + \frac{\varphi_{;\alpha}\varphi_{;\beta}}{2\varphi^2}) - \frac{\delta}{\delta x^\alpha} \left[\sqrt{|g|} g^{\alpha\beta} \left(\frac{\partial N_\beta^+}{\partial y^+} + \frac{\partial N_\beta^-}{\partial y^-} \right) \right] = 0. \tag{4.13}$$

The second set of equations is taken into the following form:

$$\begin{aligned} &\varphi\sqrt{|g|} \left(\frac{1}{2} R g^{\alpha\beta} - R^{\alpha\beta} \right) + \frac{\sqrt{|g|}}{2} g^{\alpha\beta} g^{\mu\nu} \frac{\partial \varphi}{\partial x^\nu} \left(\frac{\partial N_\mu^+}{\partial y^+} + \frac{\partial N_\mu^-}{\partial y^-} \right) \\ &\qquad - \frac{\sqrt{|g|}}{2} \left(g^{\alpha\mu} g^{\beta\nu} + g^{\beta\mu} g^{\alpha\nu} \right) \frac{\partial \varphi}{\partial x^\nu} \left(\frac{\partial N_\mu^+}{\partial y^+} + \frac{\partial N_\mu^-}{\partial y^-} \right) = 0. \end{aligned} \tag{4.14}$$

The equations (4.14) give the generalized form of the gravitational field equations in the considered fibre bundle $M \times \{y^+\} \times \{y^-\}$.

Discussion

In this paper, we developed a generalized scalar-tensor theory of the Finslerian gravitational field in the form of the unified description of the field. This standpoint was based over the total space of the fibre bundle, where its two fibres y^+, y^- play an essential role in our approach. The two G-numbers y^+, y^- were introduced in order to give rise to the noncommutative structures in the metrical and connection structures of the gravitational field. Those noncommutative structures have not been considered within the framework of Finslerian fibre bundle.

Concerning the physical meaning of y^+ and y^-, we can consider the following examples: y^+ and y^- can be compared with the discrete coordinates in the two-dimensional discrete space, which are introduced to consider a unification of the gauge and Higgs fields; or y^+ and y^- can be two different kinds of (isotopic) spin components associated with its point, which show their own intrinsic rotations; or y^+ and y^- can be regarded as two spinorial components of one (two-dimensional) spinor, which obey their own intrinsic spin transformations.

These interpretations are all related with the noncommutative structures of our vector bundle mentioned above.

In a more general point of view, these two G-numbers y^+ and y^- are necessary to consider the supersymmetric aspects of the gravitational field.

Also, the gravitational field equations were derived in the framework of a sufficiently generalized form (4.9) or (4.14) by virtue of the variational principle, as an outcome of this approach.

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