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FINSLERIAN METRIC BASED ON THE GRAVITATIONAL AND ELECTROMAGNETIC FIELDS

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On discute la métrique de Randers et ses conséquences. On continue l'étude des interactions gravitationnées par les fluctuations des deux champs électromagnétique qui se propagent. On propose une nouvelle métrique qui décrit la procédure et échaque terme est interprété du point de vue de la Physique. La dynamique d'une particule chargée et des champs électromagnétique dans la présence des fluctuations gravitationnelles est étudiée.

Over the years the question as to whether gravitation and electromagnetism can be unified in a single geometrical framework and arise as the two faces of the same geometrical quantity has been studies extensively but without important results. As it is known Albert Einstein spent a lot of years in such a program but his final theory "The relativistic theory of the non-symmetric field" was not considered by the majority of physicusts as the acceptable ultimate of gravitation and electromagnetism [6]. Also the progress of quantum field theory and its very succesful applications, especially QED and later the electroweak theory of Weinberg, Salam and Glashow, showed that it would be much more fruitful to seek the unified theory of gravitation and electromagnetism in the framework of the quantum field theory approach.

However, a lot of physicits continued to pursue the geometrical unification of the two forces, mainly because quantum field theory has its own problems and also bears a lot of conceptional difficulties associated with quantum mechanics. One of the major steps in this effort was the Randers approach, which was studied in the framework of Finsler spaces [12]. In this approach the equation of motion of a charged particle in a gravitational and electromagnetic fields results naturally as the geodesic of a Finsler spacetime. However, the electromagnetic field does not occur from the metric structure of the spacetime and has to be imposed by hand. So the Randers approach cannot be considered as a completed unified theory of gravitation and electromagnetism, given the fact that for each type of particle a different space is defined [5].

Among the authors that have studied effectively the Randers approach and its generalisations are: G. Asanov [1, 2], R. Beil [3, 5], R. Miron [11], J. Hor-

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vath [8], H. Eliopoulos [7], R. Ingarden [10] and J. Hsu [9]. We discuss now the Randers approach in brief.

As it is known, the Lagrangian of a charged particle of mass m and charge q in an electromagnetic field $F_{\mu\nu}$ is:

$$L(x,\dot{x}) = \sqrt{g_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu}} + \frac{q}{mc^2}A_i(x)\dot{x}^i, \tag{1}$$

where c is the speed of light in vacuum, $g_{\mu\nu}$ is the Riemannian metric tensor of spacetime which incorporates the effects of gravity and A_i is the electromagnetic potential, $\dot{x}^i = dx/d\tau$ is the 4-velocity where $d\tau$ is the Riemannian proper time $d\tau \equiv \sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}$. The first variation of the action corresponding to the Lagrangian (1), gives the Euler-Lagrange equations:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^{\mu}}\right) - \frac{\partial L}{\partial x^{\mu}} = 0. \tag{2}$$

If we substitute the explicit form of the Lagrangian (1) in (2) we get the Lorentz equation of motion:

$$\frac{d^2x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = \frac{q}{mc}f^{\alpha\beta}F_{\beta\gamma}\frac{dx^{\gamma}}{d\tau}, \tag{3}$$

where $F_{\alpha\beta}$ is the electromagnetic field tensor. The Christoffel symbol in the left side of (3) is the effect of gravity on the motion of the particle. Of course when there does not exist electromagnetic field in the area under consideration the $F\alpha\beta$ is zero and the equation of motion (3) corresponds to the geodesic motion of the particle only due to gravity.

If we move the right hand term of equation (3) on the left, it would be reasonable to seek a more general connection so that (3) would be the geodesic equation in a more complicated spacetime, of course, than the original Riemannian one. The Randers approach is exactly this idea. The metric that produces the appropriate connection coefficients has to be a Finslerian one and so the spacetime becomes a Finsler space. The advantage of this consideration is that the equation of motion occurs physically from the geometry of spacetime and has not to be imposed as an independent axiom.

In order to do this, we identify the metric function $F(x, x')$, of a Finsler space with the Lagrangian (1). So we get a Finsler space with the following metric function:

$$F(x,x') = \sqrt{g_{\mu\nu}(x)x'^{\mu}x'^{\nu}} + \frac{q}{mc^2}A_i(x)x'^i. \tag{4}$$

In this case the the 4-velocity is $x'^i = dx/ds$, where ds is the Finslerian proper time. We emphasize this fact, because the measurable quantity in this spacetime is ds and not $d\tau$. When no electromagnetic field exists, then ds equals $d\tau$. From the metric function (4) it is straightforward to find the metric tensor using the following definition:

$$f_{\mu\nu} = \frac{1}{2}\frac{\partial^2 F^2}{\partial x'^{\mu}\partial x'^{\nu}}. \tag{5}$$

The metric tensor is found to be:

$$f_{ij} = g_{ij} + \frac{2\beta}{\sigma}x'^{s}g_{s(i}A_{j)} + \beta^2 A_i A_j + \frac{\beta}{\sigma}x'^{l}A_l h_{ij}, \tag{6}$$

where $\beta \equiv q/mc^2$, $\sigma \equiv \sqrt{g_{ij}x'^i x'^j}$, $h_{ij} \equiv g_{ij} - \sigma^{-2}g_{si}g_{jl}x'^s x'^l$ and $a_{(ij)} = 1/2(a_{ij} + a_{ji})$. A Finsler space endowed with the metric tensor (6) is called a Randers space. We observe that whenever an electromagnetic field exists in a region of spacetime the geometry becomes Finslerian and the isotropy breaks. The geodesic equation for this space is found to be:

$$\frac{d^2x^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = \frac{q}{mc}f^{\alpha\beta}F_{\beta\gamma}\frac{dx^{\gamma}}{ds}, \tag{7}$$

where the $\Gamma^{\alpha}_{\mu\nu}$ are the Christoffel symbols of the metric $g_{ij}(x)$.

A generalisation of the above standard Randers metric was studied by Miron, where he considers $g_{ij}(x)$ to be a Finslerian metric [11]. Beil has also studied some other Finsler metrics produce the Lorentz equation of motion as geodesic and has also studied in parallel the Kaluza-Klein and Weyl theories where he overcomes the problem of dependence of the spacetime structure on the mass and the charge of the test particle via the Finsler Kaluza-Klein theory or Beil theory [4, 5].

We discuss now the geodesic deviation and its meaning in the Randers space. We may associate a curvature tensor analogue to the Berwald curvature tensor to the spacetime when an electromagnetic field exists. This tensor has been calculated by Asanov [2] and can be written in the following form:

$$\overline{H}^{i}_{h.jk} = R^{i}_{h.jk} + \Delta^{i}_{h.jk}, \tag{8}$$

where the $R^i_{h.jk}$ is the Riemannian curvature from the metric $g_{\mu\nu}$ while the other part depends on the geometrical and the electromagnetic characteristics of the Randers spacetime. The second part of (8) can be written explicitly as:

$$\Delta^{i}_{h.jk} = \frac{1}{2}(F^{i}_{h}F_{jk} + g_{h[k}F^{m}_{j]}F^{i}_{m} - F_{h[j}F^{i}_{k]} + \tag{9}$$
$$+(u_h\nabla_{[k}F^{i}_{j]} + \dot{x}^{m}g_{h[j}\nabla_{k]}F^{i}_{m} + u_{[j}\nabla_{k]}F^{i}_{h})\sigma^{-1} - \dot{x}^{m}u_h u_{[j}\sigma^{-3}\nabla_{k]}F^{i}_{m},$$

where $u_i \equiv g_{ij}\dot{x}^j/\sigma$.

The equation of deviation of geodesics can be written in the following form:

$$\frac{\delta^2 z^i}{\delta u^2} + \overline{H}^i_{j.hk}(x,\zeta)\zeta^j\zeta^h z^k = 0, \tag{10}$$

where z^i represents the deviation vector and ζ^i the tangent vectors of a geodesic surface included in the spacetime of our consideration. We observe that the deviation has two terms: one pure gravitational which corresponds to the gravitational deviation that we would observe if there was no electromagneticv field and it is associated with the $R^i_{h.jk}$ part of the curvature tensor and the other which corresponds to a mixed geometrical and electromagnetical deviation and it is associated with the $\Delta^i_{h.jk}$ part of the $\overline{H}^i_{h.jk}$ tendor. It would be very reasonable to study this second term of deviation trying to connect it with the electromagnetic force that two freely falling charged particles would exert each other. In such a case this force would result naturally as a geometrical effect and would not be necessary to impose it additionally.

If the second part $\Delta^i_{j.kl}$ of (8) is zero, then the deviation equation (10), becomes the Riemannian one:

$$\frac{\delta^2 z^i}{\delta u^2} + R^i_{j.hk}(x,\zeta)\zeta^j\zeta^h z^k = 0. \tag{11}$$

Physically, this means that we have two particles freely falling in the spacetime.

In the second case, the vanishing of the first $R^i_{j.hk}$ of (8) enatils the fact that the first term of the Randers metric corresponds to a Minkowskian metric. Therefore, the only force that influences the two particles is due to the presence of the electromagnetic field. So, the metric function $F(x, x)$ takes the form:

$$F(x,\dot{x}) = \sqrt{n_{\mu\nu}dx^\mu dx^\nu} + \frac{q}{mc^2}A_i(x)\dot{x}^i \tag{12}$$

and the deviation equation is:

$$\frac{\delta^2 z^i}{\delta u^2} + \Delta^i_{j.hk}(x,\zeta)\zeta^j\zeta^h z^k = 0. \tag{13}$$

Hereafter, we consider now a test particle with charge q and mass m, coordinates x^i and 4-velocity $\dot{x}^i$, which moves in a Riemannian gravitational field $g_{ij}(x)$ and feels the interaction of two electromagnetic fields with 4-potential vectors $A^{(1)i}$ and $A^{(2)i}$, respectively. Their 1-forms $A^{(1)i}$ and $A^{(2)i}$, respectively, not yet connected with their vectors $A^{(1)i}$ and $A^{(2)i}$ through a metric tensor, are indispensable, since this Finslerian metric tensor is to be introduced in the context. Thus, we use four tensor fields: $A^{(1)i}$, $A^{(2)i}$, $A^{(1)}_i$ and $A^{(2)}_i$ at this stage. As it is known, from the General Theory of Relativity, these electromagnetic fields also interact each other through gravitational fluctuations on the background spacetime, since their propagating energy-momentum contents cause gravitation waves. Thus, the information of the electromagnetic fields and their interaction have been incorporated in a Finslerian metric function $F(x, \dot{x})$. The meaning of this approach is that we intrinsically introduce the physical content of both gravitation, electromagnetism and gravitational fluctuations in the connection of a Finslerian manifold $2-FR$, called the 2-Field-modified Randers manifold, defined by the following metric function:

$$F(x,x') = \sqrt{g_{ij}(x)x'^i x'^j} + \frac{q}{mc^2}x'^i(A^{(1)}_i + A^{(2)}_i) + \phi\Lambda(x,x'), \tag{14}$$

where $\phi \equiv kA^{(1)i}A^{(2)}_i$, k = a real constant, $\Lambda(x, x')$ ≡ an homogeneous function of degree one. The object ϕ is assumed to be scalar in this Finslerian manifold. The last term ϕ corresponds to direct gravitational interaction between the electromagnetic fields, whereas the function $\Lambda(x, x')$ contains the information of the gravitational field cause by the interaction $\phi(x)$ of the electromagnetic fields. Consequently, $\Lambda(x, x')$ itself induces a gravitational field, which affects the motion of every physical object in the spacetime. It is significant to stress the fact that the interaction term $\phi\Lambda(x, x')$ is corporated in the metric structure intrinsically.

Applying definition (5), in the case we consider, we obtain the following Finslerian metric tensor:

$$\begin{aligned} f_{ij} = &\left\{ g_{ij} + \frac{2\beta}{\sigma}x'^s g_{s(i}A_{j)} + \beta^2 A_i A_j + \frac{\beta}{\sigma}x'^l A_l h_{ij} \right\} + \\ &+ \frac{2\phi}{\sigma}x'^s g_{s(i}\partial'_{j)}\Lambda + 2\phi^2\beta A_{(i}\partial'_{j)}\Lambda + \frac{\phi\Lambda}{\sigma}h_{ij} + \\ &+ (\sigma\phi + \beta\phi x'^l A_l)\partial'^2_{ij}\Lambda + \phi^2\lambda_{ij}, \end{aligned} \tag{15}$$

where, ∂'_i denotes partial differentiation with respect to x'^i, $A_i \equiv A_{(1)i} + A_{(2)i}$ and $\lambda_{ij} \equiv 1/2(\partial'^2_{ij}\Lambda)$. As we can observe this metric tensor contains a Randers part, between the curly braces, which describes the direct electromagnetic interaction of the test particle with each electromagnetic field. As we have already discussed, there is a region U on the manifold "bent" by the gravitational fluctuations between the electromagnetic waves. This additional curvature causes other gravitational effects like: test particle-region U, electromagnetic waves-region U and electromagnetic waves-test particle-region U effects. Consequently, the rest of the terms in (15) is associated with such effects. Therefore, it is obvious that the equation of motion for this particle contains additional gravitational terms, relevant to the gravitational interaction of the electromagnetic fields:

$$x''^i + a^{jk}\phi_{,k}x''' + \Gamma^i_{jl}x'^j x'^l + \widetilde{\gamma}(x,x')x'^j + \sigma\beta x'^l \widetilde{F}^{\,j}_l + \sigma\widetilde{\Lambda}^i(x,x') = 0 \tag{16}$$

were $\phi_{st} = f_{st} - g_{st}$, $\widetilde{\gamma}(x, x') \equiv -\dfrac{g_{mn,l}x'^m x'^n x'^l + g_{mn}x'^m x''^n}{2\sigma^2}$, $F_{ij} \equiv 2A_{[j,i]}$, with

$a_{[ij]} \equiv \dfrac{1}{2}(a_{ij} - a_{ji})$, $\widetilde{F}_i^{\ j} \equiv F_{ik}g^{kj}$, $\widetilde{\Lambda}^j(x, x') \equiv \phi_{,l}x'^l g^{kj}\partial'_k\Lambda + \phi g^{kj}\dfrac{d(\partial'_k\Lambda)}{ds} - g^{kj}(\phi\Lambda)_{,k}$.

The tensor F_{ij} is the electromagnetic tensor for both electromagnetic fields. Γ^j_{il} are the Christoffel symbols for the Riemannian metric $g_{ij}(x)$. The object $\widetilde{\gamma}(x, x')$ contains Finslerian terms of the Finslerian connection $\gamma^j_{i,l}$ not depending on the Λ. whereas the $\widetilde{\Lambda}^j(x, x')$ is a Finslerian term of the connection $\gamma^j_{i,l}$ depending on the function Λ.

In order to clarify the role of each term containing $\Lambda\ (x, x')$ in (15), we switch off the field A_j, but leave the fluctuation term $\phi\Lambda(x, x')$ (ϕ being a general function of x, now). We obtain from (15):

$$f_{ij} = g_{ij} + \frac{2\phi}{\sigma}x'^s g_{s(i}\partial'_{j)}\Lambda + \frac{\phi\Lambda}{\sigma}h_{ij} + \sigma\phi\partial'^2_{\ ij}\Lambda + \phi^2\lambda_{ij}. \tag{17}$$

f_{ij} is a metric tensor which describes the gravitational effect of the fluctuation on the test particle. There are zero-order terms like $(\phi\Lambda/\sigma)h_{ij}$, first-order terms like $(2\phi/\sigma)x'^s g_{s(i}\partial'_{j)}\Lambda$ and second-order terms like $\sigma\phi\partial'^2_{ij}\Lambda$. The term $\phi^2\lambda_{ij}$ is itself a general Finslerian metric tensor, which is related with the gravitational waves. Under this point of view, the gravitational waves cause their own metric tensor fields. It is this metric which describes the propagation of gravitational waves in the empty spacetime. Comparing equations (15) and (17), we infer that the remaining terms in (15) (except those in the curly braces) are considered as gravitational interaction terms of the region U with each electromagnetic field alone. We may classify them in first-order ones like $2\phi^2\beta A_{(i}\partial'_{j)}\Lambda$ and second-order ones like $\beta\phi x'^l A_l\partial'^2_{ij}\Lambda$. Of course, due to the 1-degree homogeneity of the model function $\Lambda\ (x, x')\Lambda$, cannot be constant with respect to x'. Therefore, we cannot eliominate higher-order terms without eliminating the zero-order term.

In the following, we may introduce thye independent internal spinorial variables $\xi, \bar{\xi}$ in the Riemannian part of our metric function increasing the internal; degrees of freedom of the space. In this case the metric function has the form:

$$F(x, x', \xi, \bar{\xi}) = G + \frac{q}{mc^2}x'^i A_i, \tag{18}$$

where $G \equiv g_{\mu\nu}(x, \xi\ \bar{\xi})dx^\mu dx^\nu + g_{\alpha\beta}(x, \xi\ \bar{\xi})D\xi^\alpha D\bar{\xi}^{*\beta} + g^{\alpha\beta}(x, \xi\ \bar{\xi})D\xi_\alpha D\xi^*_\beta$ is the metric in the space $M \times C^4 \times C^4$ as has been given in [13].

If the function $\Lambda(x, x') = 0$ and the vectorial potential $A_{(1)i} = A_{(2)i}$, then the metric will be a type of general Randers metric [11]. We do not discuss these forms of metrics now. This will be the subject of a further study in the future.

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