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VARIATIONAL PRINCIPLE TO THE GENERALIZED SCALAR-TENSOR THEORY OF GRAVITATION II

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Abstract. An extension of the previous work to the Finslerian Scalar-Tensor Theory of the gravitational field is being studied (Stavirnos and Ikeda). A generalised Brans-Dicke theory of gravitation is constructed on the basis of the fibered bundle with the metric structure.

$$\hat{G} = g_{\alpha\beta}dx^{\alpha} \otimes dx^{\beta} + \varphi(\delta y^{+} \otimes \delta y^{+} + \delta y^{-} \otimes \delta y^{-}) + 2\psi \delta y^{+} \wedge \delta y^{-}$$

of a base manifold and consists of two non-commutative G-numbers (Grassmannian) or scalars y^{+}, y^{-} playing the role of fibres.

In the framework of our approach, we study : curvatures and the gravitational field equations derived from a variational principle of appropriate Lagrangian densities.

1. Introduction In a previous paper presented by us in the 30th International Conference on Mathematical Physics held in Torun, May 26–30, 1998 (Stavrinos and Ikeda) we developed a new generalized scalar tensor theory of the Finslerian gravitational field on the basis of the fibered bundle. In that approach we considered a base manifold and two G-numbers (Grassmannian) non-commutative y^{+}, y^{-} playing the role of fibres.

In this paper our basic idea is to give the proper Lagrangian in a sufficiently generalized form.

The non-commutative structure on the fibered bundle is dominated by some components of the torsion (Ikeda 1995, Ikeda, Stavrinos and Ikeda) of the Finslerian gravitational field. It is studied without using the noncommutative geometry.

By utilizing the vector budle consideration, the Finslerian gravitational field can be regarded as the unified field over the total space of the vector bundle whose base manifold is the (x)-field and fibre at each point x is the (y)-field.

The inner product of two G-numbers which is a scalar field plays a dominant role in the metric structure of the budle, analogous of the scalar field φ of Brans-Dicke's theory (1961). This field that is introduced in our bundle $\mathcal{M} \times \{y^{+} \times y^{-}\}$ influences the external form of the field equations.

The form of connections and torsions which are used in the present work have been defined in the previous paper (Stavrinos and Ikeda).

In that consideration the adapted frame is set as follows :

$$\begin{aligned} \frac{\partial}{\partial Z^{M}} &\equiv \left(\frac{\delta}{\delta x^{\alpha}} = \frac{\partial}{\partial x^{\alpha}} - N^{+}_{\alpha}\frac{\partial}{\partial y^{+}} - N^{-}_{\alpha}\frac{\partial}{\partial y^{-}}, \frac{\partial}{\partial y^{+}}, \frac{\partial}{\partial y^{-}}\right) \\ dZ^{M} &\equiv \left(dx^{\alpha}, \delta y^{+} = dy^{+} + N^{+}_{\alpha}dx^{\alpha}, \delta y^{-} = dy^{-} + N^{-}_{\alpha}dx^{\alpha}\right) \end{aligned} \tag{1.1}$$

where, we introduced two kinds of non-linear connections, N^{+}_{α} and N^{-}_{α}.

The connection structure is gviven by

$$\nabla_{\frac{\delta}{\delta x^K}} \frac{\delta}{\delta x^M} = \Gamma^N_{MK} \frac{\delta}{\delta x^N} \quad K, M = \{\alpha, \beta, \cdots, +, -\}$$

with $\Gamma^N_{MK} = \{L^N_{M\alpha}, C^N_{M+}, E^N_{M-}\}$.

$$\nabla_{\frac{\delta}{\delta x^\alpha}} \frac{\delta}{\delta x^\beta} = L^\gamma_{\beta\alpha} \frac{\delta}{\delta x^\beta} \quad \nabla_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial y^+} = L^+_{+\alpha} \frac{\partial}{\partial y^+} \quad \nabla_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial y^-} = L^-_{-\alpha} \frac{\partial}{\partial y^-} \tag{1.2a}$$

$$\nabla_{\frac{\partial}{\partial y^+}} \frac{\delta}{\delta x^\alpha} = C^\beta_{\alpha+} \frac{\partial}{\partial x^\beta} \quad \nabla_{\frac{\partial}{\partial y^+}} \frac{\partial}{\partial y^+} = C^+_{++} \frac{\partial}{\partial y^+} \quad \nabla_{\frac{\partial}{\partial y^+}} \frac{\partial}{\partial y^-} = C^-_{-+} \frac{\partial}{\partial y^-} \tag{1.2b}$$

$$\nabla_{\frac{\partial}{\partial y^-}} \frac{\delta}{\delta x^\alpha} = E^\beta_{\alpha-} \frac{\partial}{\partial x^\beta} \quad \nabla_{\frac{\partial}{\partial y^-}} \frac{\partial}{\partial y^+} = E^+_{+-} \frac{\partial}{\partial y^+} \quad \nabla_{\frac{\partial}{\partial y^-}} \frac{\partial}{\partial y^-} = E^-_{--} \frac{\partial}{\partial y^-} \tag{1.2c}$$

For an arbitrary vector $V^M \equiv (V^\alpha, V^+, V^-)$ the covariant derivatives can be defined as follows :

$$\begin{aligned} V^\alpha_{|\gamma} &= \frac{\delta V^\alpha}{\delta x^\gamma} + L^\alpha_{\beta\gamma} V^\beta \\ V^+|_+ &= \frac{\partial V^+}{\partial y^+} + C^+_{++} V^+ \\ V^-\|_- &= \frac{\partial V^-}{\partial y^-} + E^-_{--} V^- \end{aligned} \tag{1.3}$$

Here, the metrical conditions

$$g_{\alpha\beta|\gamma} = 0, \quad g_{++}|_+ = 0 \quad \text{and} \quad g_{--}\|_- = 0$$

were assumed.

2. Torsions and curvatures. We assign the torsion tensors and curvatures for the case of our vector bundle. The torsion tensor field is given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for each differentiable vector fields X, Y, which are defined on the bundle ($X, Y \in \mathcal{X}(T, M)$).

Consequently, we have for the components of the torsion the following relations

$$
\begin{aligned}
T\left(\frac{\delta}{\delta x^{\beta}},\frac{\delta}{\delta x^{\alpha}}\right) &= T^{\gamma}_{\alpha\beta}\frac{\delta}{\delta x^{\gamma}}+T^{+}_{\alpha\beta}\frac{\partial}{\partial y^{+}}+T^{-}_{\alpha\beta}\frac{\partial}{\partial y^{-}}\\
T\left(\frac{\partial}{\partial y^{+}},\frac{\delta}{\delta x^{\alpha}}\right) &= T^{\gamma}_{\alpha+}\frac{\delta}{\delta x^{\gamma}}+T^{+}_{\alpha+}\frac{\partial}{\partial y^{+}}+T^{-}_{\alpha+}\frac{\partial}{\partial y^{-}}\\
T\left(\frac{\partial}{\partial y^{-}},\frac{\delta}{\delta x^{\alpha}}\right) &= T^{\gamma}_{\alpha-}\frac{\delta}{\delta x^{\gamma}}+T^{+}_{\alpha-}\frac{\partial}{\partial y^{+}}+T^{-}_{\alpha-}\frac{\partial}{\partial y^{-}}\\
T\left(\frac{\partial}{\partial y^{-}},\frac{\partial}{\partial y^{+}}\right) &= T^{\gamma}_{+-}\frac{\delta}{\delta x^{\gamma}}+T^{+}_{+-}\frac{\partial}{\partial y^{+}}+T^{-}_{+-}\frac{\partial}{\partial y^{-}}\\
T\left(\frac{\partial}{\partial y^{+}},\frac{\partial}{\partial y^{+}}\right) &= T^{\gamma}_{++}\frac{\delta}{\delta x^{\gamma}}+T^{+}_{++}\frac{\partial}{\partial y^{+}}+T^{-}_{++}\frac{\partial}{\partial y^{-}}\\
T\left(\frac{\partial}{\partial y^{-}},\frac{\partial}{\partial y^{-}}\right) &= T^{\gamma}_{--}\frac{\delta}{\delta x^{\gamma}}+T^{+}_{--}\frac{\partial}{\partial y^{+}}+T^{-}_{--}\frac{\partial}{\partial y^{-}}
\end{aligned}
\tag{2.1}
$$

The explicit forms of (2.1) have been given in (Stavrinos and Ikeda)

$$T^{\gamma}_{\alpha\beta}=L^{\gamma}_{\alpha\beta}-L^{\gamma}_{\beta\alpha} \qquad T^{+}_{\alpha\beta}=-R^{+}_{\beta\alpha} \qquad T^{-}_{\alpha\beta}=-V^{-}_{\beta\alpha} \tag{2.2a}$$

$$T^{\gamma}_{\alpha+}=C^{\gamma}_{\alpha+} \qquad T^{+}_{\alpha+}=\frac{\partial N^{+}_{\alpha}}{\partial y^{+}}-L^{+}_{\alpha+} \qquad T^{-}_{\alpha+}=-\frac{\partial N^{-}_{\alpha}}{\partial y^{+}} \tag{2.2b}$$

$$T^{\gamma}_{\alpha-}=-E^{\gamma}_{-\alpha} \qquad T^{+}_{\alpha-}=-\frac{\partial N^{+}_{\alpha}}{\partial y^{-}} \qquad T^{-}_{\alpha-}=L^{-}_{-\alpha}-\frac{\partial N^{-}_{\alpha}}{\partial y^{-}} \tag{2.2c}$$

$$T^{\gamma}_{+-}=0 \qquad T^{+}_{+-}=-E^{+}_{-+} \qquad T^{-}_{+-}=C^{-}_{-+} \tag{2.2d}$$

$$T^{\gamma}_{++}=0 \qquad T^{+}_{++}=C^{+}_{++}=0 \qquad T^{-}_{++}=0 \tag{2.2e}$$

$$T^{\gamma}_{--}=0 \qquad T^{+}_{--}=0 \qquad T^{-}_{--}=0 \tag{2.2f}$$

For the derivation of the above relations, we have taken into account the brackets :

$$
\begin{aligned}
\left[\frac{\delta}{\delta x^{\alpha}},\frac{\delta}{\delta x^{\beta}}\right] &= R^{+}_{\alpha\beta}\frac{\partial}{\partial y^{+}}+V^{-}_{\alpha\beta}\frac{\partial}{\partial y^{-}}\\
\left[\frac{\partial}{\partial y^{+}},\frac{\delta}{\delta x^{\alpha}}\right] &= \left[\frac{\partial}{\partial y^{-}},\frac{\delta}{\delta x^{\alpha}}\right]=0\\
\left[\frac{\partial}{\partial y^{+}},\frac{\partial}{\partial y^{-}}\right] &= \left[\frac{\partial}{\partial y^{+}},\frac{\partial}{\partial y^{+}}\right]=\left[\frac{\partial}{\partial y^{-}},\frac{\partial}{\partial y^{-}}\right]=0
\end{aligned}
\tag{2.3}
$$

ıere we have put $R^{+}_{\alpha\beta}=\dfrac{\delta N^{+}_{\beta}}{\delta x^{\alpha}}-\dfrac{\delta N^{-}}{\delta x^{\beta}}$, $V^{-}_{\alpha\beta}=\dfrac{\delta N^{-}_{\beta}}{\delta x^{\alpha}}-\dfrac{\delta N^{+}_{\alpha}}{\delta x^{\beta}}$.

In the relation (2.2d) the two last terms, T^+_{-+} and T^-_{-+} represent the non-cummutative structure. The connection has no torsion if and only if the coefficients (2.2) are equal to zero.

With the above mentioned connection coeffcients we can associate the curvature tensor with the adopted basis $X_M = \{X_\alpha, X_+, X_-\}$, $X_\alpha = \frac{\delta}{\delta x^\alpha}$, $X_+ = \frac{\partial}{\partial y^+}$, $X_- = \frac{\partial}{\partial y^-}$.

$$\begin{aligned} K(X_M,X_N)X_L &= \mathcal{K}^K_{LNM}X_K \\ \mathcal{K}^K_{LNM} &= X_L\Gamma^K_{NM} - X_N\Gamma^K_{LM} + \Gamma^Z_{LN}\Gamma^K_{ZM} - \Gamma^Z_{LM}\Gamma^K_{ZN} + \Gamma^K_{LZ}W^Z_{NM} \end{aligned} \tag{2.4}$$

where W^Z_{NM} are the non-holonomy coefficients in the local basis X_M.

$$[X_M, X_N] = W^Z_{NM}X_Z$$

Twelve kinds of curvature tensors appear,

$$\mathcal{K}^K_{LMN} \equiv (K^K_{L\alpha\beta}, K^K_{L\alpha+}, K^K_{L\alpha-}, K^K_{L+-}) \tag{2.5}$$

where $K^K_{L\alpha\beta} = \left(K^\gamma_{\delta\alpha\beta}, K^+_{+\alpha,\beta}, K^-_{-\alpha\beta}\right)$ etc. The explicit form of these curvatures can be derived in a similar way to that developed in (Stavrinos and Ikeda).

The Ricci-tensors are given by

$$\mathcal{K}_{MN} \equiv K^L_{MLN} \equiv (K_{\alpha\beta}, K_{\alpha+}, K_{\alpha-}, \cdots, K_{+-}, -K_{-+}, -K_{+\alpha}, -K_{-\alpha}) \tag{2.6}$$

or in the form

$$\mathcal{K}_{MN} = \{K_{MN}{}^\alpha_\alpha, K_{MN}{}^+_+, K_{MN}{}^-_-\}$$

where we have put

$$\begin{aligned} K_{MN}{}^\alpha_\alpha &= \frac{\delta L^\alpha_{MN}}{\delta x^\alpha} - \frac{\delta L^\alpha_{M\alpha}}{\delta x^N} + L^\alpha_{K\alpha}L^K_{MN} - L^\alpha_{KN}L^K_{M\alpha} + L^\alpha_{MZ}W^Z_{N\alpha} \\ K_{MN}{}^+_+ &= \frac{\partial C^+_{MN}}{\partial y^+} - \frac{\partial C^+_{M+}}{\partial y^N} + C^+_{K+}C^K_{MN} - C^+_{KN}C^K_{M+} + C^+_{MZ}W^Z_{N+} \\ K_{MN}{}^-_- &= \frac{\partial E^-_{MN}}{\partial y^-} - \frac{\partial E^-_{M-}}{\partial y^N} + E^-_{K-}E^K_{MN} - E^-_{KN}E^K_{M-} + E^-_{MZ}W^z_{N-} \end{aligned}$$

The scalar curvature has the form

$$\begin{aligned} \mathcal{K} &= M_{MN}G^{MN} = K_{\alpha\beta}g^{\alpha\beta} + K_{++}g^{++} + K_{--}g^{--} \\ &= K + K^{(+)} + K^{(-)} \end{aligned} \tag{2.7}$$

$$K_-{}^-{}_{+\alpha} = \frac{\delta C_-{}^-{}_+}{\delta x^\alpha} - \frac{\partial L_-{}^-{}_\alpha}{\partial y^+} - C_-{}^-{}_+\frac{\partial N^+_\alpha}{\partial y^+} - E_-{}^-{}_-\frac{\partial N^-_\alpha}{\partial y^-} \tag{2.8}$$

$$K_-{}^-{}_{-\alpha} = \frac{\partial E_-{}^-{}_-}{\delta x^\alpha} - \frac{\partial L_-{}^-{}_\alpha}{\partial y^-} - C_-{}^-{}_-\frac{\partial N^+_\alpha}{\partial y^-} - E_-{}^-{}_-\frac{\partial N^-_\alpha}{\partial y^-} \tag{2.9}$$

$$K_{+}{}^{+}{}_{+-} = \frac{\partial C_{+}{}^{+}{}_{+}}{\partial y^{-}} - \frac{\partial E_{+}{}^{+}{}_{-}}{\partial y^{+}} \tag{2.10}$$

$$K_{-}{}^{-}{}_{+-} = \frac{\partial C_{-}{}^{-}{}_{+}}{\partial y^{-}} - \frac{\partial L_{-}{}^{-}{}_{+}}{\partial y^{+}} \tag{2.11}$$

$$K_{+}{}^{+}{}_{++} = K_{-}{}^{+}{}_{++} = K_{+}{}^{-}{}_{++} = K_{-}{}^{-}{}_{++} = K_{+}{}^{+}{}_{--} = K_{-}{}^{+}{}_{--} = 0 \tag{2.12}$$

$$= K_{+}{}^{-}{}_{--} = K_{-}{}^{-}{}_{--} = K_{-}{}^{+}{}_{+-} = K_{+}{}^{-}{}_{+-} = K_{-}{}^{+}{}_{-+} = K_{+}{}^{-}{}_{-+} = 0 \tag{2.13}$$

The components of Ricci tensor with respect to a basis

$$(\tilde{e}_M) \equiv \left(\frac{\delta}{\delta x^{\mu}}, \frac{\partial}{\partial y^{+}}, \frac{\partial}{\partial y^{-}}\right) \tag{2.14}$$

of a local coordinate system of the bundle will be given by

$$K_{\alpha\beta} = \frac{\delta L_{\alpha}{}^{\mu}{}_{\beta}}{\delta x^{\mu}} - \frac{\delta L_{\alpha}{}^{\mu}{}_{\mu}}{\delta x^{\beta}} + L_{\alpha}{}^{\nu}{}_{\beta} L_{\nu}{}^{\mu}{}_{\mu} - L_{\alpha}{}^{\nu}{}_{\mu} L_{\nu}{}^{\mu}{}_{\beta}$$
$$+ C_{\alpha}{}^{\mu} + V_{\beta}{}^{+}{}_{\mu} + E_{\alpha}{}^{\mu}{}_{-} V_{\beta}{}^{-}{}_{\mu} \tag{2.15}$$

$$K_{+\alpha} = \frac{\partial L_{+}{}^{+}{}_{\alpha}}{\partial y^{+}} - \frac{\delta C_{+}{}^{+}{}_{+}}{\delta x^{\alpha}} + C_{+}{}^{+}{}_{+} \frac{\partial N_{\alpha}^{+}}{\partial y^{+}} + E_{+}{}^{+}{}_{-} \frac{\partial N_{\alpha}^{-}}{\partial y^{+}} \tag{2.16}$$

$$K_{-\alpha} = \frac{\partial L_{-}{}^{-}{}_{\alpha}}{\partial y^{-}} - \frac{\delta E_{-}{}^{-}{}_{-}}{\delta x^{\alpha}} + C_{-}{}^{-}{}_{+} \frac{\partial N_{\alpha}^{+}}{\partial y^{-}} - E_{-}{}^{-}{}_{-} \frac{\partial N_{\alpha}^{-}}{\partial y^{-}} \tag{2.17}$$

$$K_{\alpha+} = \frac{\delta C_{\alpha}{}^{\mu}{}_{+}}{\delta x^{\mu}} - \frac{\partial L_{\alpha}{}^{\mu}{}_{\mu}}{\partial y^{+}} + C_{\alpha}{}^{\nu}{}_{+} L_{\nu}{}^{\mu}{}_{\mu} - L_{\alpha}{}^{\nu}{}_{\mu} C_{\nu}{}^{\mu}{}_{+} +$$
$$+ C_{\alpha}{}^{\mu}{}_{+} \frac{\partial N_{\mu}^{+}}{\partial y^{+}} - E_{\alpha}{}^{\mu}{}_{+} \frac{\partial N_{\mu}^{-}}{\partial y^{+}} \tag{2.18}$$

$$K_{\alpha-} = \frac{\delta E_{\alpha}{}^{\mu}{}_{-}}{\delta x^{\mu}} - \frac{\partial L_{\alpha}{}^{\mu}{}_{\mu}}{\partial y^{-}} + E_{\alpha}{}^{\nu}{}_{-} L_{\nu}{}^{\mu}{}_{\mu} - L_{\alpha}{}^{\nu}{}_{\mu} - L_{\alpha}{}^{\nu}{}_{\mu} E_{\nu}{}^{\mu}{}_{-}$$
$$- C_{\alpha}{}^{\mu}{}_{+} \frac{\partial N_{\mu}^{+}}{\partial y^{-}} - E_{\alpha}{}^{\mu}{}_{-} \frac{\partial N_{\mu}^{-}}{\partial y^{-}} \tag{2.19}$$

$$K_{+-} = \frac{\partial E_{+}{}^{+}{}_{-}}{\partial y^{+}} - \frac{\partial C_{+}{}^{+}{}_{+}}{\partial y^{-}} \tag{2.20}$$

$$K_{-+} = \frac{\partial C_{-}{}^{-}{}_{+}}{\partial y^{+}} - \frac{\partial E_{-}{}^{-}{}_{-}}{\partial y^{-}} \tag{2.21}$$

$$K_{--} = K_{++} = 0 \tag{2.22}$$

3. The metric structure. We consider a metric $\tilde{g}_{AB}$ that is presented by a matrix of a block-diagonal form :

$$(\tilde{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} & \bigcirc \\ \bigcirc & A \end{pmatrix} \tag{3.1}$$

where

$$A = \begin{pmatrix} \tilde{g}_{++} & \tilde{g}_{+-} \\ \tilde{g}_{-+} & \tilde{g}_{--} \end{pmatrix}$$

the matrix A has the form $A = \begin{pmatrix} \lambda^2 & \psi \\ -\psi & \lambda^2 \end{pmatrix} = \begin{pmatrix} \varphi & \psi \\ \psi & \varphi \end{pmatrix}$. The metric $g_{\alpha\beta}$ represents the usual one of the 4-dimensional space-time. Consequently, we will have the following components :

$$\tilde{g}_{++} = \tilde{g}_{--} = \varphi(x) \tag{3.2}$$
$$\tilde{g}_{+-} = -\tilde{g}_{-+} = \psi(x) \tag{3.3}$$
$$\tilde{g}_{\alpha+} = \tilde{g}_{+\alpha} = \tilde{g}_{-\alpha} = \tilde{g}_{-\alpha} = 0 \tag{3.4}$$

Under these circumstances the components $g_{\alpha\beta}$ as well as the scalar fields φ, ψ, are independent of y^+, y^-; these depend on the space-time coordinates $x^\mu, \mu = (0,1,2,3)$.

So, we generally have

$$\frac{\partial \tilde{g}_{AB}}{\partial y^+} = \frac{\partial \tilde{g}_{AB}}{\partial y^-} = 0 \tag{3.5}$$

The inverse of $\tilde{g}_{AB}$ is written in the form $\tilde{g}^{AB}$

$$(\tilde{g}^{AB}) = \begin{pmatrix} (g_{\alpha\beta})^{-1} & \bigcirc \\ \bigcirc & A^{-1} \end{pmatrix} \tag{3.6}$$

where

$$A^{-1} = \begin{pmatrix} \tilde{g}^{++} & \tilde{g}^{+-} \\ \tilde{g}^{-+} & \tilde{g}^{++} \end{pmatrix}$$

the metric tensors $\tilde{g}^{++}$, $\tilde{g}^{+-}$, $\tilde{g}^{+-}$, $\tilde{g}^{++}$, will have the form

$$\tilde{g}^{++} = \tilde{g}^{--} = \frac{\varphi}{\varphi^2 + \psi^2} \tag{3.7}$$
$$\tilde{g}^{+-} = \tilde{g}^{-+} = \frac{\psi}{\varphi^2 + \psi^2} \tag{3.8}$$
$$\tilde{g}^{\alpha+} = \tilde{g}^{+\alpha} = \tilde{g}^{\alpha-} = \tilde{g}^{-\alpha} = 0 \tag{3.9}$$

The metric structure in this case because of the previous relations will take the form :

$$\hat{G} = g_{\alpha\beta} dx^\alpha \otimes dx^\beta + \varphi(\delta y^+ \otimes \delta y^+ + \delta y^- \otimes \delta y^-) + 2\psi \delta y^+ \wedge \delta y^- \tag{3.10}$$

with the determinant $\hat{G} = \det(\tilde{g}_{AB}) = \det(g_{\alpha\beta}), \det(A) = g(\varphi^2 + \psi^2) \neq 0$.

4. The variational problem.

4.1 The Lagrangian and the field equations. Our Lagrangian has analogous form of the Lagrangian $\mathcal{L} = \sqrt{|g|} g^{\alpha\beta} R_{\alpha\beta}$, that leads to the Einstein equations and describe the gravitational field for the empty space of a Riemannian space-time. The counterpart Lagrangian of our vector bundle $\mathcal{N} = M \times \{y^+\} \times \{y^-\}$ takes the following form :

$$L\sqrt{|\tilde{G}|}\tilde{G}^{AB}K_{AB} = \sqrt{|\tilde{G}|}K \tag{4.1}$$

where $K = \tilde{G}^{AB}K_{AB}$, and $K_{AB} = K_{\alpha\beta}+K_{+\alpha}+K_{-\alpha}+K_{\alpha+}+K_{\alpha-}+K_{+-}$. Using the variation of the Lagrangian :

$$L = L\left(G_{AB}; \Gamma_B{}^A{}_\Delta; \frac{\delta\Gamma_B{}^A{}_\Delta}{\delta Z^M}\right) \tag{4.2}$$

in which the variables $G_{AB}, \Gamma_B{}^A{}_\Delta, \frac{\delta\Gamma_B{}^A{}_\Delta}{\delta Z^M}$ are considered independent each other (Palatini method) we can obtain the gravitational field equations.

The integral of action I is given by

$$I = \int_D L\left(G_{AB}; \Gamma_B{}^A{}_\Delta; \frac{\delta\Gamma_B{}^A{}_\Delta}{\delta Z^M}\right) \tag{4.3}$$

The variation of the integral I

$$\delta I = \delta \int_D L\delta(Z) = 0 \tag{4.4}$$

After straightforward calculations we get

$$\delta I = \int_D \left(\Lambda^{AB}\delta G_{BA} + Q^A{}_M{}^B \delta\Gamma_A{}^M{}_B\right)\delta Z \tag{4.5}$$

In order to acquire $\delta I = 0$ for arbitrary δG_{BA}, $\delta\Gamma_A{}^M{}_B$, we must have the following conditions :

$$\Lambda^{AB} = 0 \quad \text{and} \quad Q^A{}_M{}^B = 0 \tag{4.6}$$

where

$$\Lambda^{AB} = K^{AB} - \frac{1}{2}KG^{AB} = 0 \tag{4.7}$$

$$Q^A{}_M{}^B = \sqrt{|G|}G^{AN}T_N{}^B{}_M + \nabla_N\left(\sqrt{|G|}G^{AN}\right)\delta_M{}^B - \nabla_M\left(\sqrt{|G|}G^{AB}\right) = 0. \tag{4.8}$$

The above condensed form of the equations represent the gravitational field equations in a sufficiently generalized form of our space and is valid for arbitrary choice of a metric and of connection.

The explicit form of the components of $Q^A{}_M{}^B$ after long calculations is given by the following relations.

$$Q^{-+}{}_\alpha = \frac{\psi}{\sqrt{\varphi^2+\psi^2}}\frac{\delta\sqrt{|g|}}{\delta x^\alpha} - \frac{\sqrt{|g|}\psi}{\varphi^2+\psi^2}\frac{\delta}{\delta x^\alpha}\sqrt{\varphi^2+\psi^2}$$

$$
\begin{aligned}
&-\sqrt{|g|(\varphi^2+\psi^2)}\frac{\delta}{\delta x^\alpha}\left(\frac{\psi}{\varphi^2+\psi^2}\right)\\
&+\sqrt{\frac{|g|}{\varphi^2+\psi^2}}\left[\psi\left(L_\alpha{}^\mu{}_\mu - L_-{}^-{}_\alpha - \frac{\partial N_\alpha^-}{\partial y^+}\right) - \varphi\frac{\partial N_\alpha^+}{\partial y^-}\right] \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
Q^{+-}{}_- = &\frac{\psi}{\sqrt{\varphi^2+\psi^2}}\frac{\delta\sqrt{|g|}}{\delta x^\alpha} + \frac{\sqrt{|g|}\psi}{\varphi^2+\psi^2}\frac{\delta}{\delta x^\alpha}\sqrt{\varphi^2+\psi^2}\\
&+\sqrt{|g|(\varphi^2+\psi^2)}\frac{\delta}{\delta x^2}\left(\frac{\psi}{\varphi^2+\psi^2}\right)\\
&-\sqrt{\frac{|g|}{\varphi^2+\psi^2}}\left[\varphi\frac{\partial N_\alpha^-}{\partial y^+} + \psi\left(L_\alpha{}^\mu{}_\mu - L_+{}^+{}_\alpha - \frac{\partial N_\alpha^-}{\partial y^-}\right)\right] \qquad (4.10)
\end{aligned}
$$

$$
\begin{aligned}
Q^{\alpha+}{}_+ = &\sqrt{\varphi^2+\psi^2}g^{\alpha\mu}\frac{\delta\sqrt{|g|}}{\delta x^\mu} + \sqrt{|g|}g^{\alpha\mu}\frac{\delta}{\delta x^\mu}\sqrt{\varphi^2+\psi^2}\\
&+\sqrt{|g|(\varphi^2+\psi^2)}\left(\nabla_\mu g^{\alpha\mu} - g^{\alpha\mu}L_\mu{}^\nu{}_\nu + g^{\alpha\mu}\frac{\partial N_\mu^+}{\partial y^+} + g^{\alpha\mu}L_+{}^+{}_\mu\right) \qquad (4.11)
\end{aligned}
$$

$$
Q^{\alpha+}{}_- = \sqrt{|g|(\varphi^2+\psi^2)}g^{\alpha\mu}\frac{\partial N_\mu^+}{\partial y^-} \qquad (4.12)
$$

$$
Q^{\alpha-}{}_+ = \sqrt{|g|(\varphi^2+\psi^2)}g^{\alpha\mu}\frac{\partial N_\mu^-}{\partial y^+} \qquad (4.13)
$$

$$
\begin{aligned}
Q^{\alpha-}{}_- = &\sqrt{\varphi^2+\psi^2}g^{\alpha\mu}\frac{\delta\sqrt{|g|}}{\delta x^\mu} + \sqrt{|g|}g^{\alpha\mu}\frac{\delta}{\delta x^\mu}\sqrt{\varphi^2+\phi^2}\\
&+\sqrt{|g|(\varphi^2+\psi^2)}\left(\nabla_\mu g^{\alpha\mu} - g^{\alpha\mu}L_\mu{}^\nu{}_\nu + g^{\alpha\mu}\frac{\partial N_\mu^-}{\partial y^-} - g^{\alpha\mu}L_-{}^-{}_\mu\right) \qquad (4.14)
\end{aligned}
$$

$$
\begin{aligned}
Q^{+\alpha}{}_\beta = &\frac{1}{\sqrt{\varphi^2+\psi^2}}\left(\varphi\frac{\partial\sqrt{|g|}}{\partial y^+} - \psi\frac{\partial\sqrt{|g|}}{\partial y^-}\right)\delta^\alpha_\beta\\
&+\frac{\sqrt{|g|}}{\varphi^2+\psi^2}\left[\varphi\frac{\partial}{\partial y^+}\left(\sqrt{\varphi^2+\psi^2}\right) - \psi\frac{\partial}{\partial y^-}\left(\sqrt{\varphi^2+\psi^2}\right)\right]\delta^\alpha_\beta\\
&+\sqrt{|g|(\varphi^2+\psi^2)}\left[\frac{\partial}{\partial y^+}\left(\frac{\varphi}{\varphi^2+\psi^2}\right) - \frac{\partial}{\partial y^-}\left(\frac{\psi}{\varphi^2+\psi^2}\right)\right]d^\alpha_\beta\\
&+\frac{\sqrt{|g|}}{\sqrt{\varphi^2+\psi^2}}\left[\varphi\left(C_+{}^+{}_+\delta^\alpha_\beta - C_\beta{}^\alpha{}_+\right) - \psi(E_+{}^+{}_-\delta^\alpha_\beta - E_\beta{}^\alpha{}_-)\right] \qquad (4.15)
\end{aligned}
$$

$$
Q^{-\alpha}{}_\beta = \frac{1}{\sqrt{\varphi^2+\psi^2}}\left(\psi\frac{\partial\sqrt{|g|}}{\partial y^+} - \varphi\frac{\partial\sqrt{|g|}}{\partial y^-}\right)\delta^\alpha_\beta
$$

$$+\frac{\sqrt{|g|}}{\varphi^2+\psi^2}\left[\psi\frac{\partial}{\partial y^+}(\sqrt{\varphi^2+\psi^2})+\varphi\frac{\partial}{\partial y^-}(\sqrt{\varphi^2+\psi^2})\right]\delta^\alpha_\beta$$
$$+\sqrt{|g|(\varphi^2+\psi^2)}\left[\frac{\partial}{\partial y^+}\left(\frac{\psi}{\varphi^2+\psi^2}\right)+\frac{\partial}{\partial y^-}\left(\frac{\varphi}{\varphi^2+\psi^2}\right)\right]\delta^\alpha_\beta$$
$$+\frac{\sqrt{|g|}}{\sqrt{\varphi^2+\psi^2}}\left[\psi\left(C_-{}^-{}_+\delta^\alpha_\beta-C_\beta{}^\alpha{}_+\right)-\varphi\left(E_-{}^-{}_-\delta^\alpha_\beta-E_\beta{}^\alpha{}_-\right)\right] \quad (4.16)$$

$$Q^{\alpha\beta}{}_\gamma=\sqrt{\varphi^2+\psi^2}\left(g^{\alpha\mu}\delta^\beta_\gamma\frac{\delta\sqrt{|g|}}{\delta x^\mu}-g^{\alpha\beta}\frac{\delta\sqrt{|g|}}{\delta x^\gamma}\right)$$
$$+\sqrt{|g|}\left[g^{\alpha\mu}\delta^\beta_\gamma\frac{\delta}{\delta x^\mu}\left(\sqrt{\varphi^2+\psi^2}\right)+g^{\alpha\beta}\frac{\delta}{\delta x^\gamma}\left(\sqrt{\varphi^2+\psi^2}\right)\right]$$
$$+\sqrt{|g|(\varphi^2+\psi^2)}\left(\nabla_\mu g^{\alpha\mu}\delta^\beta\gamma-\nabla_\nabla g^{\alpha\beta}\right)$$
$$+\sqrt{|g|(\varphi^2+\psi^2)}\left[g^{\alpha\beta}L_\gamma{}^\mu{}_\mu+g^{\alpha\mu}\left(L_\mu{}^\beta{}_\gamma-L_\gamma{}^\beta{}_\mu-L_\mu{}^\nu{}_\nu d\delta^\beta_\gamma\right)\right] \quad (4.17)$$

$$Q^{\alpha\beta}{}_+=-\sqrt{\varphi^2+\psi^2}g^{\alpha\beta}\frac{\partial\sqrt{|g|}}{\partial y^+}-\sqrt{|g|}g^{\alpha\beta}\frac{\partial}{\partial y^+}\left(\sqrt{\varphi^2+\psi^2}\right)$$
$$+\sqrt{|g|(\varphi^2+\psi^2)}\left(g^{\alpha\beta}C_+{}^+{}_++g^{\alpha\mu}C_\mu{}^\beta{}_+-\nabla_+g^{\alpha\beta}\right) \quad (4.18)$$

$$Q^{\alpha\beta}{}_-=-\sqrt{\varphi^2+\psi^2}g^{\alpha\beta}\frac{\partial\sqrt{|g|}}{\partial y^-}-\sqrt{|g|}g^{\alpha\beta}\frac{\partial}{\partial y^-}(\sqrt{\varphi^2+\psi^2})$$
$$+\sqrt{|g|(\varphi^2+\psi^2)}\left(g^{\alpha\beta}E_-{}^-{}_-+g^{\alpha\mu}E_\mu{}^\beta{}_--\nabla_-g^{\alpha\beta}\right) \quad (4.19)$$

$$Q^{\alpha+}{}_\beta=\sqrt{|g|(\varphi^2+\psi^2)}g^{\alpha\mu}V_\mu{}^+{}_\beta=\sqrt{|g|(\varphi^2+\psi^2)}g^{\alpha\mu}\left(\frac{\delta N^+_\mu}{\delta x^\beta}-\frac{\delta N^+_\beta}{\delta x^\mu}\right) \quad (4.20)$$

$$Q^{\alpha-}{}_\beta=\sqrt{|g|(\varphi^2+\psi^2)}g^{\alpha\mu}V_\mu{}^-{}_\beta=\sqrt{|g|(\varphi^2+\psi^2)}g^{\alpha\mu}\left(\frac{\delta N^-_\mu}{\delta x^\beta}-\frac{\delta N^-_\beta}{\delta x^\mu}\right) \quad (4.21)$$

Discussions. In this paper, we extend the results of the generalized scalar-tensor theory of the Finslerian gravitational field which was given by us in a previous work (Stavrinos and Ikeda). In the framework of this appoach we considered the metric structure of our space (3.10) in a sufficiently generalized form and we explicitely studied the curvatures and Lagrangian field equations (4.7).

The two scalars or Grassmanian numbers y^+, y^- were introduced within the Finslerian fibre bundle in order to give rise to the non-commuative structure in the metrical and connections structure of gravitational field.

From a physical point of view, these numbers can play a role for a unification of the Gauge and Higgs fields or can be regarded as two components of a spinor.

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