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FINSLERIAN STRUCTURE OF ANISOTROPIC GRAVITATIONAL FIELD

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We study an anisotropic model of general relativity in the framework of Finslerian geometry. The observed anisotropy of the microwave background radiation [11] is represented by an anisotropy tensor which is incorporated in the Finslerian structure of space-time [12, 13]. The Einstein equations are derived for the case of a constant curvature Finsler space-time. We also examine the electromagnetic (EM) field equations in our space. As a result, a modified wave equation of EM waves yields.

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Исследуется анизотропная модель ОТО в рамках финслеровой геометрии. Наблюдаемая анизотропия микроволнового фонового излучения [11] представляется тензором анизотропии, включенным в финслерову структуру пространства-времени [12, 13]. Выведены уравнения Эйнштейна для случая финслерова пространства-времени постоянной кривизны. Исследуются также уравнения электромагнитного поля в данном пространстве-времени. В результате получено модифицированное воновое уравнение для электромагнитных волн.

1. Introduction

In the recent years, some observational astrophysical results have shown that an anisotropic direction-dependent expansion of the Universe may be present if the underlying geometry of the Universe is anisotropic. In this case [10, 17], the isotropic Robertson-Walker metric is no longer valid. It is therefore necessary to take seriously the possibility that the Universe is anisotropic and to investigate what effect will have an anisotropic expansion on the angular distribution of the background radiation.

The direction of present research, regarding the anisotropy of the Universe, is to consider fluctuations of the homogeneous isotropic model, e.g., [5]. The fluctuations are due to anisotropic distribution of particles. The anisotropy is hidden in the particle distribution function, which do not affect the geometry [5] and have to be unlocked by interactions among the particles (at a later stage in the evolution of the Universe). It seems then natural to choose an intrinsically anisotropic geometrical model for the description of space-time [2]. For, as mentioned above, fluctuations retain the geometric concepts as in the homogeneous isotropic case. Finsler's geometry, on the other hand, has fundamentally different geometrical concepts from the homogeneous and isotropic model, i.e., it incorporates the anisotropy intrinsically in the geometry of space. It appears then as a valid candidate for the construction of such a theory.

A Finslerian geometrical structure of models which can correspond to anisotropic structures of spacetime regions (radius$\leq 10^8$ light years) can be introduced. Our work was motivated by the observed anisotropy of the microwave cosmic radiation. This anisotropy is of dipole type, i.e., the radiation intensity is maximum in one direction and minimum in the opposite direction.

In the conventional theory this anisotropy can be explained if we use the Robertson-Walker metric and take into account the motion of our galaxy with respect to distant galaxies of the Universe [14]. A small anisotropy is expected, however, due to the anisotropic distribution of galaxies in space [11].

From the above-mentioned results, it is reasonable to seek a Lagrangian which expresses this anisotropy. As such, we choose [12, 13]

$$\mathcal{L} = \sqrt{a_{ij} y^i y^j} + \varphi(x) \hat{k}_a y^a \tag{1}$$

The vector $\hat{k}_a$ expresses the observed anisotropy of the microwave background radiation.

The intrinsic behavior of the internal vector $\varphi(x)\hat{k}^i$ in which the matter density is hidden (cf. rel. (45)) can be considered as a property of the field itself. This standpoint can be thought of as a unified description between the external x^i-field and the internal $\varphi(x)\hat{k}^i$-field. Therefore the framework of the geometry of total

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space of the tangent bundle is a Finsler space, which is a convenient tool for the description of this field.

In Sec. 2. we give the necessary mathematical formalism, upon which we develop our theory.

In Sec. 3. we develop the geometric anisotropic structure of space-time based on the tangent bundle. Some physical interpretations are given.

In Sec. 4. we derive the Einstein equations for the case of constant Finsler curvature.

In Sec. 5. we study the changes imposed on the electromagnetic field as a result of the anisotropic geometry. It is shown that the EM field tensor remains unchanged in our approach. The wave equation of EM waves is modified ($\Box A^i(x)$) in such a way that it expresses an anisotropy of the electromagnetic field, i.e., in the generalized d'Alambertian there exist terms of anisotropy (namely, the Ricci tensor $R_i{}^j$ of the Finsler curvature $R^i{}_{jkl}$, the curvature of the non-linear connection $R^i{}_{jk}$ and the Cartan coefficients C^i_{jk}) which affect the conventional form of the wave equation.

2. Preliminaries

The framework in which we develop our present work is a Finsler tangent bundle. For this purpose, we consider a smooth 4-dimensional pseudo-Riemannian manifold M, (TM, π, M), its tangent bundle, and $\tilde{TM} = TM \setminus \{0\}$, where 0 means the image of the null cross-section of the projection $\pi : TM \to M$. We also consider a local system of coordinates (x^i), $i = 0,1,2,3$ and U, a chart of M. Then the couple (x^i, y^a) is a local coordinate system on $\pi^{-1}(U)$ in TM. A coordinate transformation on the total space TM is given by

$$\begin{aligned} \tilde{x}^i &= \tilde{x}^i(x^0, \dots, x^3), \qquad \det\left\| \frac{\partial \tilde{x}^i}{\partial x^j} \right\| \neq 0, \\ \tilde{y}^a &= \frac{\partial \tilde{x}^a}{\partial x^b} y^b, \qquad x^a = \delta^a_i x^i. \end{aligned} \tag{2}$$

By definition [7], a Finsler metric on M is a function $F : TM \to \mathbb{R}$ having the properties:

1. The restriction of F to $\tilde{TM}$ is of class C^∞, and F is only continuous on the image of the null cross section in the tangent bundle to M.

2. The restriction of F to $\tilde{TM}$ is positively homogeneous of degree 1 with respect to (y^a),

$$F(x, ky) = kF(x,y), \qquad k \in \mathbb{R}^*_+.$$

3. The quadratic form on $\mathbb{R}^n$ with the coefficients

$$f_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{3}$$

defined on $\tilde{TM}$ is non-degenerate ($\det(f_{ij}) \neq 0$), with $\mathrm{rank}\,(f_{ij}) = 4$.

A non-linear connection N on TM is a distribution on TM, supplementary to the vertical distribution V on TM:

$$T_{(x,y)}(TM) = N_{(x,y)} \oplus V_{(x,y)}$$

In our case, a non-linear connection can be defined by

$$N^a_j = \frac{\partial G^a}{\partial y^j} \tag{4}$$

where G^a are defined from

$$G^a = \frac{1}{4} f^{aj} \left(\frac{\partial^2 F}{\partial y^j \partial x^k} y^k - \partial_j F \right), \tag{5}$$

and the relation

$$\frac{dy^a}{ds} + 2G^a(x,y) = 0 \tag{6}$$

follows from the Euler-Lagrange equations

$$\frac{d}{ds}\left(\frac{\partial F}{\partial y^a} \right) - \frac{\partial F}{\partial x^a} = 0. \tag{7}$$

The transformation rule of the non-linear connection coefficients is

$$\tilde{N}^a_i = \frac{\partial \tilde{x}^a}{\partial x^b} \frac{\partial x^j}{\partial \tilde{x}^i} N^b_j(x,y) + \frac{\partial \tilde{x}^a}{\partial x^h} \frac{\partial^2 x^h}{\partial \tilde{x}^i \partial \tilde{x}^b} y^b; \tag{8}$$

also,

$$\begin{aligned} \frac{\delta}{\delta \tilde{x}^i} &= \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\delta}{\delta x^j}, \qquad & \frac{\partial}{\partial \tilde{y}^a} &= \frac{\partial x^b}{\partial \tilde{x}^a} \frac{\partial}{\partial y^b}, \\ d\tilde{x}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} dx^j, \qquad & \delta \tilde{y}^a &= \frac{\partial \tilde{x}^a}{\partial x^b} \delta y^b. \end{aligned}$$

A local basis of $T_{(x,y)}(TM)$, $(\delta_i, \dot{\partial}_a)$ adapted to the horizontal distribution N is

$$\delta_i = \partial_i - N^a_i(x,y) \dot{\partial}_a, \tag{9}$$

where

$$\partial_i = \frac{\partial}{\partial x^i}, \qquad \dot{\partial}_a = \frac{\partial}{\partial y^a},$$

and $N^a_i(x,y)$ are the coefficients of the non-linear Cartan connection N, as we mentioned above.

The concept of non-linear connection is fundamental in the geometry of vector bundles and anisotropic spaces. It is a powerful tool for the unification of fields. For example, in the case of the gravitational field, the non-linear connection in the framework of tangent bundle unifies the external and internal spaces, i.e., the position space (the base manifold M) with the tangent space $T_p M$. In other words, it is connected with the local anisotropic structure of space-time (depends on the velocities).

The dual local basis is

$$\{d^i = dx^i, \; \delta^a = \delta y^a = dy^a + N^a_j dx^j\}_{i,a=\overline{0,3}} \equiv \equiv \{\delta^\beta\}_{\beta=\overline{0,7}}.$$

A d-connection on the tangent bundle TM of space-time is a linear connection on TM which preserves, by parallelism, the horizontal distribution N and the vertical distribution V on TM. A covariant derivative associated with a d-connection becomes d-covariant. In our study, we use the d-connection in order to preserve the horizontal and vertical distribution of the anisotropy field with respect to the anisotropic axis.

Generally, an h-v metric on the tangent bundle (TM, π, M) is given by

$$G = f_{ij}(x,y)dx^i \otimes dx^j + h_{ab}\delta y^a \otimes \delta y^b. \tag{10}$$

We consider a metrical d-connection $C\Gamma = (N^a_j, L^i_{jk}, C^i_{jk})$ with the property

$$f_{ij|k} = \delta_k f_{ij} - L^h_{ik}f_{hj} - L^h_{jk}f_{ih} = 0, \tag{11}$$

$$f_{ij}|_k = \dot{\partial}_k f_{ij} - C^h_{ik}f_{hj} - C^h_{jk}f_{ih} = 0, \tag{12}$$

where

$$L^i_{jk} = \frac{1}{2}f^{ir}\left(\delta_j f_{rk} + \delta_k f_{jr} - \delta_r f_{jk}\right), \tag{13}$$

$$C^i_{jk} = \frac{1}{2}f^{ir}(\dot{\partial}_j f_{rk} + \dot{\partial}_k f_{jr} - \dot{\partial}_r f_{jk}). \tag{14}$$

The coordinate transformation of the objects L^i_{jk} and C^i_{jk} is

$$\tilde{L}^i_{jk} = \frac{\partial \tilde{x}^i}{\partial x^h}\frac{\partial x^l}{\partial \tilde{x}^j}\frac{\partial x^r}{\partial \tilde{x}^k}L^h_{lr}(x,y) + \frac{\partial \tilde{x}^i}{\partial x^r}\frac{\partial^2 x^r}{\partial \tilde{x}^j \partial \tilde{x}^k}, \tag{15}$$

$$\tilde{C}^i_{jk} = \frac{\partial \tilde{x}^i}{\partial x^h}\frac{\partial x^l}{\partial \tilde{x}^j}\frac{\partial x^r}{\partial \tilde{x}^k}C^h_{lr}(x,y). \tag{16}$$

The Cartan torsion coefficients C_{ijk} are given by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k f_{ij}, \tag{17}$$

while the Christoffel symbols of the first and second kind for the metric f_{ij} are

$$\gamma_{ijk} = \frac{1}{2}\left(\frac{\partial f_{kj}}{\partial x^i} + \frac{\partial f_{ik}}{\partial x^j} - \frac{\partial f_{ij}}{\partial x^k}\right), \tag{18}$$

$$\gamma^l_{ij} = \frac{1}{2}f^{lk}\left(\frac{\partial f_{kj}}{\partial x^i} + \frac{\partial f_{ik}}{\partial x^j} - \frac{\partial f_{ij}}{\partial x^k}\right), \tag{19}$$

respectively. The torsions and curvatures which we use are given by [7, 6]

$$T^i_{kj} = 0, \qquad S^i_{kj} = 0, \tag{20}$$

$$R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k, \qquad P^i_{jk} = \dot{\partial}_k N^i_j - L^i_{kj}, \tag{21}$$

$$P^i_{jk} = f^{im}P_{mjk}, \qquad P_{ijk} = C_{ijk|l}\,y^l; \tag{22}$$

$$R^i_{jkl} = \delta_l L^i_{jk} + \delta_k L^i_{jl} + L^h_{jk}L^i_{hl} - L^h_{jl}L^i_{hk} + C^i_{jc}R^c_{kl}, \tag{23}$$

$$S_{jikh} = C_{iks}C^s_{jh} - C_{ihs}C^s_{jk}, \tag{24}$$

$$P_{ihkj} = C_{ijk|h} - C_{hjk|i} + C^r_{hj}C_{rik|l}\,y^l - C^r_{ij}C_{rkh|l}\,y^l, \tag{25}$$

$$S^l_{ikh} = f^{lj}S_{jikh}, \tag{26}$$

$$P^l_{ikh} = f^{lj}P_{jikh}. \tag{27}$$

The Ricci identities for the d-connection are

$$X^i{}_{|k|h} - X^i{}_{|h|k} = X^r R_r{}^i{}_{kh} - X^i|_r R^r{}_{kh}, \tag{28}$$

$$X^i{}_{|k}|_h - X^i|_{h|k} = X^r P_r{}^i{}_{kh} - X^i{}_{|r}C^r_{kh} - X^i|_r P^r{}_{kh}, \tag{29}$$

$$X^i|_k|_h - X^i|_h|_k = X^r S_r{}^i{}_{kh}. \tag{30}$$

3. Geometrical structure of the anisotropic model

In what follows, lowering and raising of the indices of the objects $\hat{k}_a, y^a$ and all related Riemannian tensors will be performed with the metric a_{ij}. For the related Finslerian tensors we shall use the Finsler metric f_{ij}.

The Lagrangian which gives the equation of geodesics in the case of a (pseudo)-Riemannian space-time is given by

$$L = \sqrt{a_{ij}y^i y^j}, \qquad y^i = \frac{dx^i}{ds}. \tag{31}$$

or, equivalently, we may write for the line element:

$$ds_R = \sqrt{a_{ij}dx^i dx^j}, \tag{32}$$

where a_{ij} is the Riemannian metric with signature $(+,-,-,-)$.

Because of the observed anisotropy, we must insert an additional term to the Riemannian line element (32). This term must fulfil the following requirements:

(a) It must give an absolute maximum contribution for direction of motion parallel to the anisotropy axis which is preserved by the d-connection of space-time.

(b) It must give zero contribution for motion in the direction perpendicular to the anisotropy axis, i.e., the new line element must coincide with the Riemannian one for the direction vertical to the anisotropy axis.

(c) It must not be symmetric with respect to the replacement $y^a \to -y^a$. This requirement is necessary in order to express the anisotropy of dipole type of the Microwave Background Radiation (MBR). We need to have a maximum (positive) contribution for the direction that coincides with that of the anisotropy axis, and a minimum (negative) contribution for the opposite direction.

We see that a term which satisfies the above conditions is $k_a(x)y^a$, where $k_a(x)$ expresses this anisotropy axis. For a constant direction of $k_a(x)$, we may consider $k_a(x) = \varphi(x)\hat{k}_a$, where $\hat{k}_a$ is the unit vector in the direction of $k_a(x)$. Then $\varphi(x)$ plays the role of "length" of the vector $k_a(x)$, $\varphi(x) \in \mathbb{R}$. Hence, we have the Lagrangian

$$\mathcal{L} = \sqrt{a_{ij}y^i y^j} + \varphi(x)\hat{k}_a y^a. \tag{33}$$

In the form (33), we consider only timelike vectors y^i, therefore $a_{ij}y^iy^j > 0$. For spacelike vectors we must simply change the sign under the root, while for null vectors we change the affine parameter in order to obtain physically acceptable results [14, 15].

Remark: Using the d-connection, the horizontal and vertical distribution of the anisotropy field with respect to the anisotropic axis is preserved.

From (33) we define the Finsler metric function $F(x,y) = \mathfrak{L}$. Setting dx^a instead of y^a, we get

$$ds_F = \sqrt{a_{ij}dx^idx^j} + \varphi(x)\hat{k}_a dx^a. \tag{34}$$

ds_F is the Finslerian line element, and ds_R is the Riemannian one. We notice that the Finslerian line element is generated by an additional increment to the Riemannian one due to the anisotropy axis. A curve in Finslerian space-time has an arc length given by (34). Now,

$$ds_F^2 = a_{ij}dx^idx^j + 2\varphi(x)\hat{k}_a dx^a\sqrt{a_{ij}dx^idx^j} + \varphi^2(x)\hat{k}_a dx^a \hat{k}_b dx^b \tag{35}$$

For the Finslerian metric to be physically consistent with general relativity theory, it must have the same signature as the Riemannian metric $(+,-,-,-)$. We have

$$ds_R = c\,d\tau = c\gamma\,dt = \gamma\,d(ct) = \gamma\,dx^0, \tag{36}$$

where $\gamma = \sqrt{1-(v/c)^2}$ and v is the 3-velocity in Riemannian space-time. From Eqs. (35) and (36) we obtain:

$$\begin{aligned} ds_F^2 \;=\; & \left(a_{00} + 2\gamma\varphi(x)\hat{k}_0 + \varphi^2\hat{k}_0\hat{k}_0\right) dx^0dx^0 \\ & + \left(a_{\alpha\beta} + \varphi^2(x)\hat{k}_\alpha\hat{k}_\beta\right) dx^\alpha dx^\beta \\ & + 2\gamma\varphi(x)\hat{k}_\alpha dx^\alpha dx^0 + 2a_{0\alpha}dx^0dx^\alpha \\ & + 2\varphi^2(x)\hat{k}_0\hat{k}_\alpha dx^0dx^\alpha, \end{aligned} \tag{37}$$

where $\alpha,\,\beta = 1,2,3$. From Eq. (37) it is evident that we must have

$$(k_0(x))^2 + 2\gamma k_0(x) + a_{00} > 0, \tag{38}$$

$$a_{\alpha\alpha} + k_\alpha(x)k_\alpha(x) < 0 \tag{39}$$

for the signature to be preserved, where we have written $\varphi(x)\hat{k}_i = k_i(x)$. (38) admits positive values for

$$\begin{aligned} & k_0(x) < -\gamma - \sqrt{\gamma^2 - a_{00}}, \\ & -\gamma + \sqrt{\gamma^2 - a_{00}} < k_0(x), \end{aligned} \tag{40}$$

while (39) yields:

$$(k_\alpha(x))^2 < -a_{\alpha\alpha}. \tag{41}$$

Then, the components of any physically acceptable vector must lie in the interval (40), (41). Relations (40), (41) are a restriction upon the anisotropy of space-time, i.e., the anisotropy vector cannot take arbitrary values.

The equation of geodesics is given by

$$\frac{d^2x^l}{ds^2} + \overset{(a)}{\Gamma}{}^l_{ij}y^iy^j + \sigma a^{lm}(\partial_j\varphi\hat{k}_m - \partial_m\varphi\hat{k}_j)y^j = 0. \tag{42}$$

We observe that in the equation of geodesics we have an additional term, namely, $\sigma a^{lm}(\partial_j(\varphi\hat{k}_m) - \partial_m(\varphi\hat{k}_j))y^j$, which expresses a rotation of the anisotropy axis.

Now, for the case of electromagnetic waves, we must modify Eq. (42). This is because the world line of an EM wave is null. In geometrical optics the direction of propagation of a light ray is determined by the wave vector tangent to the ray. Let $\overset{w}{k}{}^l = dx^l/d\lambda$ be a four-dimensional wave vector, where λ is some parameter varying along the ray. We have:

$$\frac{d\overset{w}{k}{}^l}{d\lambda} + \overset{(a)}{\Gamma}{}^l_{ij}\overset{w}{k}{}^i\overset{w}{k}{}^j + \sigma a^{lm}(\partial_j\varphi\hat{k}_m - \partial_m\varphi\hat{k}_j)\overset{w}{k}{}^j = 0. \tag{43}$$

A physical interpretation of the anisotropy axis could be that it expresses the resultant of the spin densities of the angular momenta of galaxies in a restricted region of space ($k_a(x)$ is spacelike). It is known that the mass is anisotropically distributed in regions of space with radii $\leq 10^8$ light years [8]. Then an important kind of anisotropy might result from ordering of the angular momenta of galaxies. As we move to greater distances (radii $\geq 10^8$ l.y.), the resultant of the spin densities becomes approximately zero, as is expected for an isotropic universe,

$$k_a(x) = \sum_i \overset{(i)}{k}{}_a(x) \approx 0, \tag{44}$$

where $\overset{(i)}{k}{}_a(x)$ is the spin density tensor of each rotating mass distribution.

The spin is defined through the spin density tensor [3] by the relation

$$S_{ab} = \frac{\sqrt{-g}}{4\pi}\epsilon_{abc}k^c(x). \tag{45}$$

While $\varphi(x)\hat{k}_a$ expresses spin density, the function $\varphi(x)$ is related to mass density (angular momenta depend on the angular velocity and mass distribution).

From Eq. (42) we see that, for small variation of the resultant of the spin densities vector, a deviation from the Riemannian geodesics is very small if not negligible.

From the equation of geodesics (42) we obtain for the motion y^i perpendicular to k^i:

$$\frac{d^2x^a}{ds^2} + \overset{(a)}{\Gamma}{}^l_{ij}y^iy^j + \sigma a^{lm}\partial_j\varphi\hat{k}_m y^j = 0. \tag{46}$$

From (46) it is evident that if y^i is vertical to k^i, the equation of geodesics is different from its Riemannian couterpart. If, however, $\partial_i\varphi(x)$ is parallel to $\hat{k}_i$, i.e., the increment of anisotropy takes place only along the anisotropy axis, then the equation of geodesics is identical to that of geodesics in Riemannian space-time.

Using the notation $\beta = \hat{k}_a y^a$, $\sigma = \sqrt{a_{ij}y^iy^j}$, we calculate the metric tensor from (3):

$$f_{ij} = \frac{F}{\sigma}a_{ij} + \frac{\varphi(x)}{2\sigma} \underset{ij}{\mathcal{S}} \left(y_i\hat{k}_j\right) - \frac{\beta\varphi(x)}{\sigma^3}y_iy_j + \varphi^2(x)\hat{k}_i\hat{k}_j, \tag{47}$$

where $\underset{ij}{\mathcal{S}}$ is an operator and denotes symmetrization of the indices i, j, e.g.,

$$\underset{ij}{\mathcal{S}}(A_{ikjl}) = \frac{1}{2}(A_{ikjl} + A_{jkil}).$$

Accordingly, we define the antisymmetric operator

$$\underset{ij}{\mathcal{A}}(M_{ikjl}) = \frac{1}{2}(M_{ikjl} - M_{jkil}).$$

The metric function with the inclusion of S_{ab} is (using (45))

$$F(x,y) = \sqrt{a_{ij}y^iy^j} + \frac{2\pi}{3\sqrt{-a}}\epsilon_{abl}S^{ab}y^l. \tag{48}$$

Using S_{ij} from (45), the metric is found to be

$$\begin{aligned} f_{ij} &= \frac{F}{\sigma}a_{ij} + \frac{\varphi(x)\pi}{3\sigma\sqrt{-a}}(y_i\epsilon_{abj}S^{ab} + \epsilon_{abi}S^{ab}y_j) \\ &- \frac{2\pi}{3\sigma^3\sqrt{-a}}\epsilon_{abl}S^{ab}y^ly_iy_j - \frac{4\pi^2}{9a}\epsilon_{abi}\epsilon_{mnj}S^{ab}S^{mn}, \end{aligned} \tag{49}$$

where $a = \det a_{ij}$.

The inverse metric is

$$f^{ij} = \frac{\sigma}{F}a^{ij} - \frac{\sigma\varphi}{2F}\underset{ij}{\mathcal{S}}\left(y^i\hat{k}^j\right) + \frac{\varphi(\beta + m\sigma\varphi)}{F^3}y^iy^j, \tag{50}$$

as may be verified by a direct calculation, where $m = \hat{k}_a\hat{k}^a = 0, \mp 1$ according to $\hat{k}_a$ being null, spacelike or timelike. It must be noted, however, that if y^a represents the velocity of a particle (y^i timelike), then $\hat{k}^a$ is bound to be spacelike. This follows from the fact that one possible value of $y^a\hat{k}_a$ is zero.

Remark: The anisotropy of the geometrical structure does not follow from the y dependence of the metric tensor, i.e., the direction y is not the cause of anisotropy. The y dependence of the metric is a consequence of the existance of the anisotropic field $\varphi(x)\hat{k}^i$. This is most clearly seen from Eq. (47) or (42). For the case $y = 0$ the equations differ from the Riemannian ones. If, however, y were the cause of anisotropy, then $y = 0$ would yield elimination of the anisotropy field, i.e., the geometry would be identical to the Riemannian one, which is not. If we set $\varphi(x)\hat{k}^i = 0$, then it is clear that the geometric structure becomes Riemannian, i.e., the cause of anisotropy is $\varphi(x)\hat{k}^i$.

The determinant of the metric is

$$f = \det(f_{ij}) = \left(\frac{F}{\sigma}\right)^5 \det(a_{ij}). \tag{51}$$

The Cartan torsion coefficients, given by (17), take the form

$$\begin{aligned} C_{ijl} &= \frac{3\beta\varphi}{2\sigma^5}y_iy_jy_l + \frac{3\varphi}{\sigma}\underset{ijl}{\mathcal{S}}(a_{ij}\hat{k}_l) \\ &- \frac{3\varphi}{\sigma^3}\underset{ijl}{\mathcal{S}}(y_iy_jy_l) - \frac{3\beta\varphi}{\sigma^3}\underset{ijl}{\mathcal{S}}a_{ij}y_l. \end{aligned} \tag{52}$$

We observe from (52) that an increment of the anisotropy, i.e., an increment of φ, results in a change in the values of the components of the Cartan coefficients. This is expected since the condition

$$C_{ijk} = 0 \tag{53}$$

is the condition for the Finsler metric to be Riemannian.

The Finslerian Christoffel symbols of the first kind are given by (18):

$$\gamma_{ijl} = \frac{F}{\sigma}\overset{(a)}{\Gamma}_{ijl} + \Lambda_{ijl} + M_{ijl}, \tag{54}$$

where

$$\overset{(a)}{\Gamma}_{ijl} = \frac{1}{2}(\partial_i a_{lj} + \partial_j a_{il} - \partial_l a_{ij}) \tag{55}$$

are the Christoffel symbols corresponding to the metric a_{ij},

$$\Lambda_{ijl} = \underset{ij\{l\}}{\mathcal{G}}\left[\left(\frac{3\beta\varphi}{2\sigma^5}y_iy_j - \frac{\varphi}{\sigma^3}\underset{ij}{\mathcal{S}}y_i\hat{k}_j - \frac{\varphi\beta}{4\sigma^3}a_{ij}\right)\partial_l a_{ab}y^ay^b\right] \tag{56}$$

and

$$M_{ijl} = \underset{ij\{l\}}{\mathcal{G}}\left[\left(\frac{\beta}{2\sigma}a_{ij} + \frac{1}{\sigma}\underset{ij}{\mathcal{S}}y_i\hat{k}_j - \frac{\beta}{\sigma^3}y_iy_j + 2\varphi\hat{k}_i\hat{k}_j\right)\partial_l\varphi\right]. \tag{57}$$

The operator $\underset{ij\{l\}}{\mathcal{G}}$ denotes an interchange of the indices in the same form as in the definition of the Christoffel symbols of a metric, e.g.,

$$\begin{aligned} \underset{ij\{l\}}{\mathcal{G}} A_{ijl} &= A_{lji} + A_{ilj} - A_{ijl}, \\ \underset{ij\{l\}}{\mathcal{G}} \partial_l a_{ij} &= 2\overset{(a)}{\Gamma}_{ijl}. \end{aligned}$$

The Christoffel symbols of the second kind are found from (19):

$$\begin{aligned} \gamma^l_{ij} &= \overset{(a)}{\Gamma}{}^l_{ij} + \left(\frac{\varphi(\beta + m\sigma\varphi)}{\sigma F^2}y^ay^l - \frac{2\varphi}{F}\underset{al}{\mathcal{S}}(y^a\hat{k}^l)\right)\overset{(a)}{\Gamma}_{ija} \\ &+ \frac{\sigma}{F}(\Lambda^l_{ij} + M^l_{ij}) \\ &+ (\Lambda_{ija} + M_{ija})\left(\frac{\varphi(\beta + m\sigma\varphi)}{F^3}y^ay^l - \frac{2\sigma\varphi}{F^2}\underset{al}{\mathcal{S}}(y^a\hat{k}^l)\right), \end{aligned} \tag{58}$$

where $\Lambda^i_{jl} = \Lambda_{jlk} a^{ik}$ and $M^i_{jl} = M_{jlk} a^{ik}$. In Eq. (58), it is seen that, besides the $\overset{(a)}{\Gamma}{}^{i}_{jk}$ terms, all the rest expresses ananisotropic deviation from the Riemannian Christoffel symbols. When $\varphi = 0$, i.e., in the absence of an anisotropy, the Finsler Christoffel symbols coincide with the Riemannian ones. From the above relation, for $\overset{(a)}{\Gamma}{}^{i}_{jk} = 0$ we have $\gamma^i_{jk} \neq 0$. This shows the dependence of γ^i_{jk} on the anisotropy terms.

From the Euler-Lagrange equations we find for G^l (Eq. (5)):

$$G^l = \frac{1}{2} \overset{(a)}{\Gamma}{}^{l}_{ij} y^i y^j + \sigma a^{ml} y^j \underset{jm}{\mathcal{A}}(\partial_j \varphi(x)\, \hat{k}_m). \tag{59}$$

Using Eq. (4), we calculate the non-linear connection coefficients:

$$N^l_k = \overset{(a)}{\Gamma}{}^{l}_{ik} y^i + \sigma a^{ml} \underset{km}{\mathcal{A}} (\partial_k \varphi(x)\, \hat{k}_m) + \frac{1}{\sigma} a^{ml} y^j \underset{jm}{\mathcal{A}} (\partial_j \varphi(x)\, \hat{k}_m) y_k, \tag{60}$$

or

$$N^l_k = \overset{(a)}{N}{}^{l}_{j} + \sigma a^{ml} \underset{km}{\mathcal{A}} (\partial_k \varphi(x)\, \hat{k}_m) + \frac{1}{\sigma} a^{ml} y^j \underset{jm}{\mathcal{A}} (\partial_j \varphi(x)\, \hat{k}_m) y_k. \tag{61}$$

Eq. (61) clearly shows that the deviation from the Riemannian non-linear connection is due to the anisotropic terms. In the case of an irrotational anisotropic field, $\underset{km}{\mathcal{A}} (\partial_k \varphi(x)\, \hat{k}_m) = 0$, the non-linear connection is identical to the Riemannian one.

The connection coefficients C^l_{ij} are given by (14):

$$\begin{aligned} C^l_{ij} = {} & \frac{\varphi}{2F} a_{ij} \hat{k}^l + \frac{\varphi}{F} \underset{ij}{\mathcal{S}}(\hat{k}^i \delta^l_j) - \frac{\beta\varphi}{F\sigma^2} \underset{ij}{\mathcal{S}}(\delta^l_i y_j) \\ & - \frac{\varphi(\beta + m\sigma\varphi)}{2F^2\sigma} a_{ij} y^l - \frac{\varphi}{2F\sigma^2} y_i y_j \hat{k}^l \\ & - \frac{\varphi(\sigma - \beta\varphi)}{F^2\sigma^2} y^l \underset{ij}{\mathcal{S}}(\hat{k}_i y_j) - \left(\frac{\varphi}{F}\right)^2 \hat{k}_i \hat{k}_j y^l \\ & + \frac{\varphi(3\beta + m\sigma\varphi)}{2F^2\sigma^3} y_i y_j y^l. \end{aligned} \tag{62}$$

Accordingly, using (13), we get:

$$\begin{aligned} L^i_{jk} = {} & \overset{(a)}{\Gamma}{}^{i}_{jk} + \left(\frac{\varphi(\beta + m\sigma\varphi)}{\sigma F^2} y^a y^i \right. \\ & \left. - \frac{2\varphi}{F} \underset{ai}{\mathcal{S}}(y^a \hat{k}^i) \right) \overset{(a)}{\Gamma}{}_{jka} + \frac{\sigma}{F}(\Lambda^i_{jk} + M^i_{jk}) \\ & + (\Lambda_{jka} + M_{jka}) \left(\frac{\varphi(\beta + m\sigma\varphi)}{F^3} y^a y^i \right. \\ & \left. - \frac{2\sigma\varphi}{F^2} \underset{ai}{\mathcal{S}}(y^a \hat{k}^i) \right) - (N^l_j C^i_{kl} + N^l_k C^i_{jl} - f^{ir} N^l_r C_{jkl}), \end{aligned} \tag{63}$$

where N^l_j and C^i_{kl} are given explicitly by Eqs. (60), (62). The curvature of the non-linear connection is (21):

$$\begin{aligned} R^i_{jk} = {} & \overset{(a)}{R}{}^{i}_{ajk} y^a + \frac{1}{2\sigma} \Big(\partial_k a_{mn} \underset{jb}{\mathcal{A}}(\partial_j \varphi \hat{k}_b) \\ & - \partial_j a_{mn} \underset{kb}{\mathcal{A}}(\partial_k \varphi \hat{k}_b) \Big) y^m y^n a^{bi} \\ & + \sigma \Big(\partial_k a^{bi} \underset{jb}{\mathcal{A}} \partial_j \varphi \hat{k}_b - \partial_j a^{bi} \underset{kb}{\mathcal{A}}(\partial_k \varphi \hat{k}_b) \Big) \\ & - \sigma a^{bi} \underset{jk}{\mathcal{A}}(\partial_{bj} \varphi \hat{k}_k) + \frac{1}{\sigma} a^{bi} y^c \Big(\underset{cb}{\mathcal{A}}(\partial_{kc} \varphi \hat{k}_b) y_j \\ & - \underset{cb}{\mathcal{A}}(\partial_{jc} \varphi \hat{k}_b) y_k \Big) + \Big(\frac{2}{\sigma} a^{bi} \underset{kj}{\mathcal{A}}(\partial_k y_j) \\ & - \frac{1}{\sigma^3} a^{bi} y^m y^n \underset{kj}{\mathcal{A}}(\partial_k a_{mn} y_j) \Big) \underset{ab}{\mathcal{A}}(\partial_a \varphi \hat{k}_b) y^a \\ & - \sigma \Big(\overset{(a)}{\Gamma}{}^{i}_{kb} a^{bc} \underset{cj}{\mathcal{A}}(\partial_c \varphi \hat{k}_j) + \overset{(a)}{\Gamma}{}^{i}_{jb} a^{bc} \underset{kc}{\mathcal{A}}(\partial_k \varphi \hat{k}_c) \Big) \\ & - \beta \partial^i \varphi \underset{jk}{\mathcal{A}}(\partial_j \varphi \hat{k}_k) - \frac{\beta^2 + m\sigma^2}{2\sigma^2} \partial^i \varphi \underset{jk}{\mathcal{A}}(\partial_j \varphi y_k) \\ & - \frac{\beta}{\sigma} \partial^a \varphi \underset{kj}{\mathcal{A}}(\overset{(a)}{\Gamma}{}^{i}_{ak} y_j) - \frac{1}{\sigma} \underset{kj}{\mathcal{A}}(\overset{(a)}{\Gamma}{}_{kaj}) \Big(\beta \partial^i \varphi y^a \\ & \qquad - (\partial_b \varphi y^b) y^a \hat{k}^i \Big) \\ & - \frac{1}{2}(\partial_b \varphi y^b) \Big[\partial^i \varphi \underset{kj}{\mathcal{A}}(\hat{k}_k y_j) + \hat{k}^i \underset{kj}{\mathcal{A}}(\partial_k \varphi y_j) \Big] \\ & - \frac{1}{2}(\partial_a \varphi \partial^a \varphi) \hat{k}^i \underset{jk}{\mathcal{A}}(\hat{k}_j y_k) - (\partial_a \varphi y^a) \hat{k}^i \underset{kj}{\mathcal{A}}(\partial_k \varphi \hat{k}_j) \\ & - \frac{\beta}{2\sigma^2}(\partial_a \varphi y^a) \Big[\partial^i \varphi \underset{kj}{\mathcal{A}}(\hat{k}_k y_j) + \hat{k}^i \underset{kj}{\mathcal{A}}(\partial_k \varphi y_j) \Big] \\ & - \frac{1}{\sigma}(\partial_a \varphi y^a) \hat{k}^b \underset{jk}{\mathcal{A}}(\overset{(a)}{\Gamma}{}^{i}_{bj} y_k) \\ & - \frac{1}{\sigma} \hat{k}_a y^b \partial^i \varphi \underset{jk}{\mathcal{A}}(\overset{(a)}{\Gamma}{}^{a}_{bj} y_k) - \frac{1}{\sigma} \partial_a \varphi y^b \hat{k}^i \underset{kj}{\mathcal{A}}(\overset{(a)}{\Gamma}{}^{a}_{bk} y_j) \\ & - \frac{1}{2\sigma^2}(\partial_a \varphi y^a)^2 \hat{k}^i \underset{jk}{\mathcal{A}}(y_j \hat{k}_k), \end{aligned} \tag{64}$$

where $\overset{(a)}{R}{}^{i}_{ajk}$ is the Riemannian curvature of the metric a_{ij}.

The torsion P^i_{jk} is given by (21):

$$P^i_{jk} = \overset{(a)}{\Gamma}{}^{i}_{jk} + \frac{1}{\sigma} a^{mi} \Big[\underset{km}{\mathcal{A}} (\partial_k \varphi\, \hat{k}_m) y_j + a_{jk} \underset{rm}{\mathcal{A}} (\partial_r \varphi\, \hat{k}_m) y^r \Big] - L^i_{kj}, \tag{65}$$

and then

$$\begin{aligned} P_{ijk} = {} & \frac{F}{\sigma} a_{li} \overset{(a)}{\Gamma}{}^{l}_{jk} - \frac{\beta F}{2\sigma^2} \partial_i \varphi\, a_{jk} \\ & + \Big[\frac{F + m\sigma\varphi^2}{2\sigma^2} \partial_l \varphi\, y^l - \frac{\beta\varphi^2}{2\sigma} \partial_l \varphi \hat{k}^l \Big] \hat{k}_i a_{jk} \\ & + \varphi^2 \hat{k}_i \hat{k}_l \overset{(a)}{\Gamma}{}^{l}_{jk} + (\varphi^2 \hat{k}_l + \frac{\varphi}{\sigma} y_l) \hat{k}_i \overset{(a)}{\Gamma}{}^{l}_{jk} \end{aligned}$$

$$\begin{aligned}
&- \frac{F}{2\sigma^2}\partial_i\varphi\, y_j\hat{k}_k \\
&+ \left(\frac{\sigma+2\beta\varphi+m\sigma\varphi^2}{2\sigma^2}\right)\hat{k}_i\, y_j\partial_k\varphi \\
&- \left(\frac{\varphi^2}{2\sigma}(\partial_l\varphi\,\hat{k}^l)+\frac{\varphi}{2\sigma^2}(\partial_l\varphi\, y^l)\right)\hat{k}_i y_j\hat{k}_k \\
&+ \left(\frac{\varphi}{\sigma}\hat{k}_l - \frac{\beta\varphi}{\sigma^3}y_l\right) y_i \overset{(a)}{\Gamma}{}^{l}_{\ jk} \\
&+ \left(\frac{m\varphi}{2\sigma^2}\partial_l\varphi\, y^l - \frac{\beta\varphi}{2\sigma^2}\partial_l\varphi\,\hat{k}^l\right) y_i a_{jk} \\
&+ \frac{(m\sigma^2-\beta^2)\varphi}{2\sigma^4} y_i y_j\partial_k\varphi \\
&+ \left(\frac{\beta\varphi}{2\sigma^4}\partial_l\varphi\, y^l - \frac{\varphi}{2\sigma^2}\partial_l\varphi\,\hat{k}^l\right) y_i y_j\hat{k}_k \\
&- f_{li}\gamma^l_{jk} - \underset{jk\{i\}}{\mathcal{G}}(C_{jkl}N^l_i) - f_{ih}L^h_{jk}. \qquad (66)
\end{aligned}$$

The h-covariant derivative of the C_{ijk} coefficients is

$$C_{ijk|l} = \delta_l C_{ijk} - L^h_{il}C_{hjk} - L^h_{jl}C_{ihk} - L^h_{kl}C_{ijh}. \qquad (67)$$

From Eqs. (21), (25), (52), (62), (63), (65) and (67) we can calculate the P_{ijkl} curvature.

Taking into account the relations

$$\begin{aligned}
\delta_l L^i_{jk} = {} & \delta_l f^{ir}\left(\gamma_{jkr} - \underset{jk\{r\}}{\mathcal{G}}(C_{jkh}N^h_r)\right) \\
&+ f^{ir}\Big(\delta_l\gamma_{jkr} - \Big[(\delta_l N^h_j)C_{rkh} \\
&+ N^h_j(\delta_l C_{rkh}) + (\delta_l N^h_k)C_{jrh} \\
&+ N^h_k(\delta_l C_{jrh}) - (\delta_l N^h_r)C_{jkh} - \\
&- N^h_r(\delta_l C_{jkh})\Big]\Big), \qquad (68)
\end{aligned}$$

$$\begin{aligned}
\delta_l\gamma_{jkr} = {} & \left(\frac{1}{\sigma}\delta_l F - \frac{F}{\sigma^2}\delta_l\sigma\right)\overset{(a)}{\Gamma}{}_{jkr} \\
&+ \frac{F}{\sigma}\delta_l \overset{(a)}{\Gamma}{}_{jkr} + \delta_l\Lambda_{jkr} + \delta_l M_{jkr}, \qquad (69)
\end{aligned}$$

$$\begin{aligned}
\delta_k N^i_j = {} & \frac{\partial \overset{(a)}{\Gamma}{}^{i}_{\ jr}}{\partial x^k}y^r + \left(\frac{1}{2\sigma}\frac{\partial a_{mn}}{\partial x^k}y^m y^n a^{hi} \right.\\
&\left. + \sigma\frac{\partial a^{hi}}{\partial x^k}\right)\underset{jh}{\mathcal{A}}(\partial_j\varphi\,\hat{k}_h) \\
&+ \sigma a^{mi}\left[\underset{jm}{\mathcal{A}}\left((\partial^2_{kj}\varphi)\,\hat{k}_m\right)\right] \\
&+ \frac{1}{\sigma}a^{mi}\left[\underset{rm}{\mathcal{A}}\left((\partial^2_{kr}\varphi)\,\hat{k}_m\right)\right] y^r y_j \\
&+ \left(\frac{1}{\sigma}\frac{\partial a^{mi}}{\partial x^k} - \frac{1}{2\sigma^3}\frac{\partial a_{pn}}{\partial x^k}y^p y^n a^{mi}\right)\times \\
&\times \underset{rm}{\mathcal{A}}\left(\partial_r\varphi\,\hat{k}_m\right)y^r y_j - \frac{\beta}{2}\partial^i\varphi\,\underset{jk}{\mathcal{A}}\left(\partial_j\varphi\,\hat{k}_k\right) \\
&- \sigma a^{hl}\overset{(a)}{\Gamma}{}^{i}_{\ jh}\underset{kl}{\mathcal{A}}\left(\partial_k\varphi\,\hat{k}_l\right) + \frac{m}{4}\partial^i\varphi\,\partial_k\varphi\, y_j \\
&- \frac{1}{4}(\partial_a\varphi\,\hat{k}^a)\partial^i\varphi\,\hat{k}_k y_j \\
&- \frac{1}{2}\partial_h\varphi\hat{k}^i\Big[a^{hl}\underset{kl}{\mathcal{A}}(\partial_k\varphi\,\hat{k}_l)y_j \\
&+ y^h\underset{kj}{\mathcal{A}}(\partial_k\varphi\,\hat{k}_j)\Big] - \left(\frac{\beta}{2\sigma}\right)^2\partial^i\varphi\partial_j\varphi y_k \\
&- \frac{\beta}{2\sigma}\left[a_{mj}\overset{(a)}{\Gamma}{}^{m}_{\ ka}y^a\partial^i\varphi - \partial^a\varphi\overset{(a)}{\Gamma}{}^{i}_{\ ja}y_k\right] \\
&- \frac{\beta}{2\sigma^2}\partial_a\varphi\,\underset{ia}{\mathcal{A}}(\partial^i\varphi\,\hat{k}^a)y_j y_k + \frac{\beta}{4\sigma^2}(\partial_a\varphi y^a)\partial^i\varphi\,\hat{k}_j y_k \\
&- \frac{1}{\sigma}y^b\overset{(a)}{\Gamma}{}_{kba}y_j\underset{ai}{\mathcal{A}}(\partial^a\varphi\,\hat{k}^i) - \frac{1}{2\sigma}y^a\Big(\partial_a\varphi\,\hat{k}^b\overset{(a)}{\Gamma}{}^{i}_{\ jb}y_k \\
&+ (\partial_b\varphi\, y^b)a_{jl}\overset{(a)}{\Gamma}{}^{l}_{\ ka}y^a\hat{k}^i\Big) \\
&+ \frac{\beta}{4\sigma^2}(\partial_b\varphi\, y^b)\hat{k}^i\partial_j\varphi\, y_k \\
&+ \frac{m}{4\sigma^2}(\partial_b\varphi\, y^b)\partial^i\varphi\, y_j y_k - \overset{(a)}{\Gamma}{}^{l}_{\ kb}\overset{(a)}{\Gamma}{}^{i}_{\ jl}y^b \\
&- \frac{1}{2\sigma^2}(\partial_b\varphi\, y^b)\partial_a\varphi\, a^{al}\hat{k}^i y_k\underset{lj}{\mathcal{S}}(\hat{k}_l y_j), \qquad (70)
\end{aligned}$$

$$\delta_l\beta = -\hat{k}_h N^h_l, \qquad (71)$$

$$\delta_l\sigma = \frac{1}{2\sigma}\partial_l a_{ij}y^i y^j - N^h_l(\frac{1}{\sigma}a_{ih}y^i), \qquad (72)$$

$$\begin{aligned}
\delta_l F = {} & \frac{1}{2\sigma}\partial_l a_{ij}y^i y^j + \partial_l\varphi\,\beta \\
&- N^h_l(\frac{1}{\sigma}a_{ih}y^i + \varphi(x)\hat{k}_h), \qquad (73)
\end{aligned}$$

$$\delta_l y_i = -a_{ih}N^h_l, \qquad (74)$$

$$\begin{aligned}
\delta_l C_{ijk} = {} & \frac{3}{2}\left(\frac{\varphi}{\sigma^5}\delta_l\beta + \frac{\beta}{\sigma^5}\partial_l\varphi - 5\frac{\beta\varphi}{\sigma^6}\delta_l\sigma\right)y_i y_j y_k \\
&- \frac{3\beta\varphi}{2\sigma^5}\Big(a_{ih}N^h_l y_j y_k + a_{jh}N^h_l y_i y_k \\
&+ a_{kh}N^h_l y_i y_j\Big) + 3\left(\frac{1}{\sigma}\partial_l\varphi - \frac{\varphi}{\sigma^2}\delta_l\sigma\right)\underset{ijk}{\mathcal{S}}(a_{ij}\hat{k}_k) \\
&+ 3\frac{\varphi}{\sigma}\partial_l[\underset{ijk}{\mathcal{S}}(a_{ij}\hat{k}_k)] \\
&- 3\left(\frac{1}{\sigma^3}\partial_l\varphi - 3\frac{\varphi}{\sigma^4}\delta_l\sigma\right)\underset{ijk}{\mathcal{S}}(y_i y_j\hat{k}_k) \\
&\frac{3\varphi}{\sigma^3}\delta_l(\underset{ijk}{\mathcal{S}}(y_i y_j\hat{k}_k)) \\
&- 3\left(\frac{\varphi}{\sigma^3}\delta_l\beta + \frac{\beta}{\sigma^3}\partial_l\varphi - 3\frac{\beta\varphi}{\sigma^4}\delta_l\sigma\right)\underset{ijk}{\mathcal{S}}(a_{ij}y_k) \\
&- \frac{3\beta\varphi}{\sigma^3}\delta_l\left(\underset{ijk}{\mathcal{S}}(a_{ij}y_k)\right), \qquad (75)
\end{aligned}$$

$$\begin{aligned}
\delta_l\Lambda_{ijk} = {} & \frac{3}{2}\Big(\frac{\varphi}{\sigma^5}\delta_l\beta + \frac{\beta}{\sigma^5}\partial_l\varphi \\
&- 5\frac{\beta\varphi}{\sigma^6}\delta_l\sigma\Big)\underset{ij\{k\}}{\mathcal{G}}(y_i y_j\partial_k a_{ab})y^a y^b \\
&+ \frac{3\beta\varphi}{2\sigma^5}\delta_l\left[\underset{ij\{k\}}{\mathcal{G}}(y_i y_j\partial_k a_{ab})y^a y^b\right]
\end{aligned}$$

$$-\left(\frac{1}{\sigma^3}\partial_l\varphi-3\frac{\varphi}{\sigma^4}\delta_l\sigma\right)\underset{ij\{k\}}{\mathcal{S}}\Big((y_i\hat{k}_j$$
$$+y_j\hat{k}_i)\partial_k a_{ab}\Big)y^a y^b$$
$$-\frac{\varphi}{\sigma^3}\delta_l\left[\underset{ij\{k\}}{\mathcal{S}}\left((y_i\hat{k}_j+y_j\hat{k}_i)\partial_k a_{ab}\right)y^a y^b\right]$$
$$-\left(\frac{\varphi}{4\sigma^3}\delta_l\beta+\frac{\beta}{4\sigma^3}\partial_l\varphi\right.$$
$$\left.-3\frac{\beta\varphi}{4\sigma^4}\delta_l\sigma\right)\underset{ij\{k\}}{\mathcal{S}}(a_{ij}\partial_k a_{ab})y^a y^b$$
$$-\frac{\varphi\beta}{4\sigma^3}\delta_l\left[\underset{ij\{k\}}{\mathcal{S}}(a_{ij}\partial_k a_{ab})y^a y^b\right],\qquad(76)$$

$$\delta_l M_{ijk}=\frac{1}{2}\left(\frac{1}{\sigma}\delta_l\beta-\frac{\beta}{\sigma^2}\delta_l\sigma\right)\underset{ij\{k\}}{\mathcal{S}}(a_{ij}\partial_k\varphi)$$
$$+\frac{\beta}{2\sigma}\delta_l\left[\underset{ij\{k\}}{\mathcal{S}}(a_{ij}\partial_k\varphi)\right]$$
$$-\frac{1}{\sigma^2}\delta_l\sigma\underset{ij\{k\}}{\mathcal{S}}\left((y_i\hat{k}_j+y_j\hat{k}_i)\partial_k\varphi\right)$$
$$+\frac{1}{\sigma}\delta_l\left[\underset{ij\{k\}}{\mathcal{S}}\left((y_i\hat{k}_j+y_j\hat{k}_i)\partial_k\varphi\right)\right]$$
$$-\left(\frac{1}{\sigma^3}\delta_l\beta-3\frac{\beta}{\sigma^4}\delta_l\sigma\right)\underset{ij\{k\}}{\mathcal{S}}(y_i y_j\partial_k\varphi)$$
$$-\frac{\beta}{\sigma^3}\delta_l\left[\underset{ij\{k\}}{\mathcal{S}}(y_i y_j\partial_k\varphi)\right]+2\partial_l\varphi\underset{ij\{k\}}{\mathcal{S}}\left(\hat{k}_i\hat{k}_j\partial_k\varphi\right)$$
$$+2\varphi\partial_l\left[\underset{ij\{k\}}{\mathcal{S}}\left(\hat{k}_i\hat{k}_j\partial_k\varphi\right)\right],\qquad(77)$$

and (23), (62), (63), (64), we can calculate the R^i_{jkl} curvature explicitly.

The S-curvature (24) is

$$S_{jikh}=\frac{\varphi^2(m\sigma^2-\beta^2)}{2F\sigma^3}\underset{ji}{\mathcal{A}}(a_{hj}a_{ik})$$
$$+\frac{\varphi^2}{2F\sigma}\left((\underset{ij}{\mathcal{A}}(a_{ki}\hat{k}_j)\hat{k}_h+\underset{ji}{\mathcal{A}}(a_{hj}\hat{k}_i)\hat{k}_k)\right)$$
$$+\frac{\beta\varphi^2}{2F\sigma^3}\left((\underset{ji}{\mathcal{A}}(a_{kj}\hat{k}_i)y_h+\underset{kh}{\mathcal{A}}(a_{jk}\hat{k}_h)y_i)\right)$$
$$+\frac{\varphi^2}{2F\sigma^3}\left(\hat{k}_h y_k\underset{ij}{\mathcal{A}}(\hat{k}_i y_j)+\hat{k}_k y_h\underset{ji}{\mathcal{A}}(\hat{k}_j y_i)\right)$$
$$+\frac{\beta\varphi^2}{2F\sigma^3}\left(\underset{hk}{\mathcal{A}}(a_{ih}\hat{k}_k)y_j+\underset{ij}{\mathcal{A}}(a_{hi}\hat{k}_j)y_k\right)$$
$$+\frac{\varphi^2(m\sigma^2-2\beta^2)}{4F\sigma^5}\left(\underset{hk}{\mathcal{A}}(a_{ih}y_k)y_j+\underset{kh}{\mathcal{A}}(a_{jk}y_h)y_i\right),\qquad(78)$$

$$S^r_{ikh}=\frac{(m\sigma^2-\beta^2)}{2F^2\sigma^2}\underset{hk}{\mathcal{A}}(\delta^r_h a_{ki})+\frac{\varphi^2}{2F^2}\left(\hat{k}_i\underset{hk}{\mathcal{A}}(\delta^r_h\hat{k}_k)\right.$$
$$\left.+\hat{k}^r\underset{kh}{\mathcal{A}}(a_{ki}\hat{k}_h)\right)+\frac{\beta\varphi^2}{2F^2\sigma^2}\left(\delta^r_k\underset{ih}{\mathcal{S}}(\hat{k}_i y_h)\right.$$
$$\left.-\delta^r_h\underset{ik}{\mathcal{S}}(\hat{k}_i y_k)-\right)+\frac{\beta\varphi^2}{2F^2\sigma^2}\hat{k}^r\underset{hk}{\mathcal{A}}(a_{ih}y_k)$$
$$+\frac{(m\sigma^2-2\beta^2)\varphi^2}{2F^2\sigma^4}y_i\underset{kh}{\mathcal{A}}(\delta^r_k y_h)$$
$$+\frac{\varphi^2}{2F^2\sigma^2}\left(\hat{k}^r y_i\underset{kh}{\mathcal{A}}(\hat{k}_k y_h)+y^r\hat{k}_i\underset{hk}{\mathcal{A}}(\hat{k}_h y_k)\right)$$
$$+\frac{\varphi^2(\beta\sigma-\beta^2\varphi+2m\sigma^2\varphi)}{2F^3\sigma^2}y^r\underset{hk}{\mathcal{A}}(a_{ih}\hat{k}_k)$$
$$+\frac{2\beta^2\varphi^2-m\sigma^2\varphi^2+\beta m\sigma\varphi^3}{2F^3\sigma^3}y^r\underset{kh}{\mathcal{A}}(a_{ik}y_h)$$
$$+\frac{(m\sigma^2-\beta^2)\varphi^3}{F^3\sigma^4}y^r y_i\underset{kh}{\mathcal{A}}(\hat{k}_k y_h),\qquad(79)$$

$$S_{ih}=-\frac{3(m\sigma^2\varphi^2-\beta^2\varphi^2)}{4F^2\sigma^2}a_{ih}-\frac{\varphi^2}{4F^2}\hat{k}_i\hat{k}_h$$
$$+\frac{\beta\varphi^2}{2F^2\sigma^2}\underset{ih}{\mathcal{S}}(\hat{k}_i y_h)+\frac{3m\sigma^2\varphi^2-4\beta^2\varphi^2}{4F^2\sigma^4}y_i y_h,\qquad(80)$$

$$S=\frac{5(\beta^2-m\sigma^2)\varphi^2}{2\sigma F^3}.\qquad(81)$$

From a physical point of view, the S-curvature can be considered as a curvature parameter of anisotropy, as is evident from Eq. (81). In the absence of anisotropy $\varphi=0$, we have $S=0$. In other words, S represents a measure of anisotropy of matter [11].

4. Space of constant Riemannian curvature

In this section we will give an explicit form of the curvature of a Finslerian space-time with constant curvature as well as the Einstein tensor which is derived for this space with the metric (1). The conditions for a Finsler space metric to be of constant curvature are given in [16]. In this case, the associated Riemannian space should be of constant curvature $4\overset{a}{R}$ ($\overset{a}{R}$ is the Riemannian scalar curvature):

$$\underset{ijkl}{\overset{(a)}{R}}=4\overset{a}{R}(a_{ik}a_{jl}-a_{il}a_{jk}),\qquad(82)$$
$$k_i(x)_{|j}=\lambda(a_{ij}-k_i(x)k_j(x)),\quad\lambda=\text{const},\qquad(83)$$
$$\lambda^2+4\overset{a}{R}=0.\qquad(84)$$

As a consequence of the above conditions (82), (83), (84) the form of the curvature of a Finsler space of constant curvature will be given by

$$R_{ijkl}=K(f_{ik}f_{jl}-f_{il}f_{jk}),\quad K=\text{const},\qquad(85)$$

or, using (47), the R_{ijkl} curvature is written explicitly:

$$R_{ijkl}=K\Big\{(\frac{F}{\sigma})^2(a_{ik}a_{j}l-a_{il}a_{jk})+\frac{F\varphi(x)}{2\sigma^2}[a_{ik}(y_j\hat{k}_l$$
$$+\hat{k}_j y_l)-a_{il}(y_j\hat{k}_k+\hat{k}_j y_k)]$$

$$- \frac{F\beta\varphi(x)}{\sigma^4}(a_{ik}y_jy_l - a_{il}y_jy_k)$$
$$+ \frac{F\varphi^2(x)}{\sigma}(a_{ik}\hat{k}_j\hat{k}_l - a_{il}\hat{k}_j\hat{k}_l)$$
$$+ \frac{F\varphi(x)}{2\sigma^2}[(y_i\hat{k}_k + \hat{k}_iy_k)a_{jl} - (y_i\hat{k}_l + \hat{k}_iy_l)a_{jk}]$$
$$+ \frac{\varphi^2(x)}{4\sigma^2}[(y_i\hat{k}_k + \hat{k}_iy_k)(y_j\hat{k}_l + \hat{k}_jy_l)$$
$$- (y_i\hat{k}_l + \hat{k}_iy_l)(y_j\hat{k}_k + \hat{k}_jy_k)]$$
$$- \frac{\beta\varphi^2(x)}{2\sigma^4}[y_i\hat{k}_ky_jy_l - y_i\hat{k}_ly_jy_k]$$
$$+ \frac{\varphi^3(x)}{2\sigma}(\hat{k}_iy_k\hat{k}_j\hat{k}_l - \hat{k}_iy_l\hat{k}_j\hat{k}_k)$$
$$- \frac{F\beta\varphi(x)}{\sigma^4}(y_iy_ka_{jl} - y_iy_la_jk)$$
$$- \frac{\beta\varphi^2(x)}{2\sigma^4}[y_iy_k(y_j\hat{k}_l + \hat{k}_jy_l) - y_iy_l(y_j\hat{k}_k$$
$$+ \hat{k}_jy_k)] - \frac{\beta\varphi^3(x)}{\sigma^3}(y_iy_k\hat{k}_j\hat{k}_l - y_iy_l\hat{k}_j\hat{k}_k)$$
$$+ \frac{F\varphi^2(x)}{\sigma}(\hat{k}_i\hat{k}_ka_{jl} - \hat{k}_i\hat{k}_la_{jk}) + \frac{\varphi^3}{2\sigma}[(\hat{k}_i\hat{k}_ky_j\hat{k}_l$$
$$- \hat{k}_i\hat{k}_l\hat{k}_jy_k) + (\hat{k}_i\hat{k}_k\hat{k}_jy_l - \hat{k}_i\hat{k}_l\hat{k}_jy_k)]$$
$$- \frac{\beta\varphi^3(x)}{\sigma^3}(\hat{k}_i\hat{k}_ky_jy_l - \hat{k}_i\hat{k}_ly_jy_k)\Big\}. \tag{86}$$

The Ricci R_{ij} curvature is $R_{ij} = 3Kf_{ij}$, or, using (47), we have

$$R_{ij} = 3K\left[\frac{F}{\sigma}a_{ij} + \frac{\varphi(x)}{2\sigma}\underset{ij}{\mathrm{S}}\left(y_i\hat{k}_j\right) - \frac{\beta\varphi(x)}{\sigma^3}y_iy_j + \varphi^2(x)\hat{k}_i\hat{k}_j\right]. \tag{87}$$

For the case $\dim M = 4$ the constant is $K = R/12$.

The Einstein tensor is defined as

$$E^{ij} = R^{ij} - \frac{1}{2}Rf^{ij}, \tag{88}$$

where

$$E_{ij} = -\frac{R}{4}\left[\frac{F}{\sigma}a_{ij} + \frac{\varphi(x)}{2\sigma}\underset{ij}{\mathrm{S}}\left(y_i\hat{k}_j\right) - \frac{\beta\varphi(x)}{\sigma^3}y_iy_j + \varphi^2(x)\hat{k}_i\hat{k}_j\right]. \tag{89}$$

From Eq. (88) and the condition $f_{ij|k} = 0$ we observe that $E^{ij}{}_{|j} = 0$ for the case of a constant-curvature Finsler space.

5. Anisotropic electromagnetic field equations in vacuum

In what follows we present a generalization of the electromagnetic field equations in the framework of the tangent bundle based on the metric Cartan connection for Finsler spaces.

The electromagnentic field tensor in special relativity is $F_{ij} = \partial_jA_i(x) - \partial_iA_j(x)$. A generalization in our approach yields

$$\begin{aligned}\tilde{F}_{ij} &= A_{i|j}(x) - A_{j|i}(x)\\ &= \delta_jA_i(x) - \delta_iA_j(x) - L^h_{ij}A_h + L^h_{ji}A_h\\ &= (\partial_j - N^l_j\dot{\partial}_l)A_i(x) - (\partial_i - N^l_i\dot{\partial}_l)A_j(x),\end{aligned} \tag{90}$$

or

$$\tilde{F}_{ij} = \partial_jA_i(x) - \partial_iA_j(x) = F_{ij} \tag{91}$$

since $\dot{\partial}_lA_i(x) = 0$. Therefore the electromagnetic field tensor remains invariant, as in the usual electromagnetic theory in Riemannian space-time.

The first pair of Maxwell equations is

$$\partial_lF_{ik} + \partial_kF_{li} + \partial_iF_{kl} = 0. \tag{92}$$

Replacing the partial derivatives with the h-covariant derivatives of the bundle,

$$\partial_lF_{ik} \to \tilde{F}_{ik|l}, \tag{93}$$

we have

$$\tilde{F}_{ik|l} = \delta_l\tilde{F}_{ik} - L^h_{li}\tilde{F}_{hk} - L^h_{lk}\tilde{F}_{ih}, \tag{94}$$
$$\tilde{F}_{li|k} = \delta_k\tilde{F}_{li} - L^h_{kl}\tilde{F}_{hi} - L^h_{ki}\tilde{F}_{lh}, \tag{95}$$
$$\tilde{F}_{kl|i} = \delta_i\tilde{F}_{kl} - L^h_{ik}\tilde{F}_{hl} - L^h_{il}\tilde{F}_{kh}. \tag{96}$$

Using the relations

$$\delta_l\tilde{F}_{ij} = (\partial_l - N^a_l\dot{\partial}_a)\tilde{F}_{ij}, \qquad \dot{\partial}_l\tilde{F}_{ij} = 0$$

and summing of (94), (95), (96) yields:

$$\tilde{F}_{ik|l} + \tilde{F}_{li|k} + \tilde{F}_{kl|i} = \partial_lF_{ik} + \partial_kF_{li} + \partial_iF_{kl} = 0, \tag{97}$$

where we took into account the symmetry properties of L^i_{jk} and $F_{ij} = -F_{ji}$. It is seen that the first pair of Maxwell equations remains unchanged.

The second pair of Maxwell equations in vacuum is

$$\partial_kF^{ik} = 0 \tag{98}$$

As before, we consider

$$\partial_kF^{ik} = 0 \;\to\; \tilde{F}^{ik}{}_{|k} = 0, \tag{99}$$

namely,

$$\tilde{F}^{ij}{}_{|j} = \delta_j\tilde{F}^{ij} + L^i_{hj}\tilde{F}^{hj} + L^j_{hj}\tilde{F}^{ih} = 0. \tag{100}$$

From the second pair of Maxwell equations (100), inserting the expression

$$\tilde{F}_{ij} = A_{j|i} - A_{i|j},$$

one can derive the wave equation that governs the vector potential. We have

$$F^{ij}{}_{|j} = 0 \quad\Rightarrow\quad A^{i|j}{}_j - A^{j|i}{}_j = 0, \tag{101}$$

or, using the operator $\Box A^i = A^{i|j}{}_{|j}$, we find

$$\Box A^i - f^{li}A^j{}_{|l|j} = 0. \tag{102}$$

Using the commutation relation (28), we have

$$A^j{}_{|k|j} = A^r R_{rk} - A^h C^j_{hr} R^r{}_{kj}, \quad (103)$$

where we took into account vanishing of the divergence $A^j{}_{|j} = 0$, which follows from a generalization of the Lorentz condition:

$$\partial_a A^a \to A^a{}_{|a}, \quad (104)$$

or

$$\partial_a A^a + L^a_{ab} A^b = 0. \quad (105)$$

Replacing (103) with (102) yields

$$\Box A^i - A^r R_r{}^i + A^h C^j_{hr} R^{ri}{}_j = 0. \quad (106)$$

The tensor form of Eq. (106) is evident, thus ensuring invariance of the form of the equation for relative observers. Also the transformation rule of the L^i_{jk} connection implies that the Lorentz condition has the same form for any observer.

It may be possible that Eq. (106) is connected with the observed anisotropy of the electromagnetic propagation over cosmological distances [9].

Finally, we give the equation of motion of a charged particle subject to the anisotropic geometrical framework we developed and to the electromagnetic field (we consider $\sigma = c = 1$):

$$m\left(\frac{d^2x^l}{ds^2} + \overset{(a)}{\Gamma}{}^l_{ij} y^i y^j + a^{lm}(\partial_j \varphi \hat{k}_m - -\partial_m \varphi \hat{k}_j) y^j\right) = qF^l_j y^j, \quad (107)$$

or

$$\frac{d^2x^l}{ds^2} + \overset{(a)}{\Gamma}{}^l_{ij} y^i y^j + \left(a^{lm}(\partial_j \varphi \hat{k}_m - \partial_m \varphi \hat{k}_j) - \frac{q}{m} F^l_j\right) y^j = 0. \quad (108)$$

It is of interest to note that Eq. (108) is produced by a Lagrangian of the form

$$\mathcal{L} = m\left(\sqrt{a_{ij} y^i y^j} + \varphi(x) \hat{k}_a y^a\right) + qA_a y^a. \quad (109)$$

Thus one may use the Lagrangian (109) as a metric function and produce the equation of motion of a charged particle subject to an EM field in the anisotropic geometrical model, as a geodesic of the space generated by (109).

6. Conclusion

The observed anisotropy of the microwave cosmic rediation, represented by a vector $k_a(x)$, can be incorporated in the framework of Finsler geometry. The equations of geodesics are generalized in a Finsler anisotropic space-time. The calculation of a curvature parameter of anisotropy is performed explicitly by contraction of the S^i_{jkl} curvature. Also, the electromagnetic tensor F_{ij} as well as the first pair of Maxwell equations are unaffected in the transition to the anisotropic geometry. The Lorentz condition and the d'Alambertian are shown to retain their form under coordinate transformations. In our case, however, the generalized wave equation (106) includes the anisotropy vector through the Ricci curvature R_{ij}, the curvature of the non-linear connection $R^i{}_{jk}$ and the Cartan torsion coefficients C_{ijk}.

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