Gen Relativ Gravit
DOI 10.1007/s10714-007-0540-1

RESEARCH ARTICLE

Friedman-like Robertson–Walker model in generalized metric space-time with weak anisotropy

**P. C. Stavrinos · A. P. Kouretsis ·
M. Stathakopoulos**

Received: 25 May 2007 / Accepted: 17 October 2007

Abstract A generalized FRW model of space-time is studied, taking into consideration the anisotropic structure of fields which are depended on the position and the direction (velocity). The Raychaudhouri and Friedman-like equations are investigated assuming the Finslerian character of space-time. A long range vector field of cosmological origin is considered in relation to a physical geometry where the Cartan connection has a fundamental role. The Friedman equations are produced including extra anisotropic terms. The variation of anisotropy z_t is expressed in terms of the Cartan torsion tensor of the Finslerian manifold. A physical generalization of the Hubble and other cosmological parameters arises as a direct consequence of the equations of motion.

Keywords Finsler Geometry · Cosmology · Gravitation

1 Introduction

During the last few years considerable studies concerning observable anisotropies of the universe have been investigated [5,6,8,12]. These are connected to the very early

P. C. Stavrinos (✉)
Department of Mathematics, University of Athens, 15784 Athens, Greece
e-mail: pstavrin@math.uoa.gr

A. P. Kouretsis
5 Festou St., 16674 Glifada, Athens, Greece
e-mail: a.kouretsis@yahoo.gr

M. Stathakopoulos
1 Anastasiou Genadiou St., 11474 Athens, Greece
e-mail: michaelshw@yahoo.co.uk

state of the universe and related to the estimations of WMAP of CMB, the anisotropic pressure or the incorporation of a primordial vector field (e.g., a magnetic field) to the metrical spatial structure of the universe [11,24,40]. In this case the form of scale factor can be influenced by the introductory field. A geometry which may connect the Riemannian metric structure of the space-time to physical vector fields, is the class of Finsler–Randers type spaces. In these spaces an electromagnetic field, a magnetic field or a gauge vector field may emerge out by a physical source of the universe and can be incorporated into the geometry causing an anisotropic structure [1,3,33–35,37,42].

Finsler geometry or the theory of Finsler spaces may be considered as a generalized Riemannian geometry of the first order within the sphere of metrical differential geometry [31]. A Finsler space is a metric space in which the metric function is defined by a norm $\mathcal{F}$ on a tangent bundle instead of defining an inner product structure on it. The norm will be a real function $\mathcal{F}(x, y)$ of a space-time point x and a tangent vector $y \in T_xM$ which places the role of an internal variable (Appendix A). This y dependence characterizes essentially the Finslerian field and has been combined with the concept of anisotropy which causes the deviation from Remannian geometry [15–17]. All kinds of generalized metric theories and unified field theories belong to the larger class of the so called *anisotropic* field theories [15,16]. Therefore these geometrical anisotropies are caused by internal variables. Under these conditions a Finsler geometry can be considered as a physical geometry on which matter dynamics takes place while the Riemann geometry is the gravitational geometry [4,41,42].

The Cartan's torsion tensor [1,31] characterizes all the geometrical concepts of Finsler geometry and appears to all expressions of geometrical objects such as connection and curvature. In some cases it is useful from a physical point of view to consider a vector field in the form $y^i(x)/i = 1, 2, 3, 4$ and the induced Finslerian metric tensor gives rise to the osculating Riemannian metric tensor $r_{\mu\nu}(x) = f_{\mu\nu}(x, y(x))$.

In the present paper we adopt such an approach in order to obtain some results concerning to a Friedman-like Robertson–Walker cosmological model (F-LRW) and its consequences. We proceed by introducing a Randers-type Lagrangian and the induced Finslerian metric modified appropriately for a weak primordial vector field u_a [37] (Sect. 2). We construct the field equations using the osculating approach [1,31] and derive a Friedman-like equation of motion with an extra anisotropic term (Sects. 3, 4). We generalize the Hubble parameter H, the density parameter Ω and the deceleration parameter q for a weak anisotropic approach. The cosmological parameters depend directly on the anisotropy generated by the vector field defined in the Lagrangian metric function (Sect. 4). The solutions of the Friedman-like equation for both matter and radiation dominated universe, the Raychaudhuri equation initially presented in [34,35] and the CMB temperature estimation are all affected by the presence of the rate of anisotropy at the field equations (Sects. 4, 5). The anisotropic solution of the scale factor coincides with the standard ones derived under the assumptions of homogeneity and isotropy. A possible estimation of the rate of anisotropy parameter might be possible if we consider intermediate values for the cosmological constant for a generalized de-Sitter model defined by the Friedman-like equation of motion with a cosmological constant [9]. A more accurate estimation of the model's extra parameter can be made by calculating the CMB shift and the baryon oscillation acoustic peak using the Friedman equations and comparing them to the corresponding values coming from the WMAP data [10, 13,26,32,44]. The data produced by these tests depicts a flat universe confirming the observations.

2 Anisotropy and a Randers type Lagrangian metric

An alternative way of studying of physical phenomena is to incorporate the dynamics to the active geometrical background following Einstein's meaning of gravity. Our investigation is based on the introduction of a Lagrangian metric [35,37] considering the anisotropy of the universe [5] as an embodied characteristic of the geometry of space-time. A similar investigation has been applied to the case of electromagnetism [3,33] together with some recent progress in gravity, cosmology and fluid dynamics [19,34,37]. We consider the geodesics of the four-dimensional space-time to be produced by a Lagrangian identified to be the Randers-type metric function (the greek indices belong to $\{0, 1, 2, 3\}$ and the latin ones to $\{1, 2, 3\}$)

$$F(x, y) = \sigma(x, y) + \phi(x)\hat{k}_\alpha y^\alpha \tag{2.1}$$

$$\sigma(x, y) = \sqrt{a_{\kappa\lambda}(x) y^\kappa y^\lambda} \tag{2.2}$$

where $a_{\kappa\lambda}(x)$ is the Robertson–Walker metric defined as

$$a_{\kappa\lambda}(x) = \mathrm{diag}\left(1, -\frac{a^2}{1 - kr^2}, -a^2r^2, -a^2r^2 \sin^2\theta\right). \tag{2.3}$$

where $k = 0, \pm 1$ for a *flat, closed* and *hyperbolic* geometry respectively. The spatial coordinates are comoving and the time coordinate represents the proper time measured by the comoving observer. The vector $y^\mu = \frac{dx^\mu}{ds}$ represents the tangent four-velocity of a comoving observer along a preferred family of worldlines (fluid flow lines) in a locally anisotropic universe;the arclength parameter s stands for the proper time. We proceed by considering the natural Lorentzian units, i.e., $c = 1$. If we fix the direction $y = \dot{x}$ then $\sigma(x, \dot{x}) = 1$. The vector field

$$u_\alpha(x) = \hat{k}_\alpha \phi(x) \tag{2.4}$$

stands for a weak primordial vector field $|u_\alpha| \ll 1$ incorporated to the geometry of space-time as an intrinsic characteristic. This field would most naturally be expected to point in the same direction with the tangent vectors of the fluid flow lines [27]. As a result it will have only a timelike component which can be expressed as a function of the proper time $u_a = (u_0, 0, 0, 0)$. Important information about the anisotropy is encoded into the component $u_0(t)$ [27]. We consider a linearized variation of anisotropy, therefore the approximation

$$\phi(x) \approx \phi(0) + \partial_\mu \phi(0) x^\mu \tag{2.5}$$

is valid for small x.

3 The choice of the connection $A^{\kappa}_{\lambda\mu}(x)$ and the curvature

The metric of the Finsler space can be directly calculated from the metric function F. Since $f_{\mu\nu}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^\mu \partial y^\nu}(x,y)$ we derive

$$f_{\mu\nu} = g_{\mu\nu} + \frac{1}{4\sigma}(u_\mu y_\nu + u_\nu y_\mu) - \frac{\beta}{\sigma^3} y_\mu y_\nu + u_\mu u_\nu \tag{3.1}$$

where

$$g_{\mu\nu}(x,y) = \frac{F}{\sigma}(x,y) a_{\mu\nu}(x) \tag{3.2}$$

and

$$\beta(x,y) = \phi(x)\hat{k}_\alpha y^\alpha = u_\alpha(x) y^\alpha \tag{3.3}$$

Under the weak field assumption we can approximate the Finslerian metric (3.1) as a perturbation of the FRW metric since in General Relativity a weak vector field in a space (e.g., primordial magnetic field) can be treated as first order perturbation of the Riemann metric tensor. The metric is considered to have signature $(+,-,-,-)$ for any (x,y). The square of the length of an arbitrary contravariant vector X^μ is to be defined $|X|^2 = f_{\mu\nu}(x,y)X^\mu X^\nu$. The connection components of the metric are given by (A.7).

In many cases we consider a convenient Finsler metric to approximate the gravitational theories [1,31]. This metric is connected to a Riemannian one, $r_{\mu\nu}(x)$ referred as *osculating Riemannian metric* [31]

$$r_{\mu\nu}(x) = f_{\mu\nu}(x, y(x)) \tag{3.4}$$

with the following Christoffel components

$$\begin{aligned} r^{\kappa}_{\lambda\mu}(x) = {} & \gamma^{\kappa}_{\lambda\mu}(x,y(x)) + C^{\kappa}_{\mu\rho}(x,y(x))\frac{\partial y^\rho}{\partial x^\lambda}(x) \\ & + C^{\kappa}_{\lambda\rho}(x,y(x))\frac{\partial y^\rho}{\partial x^\mu}(x) - g^{\kappa\sigma}(x,y(x))C_{\lambda\mu\rho}(x,y(x))\frac{\partial y^\rho}{\partial x^\sigma}(x) \end{aligned} \tag{3.5}$$

thus the equation of geodesics is given by

$$\frac{d^2x^\mu}{ds^2} + r^{\mu}_{\rho\sigma}(x) y^\rho y^\sigma = 0. \tag{3.6}$$

Under the assumption that the vector field y^α satisfies the relation $y^{\mu}_{;\nu} = 0$ the Finslerian $\delta-$covariant derivative and the Cartan's covariant derivative of an arbitrary vector field $X^\alpha(x)$ are equal [1,31] (see Appendix B)

$$X^{a}_{;\beta}(x,y(x)) = X^{a}_{|\beta}(x,y(x)). \tag{3.7}$$

The Cartan's torsion tensor can be easily deduced from (2.1) and (A.8) the full expression is [37]

$$C_{\mu\nu\lambda} = \frac{1}{2}\left\{\frac{1}{\sigma}S_{(\mu\nu\lambda)}(a_{\mu\nu}u_{\lambda}) - \frac{1}{\sigma^3}S_{(\mu\nu\lambda)}(y_{\mu}y_{\nu}u_{\lambda}) - \frac{\beta}{\sigma^3}S_{(\mu\nu\lambda)}(a_{\mu\nu}y_{\lambda})\right\} \quad (3.8)$$

where $S_{\mu\nu\lambda}$ denotes the sum over the cyclic permutation of the indices. Every single term of (3.8) is proportional to the components of the field u_α thus $C_{\mu\nu\lambda} \approx 0$ under the condition $|u_\alpha| \ll 1$ and then we can drop all the torsion dependent terms in (3.5). Therefore the approximation for the Christoffel components becomes

$$A^{\kappa}_{\lambda\nu}(x) \approx \gamma^{\kappa}_{\lambda\nu}(x, y(x)) \quad (3.9)$$

where $A^{\kappa}_{\lambda\nu}(x)$ represent the osculating affine connection coefficients. The affine curvature tensor associated with the proper choice of the connection coefficients $A^{\kappa}_{\lambda\nu}$ gives directly the curvature which is associated with the commutation relations of the δ-derivatives

$$L^{\kappa}_{\lambda\mu\nu} = A^{\kappa}_{\lambda\nu,\mu} - A^{\kappa}_{\lambda\mu,\nu} + A^{\rho}_{\lambda\nu}A^{\kappa}_{\rho\mu} - A^{\rho}_{\lambda\mu}A^{\kappa}_{\rho\nu}. \quad (3.10)$$

The Ricci tensor is given by

$$L_{\mu\nu} = L^{\alpha}_{\mu\alpha\nu} \quad (3.11)$$

and the scalar curvature

$$L = f^{\mu\nu}L_{\mu\nu}. \quad (3.12)$$

The inverted metric $f^{\mu\nu}$ is calculated in [37]. The components of the Ricci tensor can be simplified due to the conditions

$$\ddot{u}_0 \approx 0 \quad (3.13)$$

$$\dot{u}_0^2 \approx 0. \quad (3.14)$$

The condition (3.13) is valid since $\phi(x)$ can be written at the linear form (2.5) together with (3.14) where we have considered $\dot{u}$ very small at the first stages of a highly accelerated expanding universe [30]. We arrive then at the following nonzero components

$$\begin{aligned} L_{00} &= 3\left(\ddot{a}/a + 3/4\dot{a}/a\dot{u}_0\right) \\ L_{11} &= -\left(a\ddot{a} + 2\dot{a}^2 + 2k + 11/4a\dot{a}\dot{u}_0\right)/(1-kr^2) \\ L_{22} &= -\left(a\ddot{a} + 2\dot{a}^2 + 2k + 11/4a\dot{a}\dot{u}_0\right)r^2 \\ L_{33} &= -\left(a\ddot{a} + 2\dot{a}^2 + 2k + 11/4a\dot{a}\dot{u}_0\right)r^2\sin^2\theta \end{aligned} \quad (3.15)$$

The geodesic deviation equation in the case of a perfect fluid along the neighboring world lines can be generalized within the Finslerian framework (ξ^μ is the deviation vector) [2,36]

$$\frac{\delta^2\xi^\mu}{\delta s^2} + L^\mu_{\nu\rho\sigma} y^\nu y^\rho \xi^\sigma = 0 \tag{3.16}$$

where the operator $\frac{\delta}{\delta s}$ denotes the Finslerian δ−connection along the geodesics.

Within a Finslerian space-time framework the concept of constant curvature K is formulated by [31,37]

$$L_{\kappa\lambda\mu\nu} = K(f_{\kappa\mu} f_{\lambda\nu} - f_{\kappa\nu} f_{\lambda\mu}) \tag{3.17}$$

4 Einstein's field equations for an anisotropic universe

4.1 The energy-momentum tensor and the Friedman-like equation

The energy-momentum tensor of a Finslerian perfect fluid for a comoving observer [1,29,34] is defined to be

$$T_{\mu\nu}(x, y(x)) = (\mu + P) y_\mu(x) y_\nu(x) - P f_{\mu\nu}(x, y(x)) \tag{4.1}$$

where $P \equiv P(x)$, $\mu \equiv \mu(x)$ is the pressure and the energy density of the cosmic fluid respectively. The vector $y^\alpha = \frac{dx^\alpha}{d\tau}$ is the 4-velocity of the fluid since $y^\alpha = (1, 0, 0, 0)$ with respect to comoving coordinates. Thus $T_{\mu\nu}$ becomes ($T_{\mu\nu} = \mathrm{diag}(\mu, -P f_{ij})$ in matrix form) [7,24,27,28]

$$\begin{aligned} T_{00} &= \mu \\ T_{ij} &= -P f_{ij} \\ T &= T^\mu_\mu \\ &= f^{00}\mu - 3P \end{aligned} \tag{4.2}$$

The substitution of (4.1) to the field equations

$$L_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) \tag{4.3}$$

implies the following equations at the weak field limit

$$\frac{\ddot{a}}{a} + \frac{3}{4}\frac{\dot{a}}{a}\dot{u}_0 = -\frac{4\pi G}{3}(\mu + 3P) \tag{4.4}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} + \frac{11}{4}\frac{\dot{a}}{a}\dot{u}_0 = 4\pi G(\mu - P) \tag{4.5}$$

after subtracting (4.4) from (4.5) we obtain the Friedman-like equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a} z_t = \frac{8\pi G}{3}\mu - \frac{k}{a^2} \tag{4.6}$$

where we define z_t as

$$z_t = \dot{u}_0. \tag{4.7}$$

The quantity z_t is a constant since we consider a linear approach for the $\phi(x)$ by (2.5). The previous equation is similar to the one derived from the Robertson–Walker metric in the Riemannian framework, apart from the extra term $\frac{\dot{a}}{a} z_t$. We associate this extra term to the present Universe's anisotropy. In case we study Finslerian models with a cosmological constant [9] the field equations (4.4) and (4.5) can be given in the following form

$$\frac{\ddot{a}}{a} + \frac{3}{4}\frac{\dot{a}}{a}\dot{u}_0 = -\frac{4\pi G}{3}(\mu + 3P) + \frac{\Lambda}{3} \tag{4.8}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} + \frac{11}{4}\frac{\dot{a}}{a}\dot{u}_0 = 4\pi G(\mu - P) + \Lambda \tag{4.9}$$

and we end up with the equation of motion

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a} z_t = \frac{8\pi G}{3}\mu - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{4.10}$$

4.2 The parameter z_t and the weak linearized anisotropy

The physical quantity z_t describes the variation of anisotropy which evolves linearly due to (2.5); it depends upon the scalar $\phi(x)$ which is the only quantity of the Lagrangian that gives us insight about the evolution of anisotropy. The parameter z_t is measured by the Hubble's units as (4.6) implies. It is significant that z_t depends on the geometrical properties of the Finslerian space-time manifold. Indeed, the component C_{000} can be directly calculated from (3.8) as

$$C_{000} = \frac{u_0}{2} \tag{4.11}$$

and after differentiating with respect to proper time we lead to *the direct dependence of z_t on the Cartan torsion component C_{000}*

$$z_t = 2C_{000,0} \tag{4.12}$$

hence the variation of anisotropy is closely related to the variation of the Cartan torsion tensor as an intrinsic object of the Finslerian space-time.

5 The cosmological anisotropic parameters

We list the main anisotropic parameters constructed within the Finslerian framework [1,34,35,37]

5.1 The anisotropic scale factor $\tilde{a}(\upsilon(s))$ and the Hubble parameter $\tilde{H}$

The anisotropic scale factor is defined along each world line. $S(s)$ is the length scale introduced in [12,34] defined as

$$S(s) = \tilde{a}(\upsilon(s)) \tag{5.1}$$

where $\upsilon(s)$ is the tangent vector field along the world lines. The anisotropic Hubble parameter $\tilde{H}$ is given by the relation $\left(\dot{S} = \frac{\partial \tilde{a}}{\partial \upsilon^{\mu}} \dot{\upsilon}^{\mu}(s)\right)$

$$\tilde{H} = \frac{\dot{S}}{S} = \frac{1}{3}\tilde{\Theta} \tag{5.2}$$

The term $\tilde{\Theta}$ is the expansion in the Finslerian space-time expressed as

$$\tilde{\Theta} = y^{\mu}_{;\mu} - C^{\lambda}_{\mu\lambda.}\dot{y}^{\mu} \tag{5.3}$$

The anisotropic Hubble parameter can be computed from (4.6)

$$\begin{aligned} \tilde{H}^2 &= \frac{8\pi G}{3}\mu - \frac{k}{a^2} \\ &= H^2 + H z_t \end{aligned} \tag{5.4}$$

thus Hubble's units have to be attributed to z_t. Since $\tilde{H}^2 > 0$ we should fix $H^2 \gg |H z_t|$ Therefore the lower limit of z_t is $-H$,

$$z_t \geq -H. \tag{5.5}$$

This result is in agreement with the fixing of a similar parameter in [20,25,26] for self-accelerated brane-world cosmology. Since the parameter z_t is related to the variation of anisotropy we expect it to have negative sign (*self accelerating universe*) as it may control a transition of the universe from a state of anisotropy to a smoother isotropic phase [46].

5.2 The density and the deceleration parameter

The density parameter can be defined with respect to the parameter $\tilde{H}$

$$\tilde{\Omega}_{\mu} = \frac{8\pi G}{3\tilde{H}^2}\mu = \frac{\mu}{\tilde{\mu}_{\text{crit}}} \tag{5.6}$$

where

$$\tilde{\mu}_{\text{crit}} = \frac{3\tilde{H}^2}{8\pi G}. \tag{5.7}$$

The deceleration parameter is defined in terms of the anisotropic scale factor $S(s)$

$$\tilde{q} = -S\ddot{S}/\dot{S}^2. \tag{5.8}$$

The Friedman-like equation can also be rewritten in the form

$$\tilde{\Omega}_\mu - 1 = k/(\tilde{H}^2 a^2) \tag{5.9}$$

If $\mu < \tilde{\mu}_{\text{crit}}$ then $\tilde{\Omega}_\mu < 1$ or $k < 0$ (*open universe*). If $\mu > \tilde{\mu}_{\text{crit}}$ then $\tilde{\Omega}_\mu > 1$ or $k > 0$ (*closed universe*). The latter case $\mu = \tilde{\mu}_{\text{crit}}$ corresponds to $\tilde{\Omega}_\mu = 1$ or $k = 0$ (*flat universe*). Thus the values of z_t influence the type of spatial curvature. In case we need to express $\tilde{H}$ in terms of the redshift, the present value of the Hubble parameter H_0 and $\Omega_{M_0}, \Omega_{\Lambda_0}$, we insert [7,29,30]

$$H(z) = H_0 E(z) \tag{5.10}$$

into (5.4) where the quantity $E(z) = H(z)/H_0$ is given by

$$E(z) = \sqrt{\Omega_{z_t}} + \sqrt{\Omega_{z_t} + \Omega_{M_0}(1+z)^3 + \Omega_{K_0}(1+z)^2 + \Omega_{\Lambda_0}} \tag{5.11}$$

and the parameter Ω_{z_t} is defined by

$$\Omega_{z_t} = \frac{z_t^2}{4H_0^2}. \tag{5.12}$$

Therefore we are dealing with an expression for $\tilde{H}(z)$ which depend upon the anisotropic parameter z_t, the redshift z and the Ω's.

The Friedman equations in terms of Ω's

The Friedman equations can be expressed in turns of the density parameters. The Ω's give some useful information about the parameter z_t and enable us to test if our cosmological model fits to the current data (e.g., WMAP data). Indeed, the equations of motion (4.6) and (4.10) are reduced to the form

$$1 = \Omega_{M_0} + \Omega_{K_0} + \Omega_{\Lambda_0} - z_t/H_0. \tag{5.13}$$

In case we are interested in manipulating the Friedman equation for a specific value of the redshift z we can make use of (5.11). It is difficult to distinguish the last term of (5.13) from GR dark energy, since both terms accelerate the universe. If the self-accelerating term $z_t H$ domimates over the expansion, the cosmological constant in (5.13) and (5.11) has to vanish [26].

5.3 The continuity equation

The energy density can be calculated by direct integration of the continuity equation $T^{\mu}_{0;\mu} = 0$. A proper manipulation of (4.1) and (3.9) leads to the equation

$$T^{0}_{0,0} + (\dot{a}/a + \dot{u}_0/2)\left(3T^{0}_{0} - T^{i}_{i}\right) = 0 \tag{5.14}$$

which can be simplified to the form($c = 1$)

$$\dot{\mu} + 3\dot{a}/a\,(\mu + P) + \dot{u}_0/2\,(3P + 2\mu) = 0 \tag{5.15}$$

since we have applied the approximations

$$\begin{aligned} f^{00} &= 2/(2 + u_0 + 2u_0^2) \approx 1 \\ f^{00}_{,0} &= -2\dot{u}_0(1 + 4u_0)/(2 + u_0 + 2u_0^2)^2 \\ T^{0}_{0,0} &= f^{00}\dot{\mu} + f^{00}_{,0}\mu \approx \dot{\mu} - \mu\dot{u}_0/2 \\ 3T^{0}_{0} - T^{i}_{i} &\approx 3\mu + 3P. \end{aligned} \tag{5.16}$$

A perfect fluid relevant to cosmology obeys the equation of state

$$P = w\mu \tag{5.17}$$

where $w = 0$ for a matter dominated universe and $w = 1/3$ for a radiation dominated universe. The substitution of (5.17) to (5.15) leads to

$$\dot{\mu}/\mu = -3(1 + w)\dot{a}/a + \dot{u}_0/2(2 + 3w) \tag{5.18}$$

and the integration of the differential equation (5.18) implies

$$\mu \propto a^{-3(1+w)} \exp\left(-u_0\,(3w + 2)\,/2\right) \tag{5.19}$$

therefore

$$\mu \propto \begin{cases} a^{-3}\exp(-u_0) & \text{matter dominated universe} \\ a^{-4}\exp(-3u_0/2) & \text{radiation dominated universe} \end{cases} \tag{5.20}$$

This asymptotic behavior indicates that the weak anisotropy *affects homogeneity.*

5.4 The look back time at the presence of weak anisotropy

Given the definition of the anisotropic Hubble parameter (5.4) we can generalize the concept of the lookback time $t_0 - t_*$ in [7]

$$\tilde{t}_0 - \tilde{t}_* = \int_0^{z_*} \frac{dz'}{(1+z')\tilde{H}(z')} \tag{5.21}$$

where t_0 is the age of the universe today and t_* is the age of the universe when the redshift was $z = z_*$. Therefore the general expression for the world time evolution should be [30]

$$\tilde{t}(z) = \int_z^{\infty} \frac{dz'}{(1+z')\tilde{H}(z')} \tag{5.22}$$

since $z \to \infty$ at the start of the universe. Taking into account the weak anisotropy scenario and the redshift expression of the Hubble parameter (5.10) into account we deduce the small z_t expansion for the worldtime today $\tilde{t}_0(z = 0)$

$$\tilde{t}_0 = t_0 + \sum_{k=1}^{\infty} T_k z_t^k \tag{5.23}$$

where

$$T_k = H_0^{-k-1} \binom{-1/2}{k} \int_0^{\infty} \frac{dz}{(1+z)E(z)^{k+1}}. \tag{5.24}$$

5.5 The Finslerian Raychaudhuri equation

The generalization of Raychaudhuri's equation has been given by the following formula [34]

$$\dot{\tilde{\Theta}} = -\frac{1}{3}\tilde{\Theta}^2 - \tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu} + \tilde{\omega}_{\mu\nu}\tilde{\omega}^{\mu\nu} - 4\pi G(\mu + 3P) + (y^{\mu}_{;\kappa} y^{\kappa})_{;\mu} \tag{5.25}$$

where $\tilde{\sigma}^2 = \tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu}$, $\tilde{\omega}^2 = \tilde{\omega}_{\mu\nu}\tilde{\omega}^{\mu\nu}$ are the Finslerian *shear* and *vorticity* respectively, which are defined in [1,31] and refer to a perfect fluid. Equation (5.25) is a direct application of the Finslerian Lie derivative for dust-like matter developed in [38]. It includes the anisotropic gravitational influence of the matter along the worldlines which is expressed by the tidal force of the field

$$L_{\mu\nu} y^{\mu} y^{\nu} = 4\pi G(\mu + 3P). \tag{5.26}$$

Using the expressions (5.8) and (5.3) we produce the linearized Raychaudhuri equation [$\tilde{q} = -\frac{\ddot{\tilde{a}}}{\tilde{a}}\tilde{H}^{-2}$ where $\tilde{H}$ depends on $a, \dot{a}$ in virtue of (5.4)]

$$\frac{1}{3}\tilde{\Theta}^2\tilde{q} = 4\pi G(\mu + 3P)\frac{H}{\tilde{H}} + f(a, \dot{a}, \ddot{a}, z_t) \tag{5.27}$$

where the function $f(a, \dot{a}, \ddot{a}, z_t)$ is defined as

$$f(a, \dot{a}, \ddot{a}, z_t) = -3\tilde{H}^2 - \frac{3}{2}\frac{H}{\tilde{H}}\left(\tilde{H}^2 + H^2\right) + \frac{1}{\tilde{H}}\left[2\pi G(\mu + 3P) + 3z_t H\right] z_t.$$

Thus we can expand for small values of z_t and conclude that

$$\frac{1}{3}\tilde{\Theta}^2\tilde{q} = 4\pi G(\mu + 3P) + 3\frac{\dot{a}}{a}z_t - \frac{3}{8}\frac{\ddot{a}(t)a(t) - \dot{a}(t)^2}{\dot{a}(t)^2}z_t^2 + O(z_t^3) \quad (5.28)$$

The sign of the right hand side of (5.27) determines the state of expansion. If the inequality $f(a, \dot{a}, \ddot{a}, z_t) < 0$ is valid then the term $f(a, \dot{a}, \ddot{a}, z_t)$ contributes to the acceleration of the universe(the field assist inflation), whereas if $f(a, \dot{a}, \ddot{a}, z_t) > 0$ it will slow the expansion down(the inflation domination must be longer to accelerate the universe). This specific effect is due to the kinematical reaction of the geometry of the spatial hypersurfaces, rather than an attempt to suppress inflation(it can be considered as an essential ingredient of the *Finslerian ansatz*) [24,31].

6 Einstein field equations with anisotropic term

We assume z_t to be a constant and study the differential equations (4.4), (4.5) and (4.6) both for the cases of matter and radiation dominated universe. We notice that for $z_t = 0$ the field equations reduce to the usual ones coming from a Riemannian Robertson–Walker metric [14,27,28,45]. The whole calculation is done for a homogeneous universe of constant density μ.

6.1 Solution for a matter dominated universe

The solution for the scale factor in the case of a matter dominated universe is derived by the integration of the Friedman-like equation (4.6) with initial condition

$$a(0) = 0 \quad (6.1)$$

where $z_t = \text{const}$. We set up $t = 0$ as the beginning of time without considering any quantum effects; there are different ways of handling the initial condition for example setting $a(t_{Pl}) = 0$ or considering the scale factor after the Plank scale (e.g., see [11]). Equation (4.6) is simplified if we insert the parameter (conformal proper time) [27,28]

$$\eta = \int_0^t \frac{d\omega}{a(\omega)} \quad (6.2)$$

which measures the arc in rad traveled along by a photon on a sphere of radius $a(t)$. We study the asymptotics of the scale factor

$$a(t) \equiv a(t(\eta)) = \bar{a}(\eta) \to 0 \tag{6.3}$$

where $cdt = ad\eta$. We consider a homogeneous universe of constant density μ calculated as [27]

$$\mu = \frac{M}{V} = \frac{M}{2\pi^2 a^3}. \tag{6.4}$$

We assume that the universe would take over the same volume as in the closed case ($k = 1$) since (5.20) is valid for the asymptotics of μ. The parameter k determines the kind of spatial geometry $k = 0, -1, +1$ (*flat, open, closed* universe, respectively). Since $\frac{\dot{a}}{a} = \bar{a}^{-2}\dot{\bar{a}}$ Eq. (4.6) becomes

$$\left(\dot{\bar{a}} + \frac{z_t}{2}\bar{a}^2\right)^2 = \frac{z_t^2}{4}\bar{a}^4 - k\bar{a}^2 + \frac{4GM}{3\pi}\bar{a} \tag{6.5}$$

We study a Universe that accelerates very fast at its early stages thus we can accept $\dot{\bar{a}} + \frac{z_t}{2}\bar{a}^2 > 0$. The velocity of expansion $\dot{a}$ takes on very large values. Equation (6.5) can be integrated directly for all the values of k. It is more convenient to substitute [28]

$$a_* = \frac{2GM}{3\pi} \tag{6.6}$$

and the separable Eq. (6.5) leads to

$$t = -\frac{z_t}{2}\left\{\int_0^a \frac{\sqrt{x^4 - 4k/z_t^2 x^2 + 8a_*/z_t^2 x}}{kx - 2a_*}dx + \int_0^a \frac{x^2}{kx - 2a_*}dx\right\} \tag{6.7}$$

together with the initial condition

$$a(0) = 0. \tag{6.8}$$

I Calculation for $k = 0$
We fix $k = 0$ at (6.7) and find

$$\begin{aligned} t &= \frac{z_t}{2}\left[\frac{a^3}{6a_*} + I_0\right] \\ I_0 &= \frac{1}{2a_*}\int_0^a \sqrt{x^4 + 8a_*/z_t^2 x}\,dx \end{aligned} \tag{6.9}$$

we expand for a small z_t and arrive at the solution

$$t = \frac{\sqrt{2}}{3\sqrt{a_*}} a^{3/2} + 1/(12a_*)a^3 z_t/c + O(z_t^2) \tag{6.10}$$

II Calculation for $k = -1$
A direct integration of (6.7) leads to

$$t = -\frac{z_t}{4}\left[a^2/2 + 2a_*a + 4a_*^2 \log|a - 2a_*| + I_1\right] \tag{6.11}$$

where

$$I_1 = \int_0^a \frac{\sqrt{x^4 - 4/z_t^2 x^2 + 8a_*/z_t^2 x}}{x - 2a_*} dx \tag{6.12}$$

after expanding for small z_t

$$t = \sqrt{a(a + 2a_*)} - a_* \log\left(1 + a/a_* + \sqrt{a(a + 2a_*)}/a_*\right)$$
$$- \left(a^2/4 + a_*a + 2a_*^2 \log|a - 2a_*|\right) z_t + O(z_t^2). \tag{6.13}$$

III Calculation for $k = +1$
The calculation is the same as the previous case

$$t = \frac{z_t}{2}\left[a^2/2 - 2a_*a + 4a_*^2 \log|a + 2a_*| + I_{-1}\right] \tag{6.14}$$

where

$$I_{-1} = \int_0^a \frac{\sqrt{x^4 + 4/z_t^2 x^2 + 8a_*/z_t^2 x}}{x + 2a_*} dx \tag{6.15}$$

and after expanding for small z_t we obtain the solution

$$t = -\sqrt{a(2a_* - a)} + a_* \arccos(1 - a/a_*)$$
$$+ \left(a^2/4 - a_*a + 2a_*^2 \log|a + 2a_*|\right) z_t + O(z_t^2). \tag{6.16}$$

The leading term of the solutions for all k represents the solution given by the field equations of the Robertson–Walker metric [28]. A small a expansion gives the asymptotic behavior $t \sim a^{3/2}$ or equivalently

$$a \sim t^{2/3}. \tag{6.17}$$

6.2 Solution for a radiation dominated universe

The solution for a radiation dominated universe can be deduced by inserting the equation of state $P = \frac{1}{3}\mu_{\text{rad}}$ into (4.4) and after adding this to (4.5) we end up with the equation

$$\frac{d}{dt}(a\dot{a}) = -k - \frac{7}{4}a\dot{a}z_t \qquad (6.18)$$

which we integrate and find

$$\dot{a}a + \frac{7}{4}z_t a^2 = -kt + C_1 \qquad (6.19)$$

If we substitute $z_t = 0$ to (6.19) we get back the usual solution for a radiation dominated universe $a \propto \sqrt{t}$ for all values of k. If we take into account the initial condition $a(0) = 0$ we arrive at the solution

$$a(t) = \frac{4\sqrt{2}}{7z_t}\left\{(C_0 z_t + k)\left(1 - \exp\left(-\frac{7}{4}z_t t\right)\right) - \frac{7}{4}kz_t t\right\}^{1/2}. \qquad (6.20)$$

The expansion of the solution for small z_t is

$$a(t) = \sqrt{t}\left[\sqrt{2C_0 - kt} + \frac{7t(kt - 3C_0)}{6\sqrt{4C_0 - 2kt}}\cdot z_t + O(z_t^2)\right] \qquad (6.21)$$

6.3 Solution for the de-Sitter model

The de-Sitter model for an empty anisotropic universe constructed in [34] leads to the equation of motion [$\mu = 0$ in (4.10)]

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}z_t = -\frac{k}{a^2} + \frac{\Lambda}{3} \qquad (6.22)$$

Since $\tilde{H}^2$ in (6.22) can be written as $\tilde{H}^2 = \left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}z_t \geq 0$, the cosmological constant Λ is restricted by the inequality $\Lambda \geq 3K$ where $K = k/a^2$ is the curvature of the space.

I Calculation for $k = 0$
The case of zero curvature can be integrated to give

$$a(t) = \text{const} \times \exp\left(-z_t/2t\right)\exp\left[\left(\Lambda/3 + z_t^2/4\right)^{1/2} t\right] \qquad (6.23)$$

The solution converges to the one without anisotropy if we let $z_t \to 0$ [9].

II Calculation for the special case $\Lambda = 3K$
The field equations for a space of constant spatial curvature (maximal symmetry) and an empty universe ($T_{\mu\nu} = 0$) imply the condition [7,34]

$$\Lambda = 3K \tag{6.24}$$

hence Λ can be substituted to the equation of motion (6.22)

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a} z_t = 0 \tag{6.25}$$

and the solution for the scale factor is

$$a(t) = \text{const} \times \exp(-z_t t). \tag{6.26}$$

In both cases the constant of integration can be absorbed into $a(t)$ if we choose the right scale (e.g., $a(0) = 1$).

6.4 A model for inflation with anisotropy

The idea of inflation can be incorporated into the model of Friedman-like equations with weak anisotropy if we introduce the vacuum energy density of a scalar field V_0 to the energy density μ. Indeed, we consider $G\mu = V_0/m_{Pl}^2$, i.e., $\mu = V_0$ thus (4.6) implies [7,9,21]

$$\dot{a}^2 + a\dot{a}z_t = \frac{8\pi}{3}\frac{V_0}{m_{Pl}^2}a^2 - k. \tag{6.27}$$

Since we work at an inflationary phase the size of the scale factor is such that the term $\frac{8\pi}{3}\frac{V_0}{m_{Pl}^2}a^2$ dominates over k hence we can neglect k and (6.27) can be rewritten as

$$\tilde{H}^2 = \frac{8\pi}{3}\frac{V_0}{m_{Pl}^2}. \tag{6.28}$$

Taking the positive square root of (6.28) we find the scale factor

$$a(t) = \text{const} \times \exp(-z_t/2t) \exp\left[\left(8\pi/3 \cdot V_0/m_{Pl}^2 + z_t^2/4\right)^{1/2} t\right] \tag{6.29}$$

recovering the de-Sitter solution (6.23) and the expected exponential rate of expansion for the early inflationary phase of the universe.

7 Estimation of the cosmic microwave background radiation (CMB)

The estimation of CMB can be achieved with the aid of Stefan-Boltzmann's law [22,28]

$$\mu_{\rm rad}c^2 = \sigma_{SB}g_* T(z)^4 \tag{7.1}$$

where

$$\sigma_{SB} = \frac{\pi^2 k_B^4}{30\hbar^3 c^3}. \tag{7.2}$$

The temperature $T(z)$ is the radiation temperature for a given redshift z and

$$g_* = \sum_{\rm bosons} g_i + \frac{7}{8}\sum_{\rm fermions} g_i \tag{7.3}$$

is defined as the sum of the boson and fermion spin states (e.g., for photons $g = 2$, for neutrinos $g = 1$ and for massive particles $g = 2s + 1$). The calculation could be attainable if we consider some data for the Hubble parameter $H = \dot{a}/a$ and the anisotropic constant z_t. The radiation dominated solution does not depend on the nature of the spatial geometry of the universe thus we fix $k = 0$ in (4.6) and obtain μ_{rad}

$$\begin{aligned}\mu_{\rm rad} &= \frac{3}{8\pi G}(H^2 + Hz_t)\\ &= \frac{3}{8\pi G}\tilde{H}^2\end{aligned} \tag{7.4}$$

therefore (7.1) and (7.4) yield [28]

$$T(z) = \left\{\frac{1}{\sigma_{SB}g_*}\left(\frac{3}{8\pi G}\tilde{H}^2 c^2\right)\right\}^{1/4}. \tag{7.5}$$

The calculation can be directly derived if we manipulate (7.5) into the form

$$k_B^2 T(z)^2 = \frac{3\sqrt{5}}{2\sqrt{g_*}\pi^{3/2}} m_{Pl} c^2 \hbar \tilde{H} \tag{7.6}$$

where m_{Pl} stands for the Planck mass

$$m_{Pl} = \sqrt{\frac{\hbar c}{G}}. \tag{7.7}$$

Thus we can calculate the temperature $T(z)$ for a given value of the redshift z with the aid of the formula

$$T(z) = T_H(z)\left[1 + z_t/H(z)\right]^{1/4} \tag{7.8}$$

where $T_H(z)$ is the value of the temperature for a given value of the Hubble parameter without the assumption of weak anisotropy and $H(z)$ is calculated from (5.10). Due to the adiabatic expansion of the universe we can transform the value $T(z)$ at the redshift of CMB to its present value T_0 using the formula($a_{CMB} = 1/(1+z_R)$, $z_R \approx 1090$)

$$T_0 = a_{CMB} T(z). \tag{7.9}$$

8 Discussion

The study of a FRW-model with a weak vector field incorporated in the metric structure of space-time provides us the extended Friedman-like Eq. (4.6). The contribution of the variation of anisotropy is expressed by the additional parameter z_t produced by the Finslerian character of the geometry of space-time. Especially as it is evident from Eq. (4.12) z_t has a direct dependence up on the Cartan torsion component C_{000}. We remark that our present model correspond to the ones studied in [20, 26] for a flat universe due to the correspondence of z_t to $\pm\frac{1}{r_c}$, where r_c is the extra parameter defined there. The extra parameter z_t appears to compete against the contribution of the cosmological constant due to (5.13).

We perform the model-independent and insensitive to perturbation S test [43], where S is the CMB shift parameter [20, 26]. Our model is close to the WMAP data for a flat universe since for $S = 1.70 \pm 0.03$ [32] we end up with $|\Omega_K| \ll 1$. The same result seems to be valid as we can see in Fig. 1 if we apply the baryon acoustic oscillation peak test for $A = 0.469 \pm 0.017$ [10, 13] (Table 1). The procedures of the tests and the formulas for A and S can be found in [26]. A part of a future work is the investigation of cosmological perturbations since the Finslerian approach generates deviations from homogeneity and isotropy. A more fundamental task is the comparison of our model to the data of CMB anisotropies and the matter power spectrum, which is strongly connected to the analysis of the density perturbations [18].

The initial highly compressed, thermal radiation dominated state of anisotropy is considered to be adiabatically transformed to a cooler matter dominated isotropic phase [27]. In such a case where the anisotropy energy is converted to thermal energy and large amount of entropy [23], a phase transition in the geometric structure can be regulated by the second term of Eq. (2.1), since we expect $\phi(x)$ to decrease monotonically as the universe expands, in order to obtain the standard FRW model. This ensures negative values for the parameter z_t and provides us with a self accelerating cosmological model.

The whole picture of anisotropy directed by a primordial vector field can be locally incorporated to the anisotropic metric structure of a Finslerian space-time. The osculation of the Finsler space leads to the construction of a disformal Riemann structure and can be interpreted as a model of modified gravity.

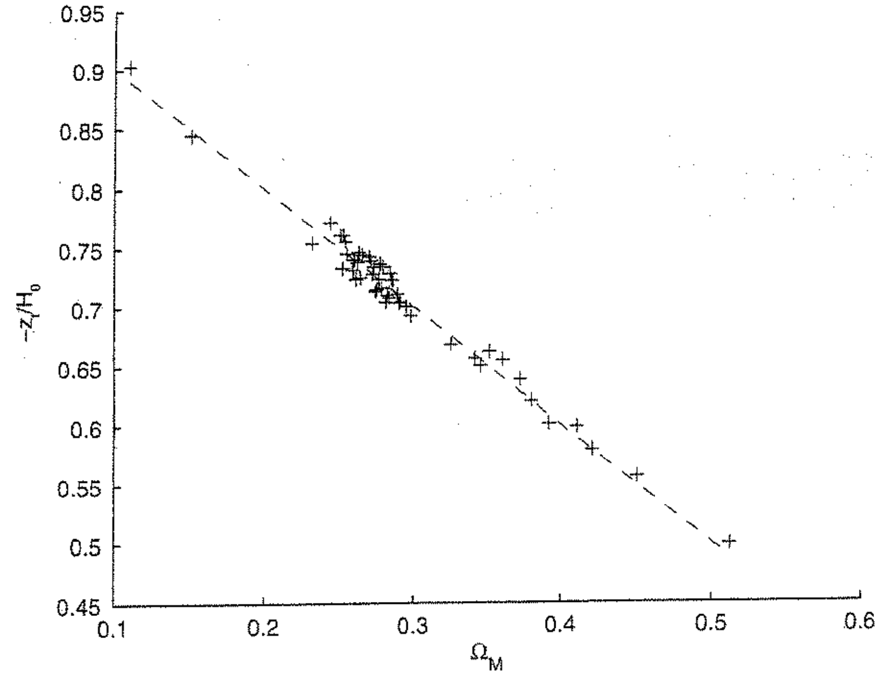

Fig. 1 The z_t parameter versus Ω_M; the A test values of $-z_t/H_0$ are depicted by the "plus sign" points and compared to the "dashed line" values coming from the S test. Both tests almost reproduces the theoretical values of z_t for a flat universe($\Omega_K = 0$) calculated by the Friedman equation $-z_t/H_0 = 1 - \Omega_M$)

Table 1 The A test estimation for the parameter z_t

Ω_M	z_t/H_0	$\Omega_K = 1 - \Omega_M + z_t/H_0$
0.110	−0.9029492854	−0.0129492854
0.151	−0.8443621881	0.0046378119
0.232	−0.7539951109	0.0140048891
0.244	−0.7710952485	−0.0150952485
0.251	−0.7608626761	−0.0118626761
0.252	−0.7327905974	0.0152094026
0.253	−0.7610339645	−0.0140339645
0.254	−0.7556133186	−0.0096133186
0.255	−0.7449999998	0.00000000002
0.259	−0.7313813921	0.096186079
0.260	−0.7400000040	−0.000000004
0.261	−0.7236653148	0.0153346852
0.262	−0.7378258680	0.0001741320
0.263	−0.7464096424	−0.0094096424
0.264	−0.7249493659	0.0110506341
0.2651	−0.7441489179	−0.0092489179
0.270	−0.7425851580	−0.012585158

Table 1 continued

Ω_M	z_t/H_0	$\Omega_K = 1 - \Omega_M + z_t/H_0$
0.271	−0.7389289128	−0.099289128
0.272	−0.7279999954	0.0000000046
0.273	−0.7337402645	−0.0067402645
0.274	−0.7135079061	0.0040405097
0.275	−0.7151439555	0.098560445
0.2761	−0.7238999999	0.0000000001
0.277	−0.7363820256	−0.0133820256
0.278	−0.7166084631	0.0053915369
0.279	−0.7339876359	−0.0129876359
0.281	−0.7038915738	0.0151084262
0.282	−0.7096732356	0.0083267644
0.2831	−0.7079203819	0.089796181
0.284	−0.7285702833	−0.0125702833
0.285	−0.7232538088	−0.083538088
0.2861	−0.7229838341	−0.090838341
0.289	−0.710999993	0.000000007
0.290	−0.7032528013	0.0067471987
0.291	−0.7039021795	0.0050978205
0.295	−0.7002433383	0.0047566617
0.2981	−0.6924644204	0.0094355796
0.325	−0.6674671885	0.0075328115
0.341	−0.6560099554	0.0029900446
0.345	−0.6501638547	0.0048361453
0.351	−0.6617634475	−0.0127634475
0.36	−0.6545617924	−0.0145617924
0.372	−0.6385195052	−0.0105195052
0.38	−0.6199999969	0.0000000031
0.392	−0.6008829186	0.0071170814
0.411	−0.5985044824	−0.0095044824
0.421	−0.5789999998	0.0000000002
0.451	−0.5563735170	−0.0073735170
0.512	−0.4989395368	−0.0109395368

We insert the values of the Ω_M to the baryon acoustic oscillation peak $A = \sqrt{\Omega_M}\left[\frac{H_0^3 d_L^2(z_1)}{H_1 z_1^2(1+z_1)^2}\right]^{1/3}$ and calculate back the corresponding values of z_t, where $A = 0.469 \pm 0.017$ [10], $z_1 = 0.35$ the typical luminous red galaxies redshift and $d_L(z)$ the luminosity distance defined in various Relativity textbooks (e.g., see [7]). The parameter Ω_K is close to zero as expected from the present observational predictions for a flat universe of WMAP [32]

Acknowledgments We wish to thank R. Maartens for the useful comments on the text and the University of Athens (Special Account for Research Grants) for the support to this work.

A Appendix

In the following we present some basic elements of Finsler geometry [6,8,31,36,42]. In 1854, B. Riemann, before arriving at Riemannian metric was concerned with the concept of a more generalized metric

$$ds^2 = \mathcal{F}(x^1, x^2, \ldots, x^n, dx^1, \ldots, dx^n) \tag{A.1}$$

where n is the dimension of the space. A Finsler structure is provided by a n-dimensional C^{∞} manifold M^n, a C^{∞} function $F \equiv F(x, y)$ defined on the tangent bundle $\tilde{TM} = TM/\{0\}$, $F : \tilde{TM} \to R$ that satisfies the conditions

$$\begin{aligned} &(F1)\ \mathcal{F}(x, y) > 0 \quad \forall\, y \neq 0 \\ &(F2)\ \mathcal{F}(x, py) = p\mathcal{F}(x, y) \quad \text{for any } p > 0 \end{aligned} \tag{A.2}$$

where y denotes the directions or velocities on the considered manifold with the previous coordinates.The metric tensor (Hessian)

$$f_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}(x, y) \tag{A.3}$$

is of $rank[(f_{ij})_{i,j}] = n$ and homogeneous of zero degree with respect to y due to the Euler's theorem. The length s of a curve $C : x^i(t), a \leq t \leq b$ on the manifold is

$$s = \int_a^b \mathcal{F}(x(t), y(t))dt. \tag{A.4}$$

The integral of the length is independent of the parameter if and only if the condition (F2) is valid. The condition of homogeneity enables us to define the line element

$$ds = \mathcal{F}(x, dx) \tag{A.5}$$

and the variation of the arclength $\delta \int ds = 0$ implies the Euler–Lagrange equations $\frac{d}{ds}\left(\frac{\partial \mathcal{F}}{\partial y}(x, y)\right) - \frac{\partial \mathcal{F}}{\partial x}(x, y) = 0$ which represent the geodesics of the Finsler space. The equation of geodesics then becomes analogous to the ones of the Riemann space

$$\frac{d^2 x^i}{ds^2} + \gamma^i_{jk} y^j y^k = 0 \tag{A.6}$$

where the Christoffel symbols are defined by the usual formula

$$\gamma^i_{jk}(x, y) = \frac{1}{2} f^{ir}(x, y)\left(f_{rj,k}(x, y) + f_{rk,j}(x, y) - f_{jk,r}(x, y)\right). \tag{A.7}$$

The notion of torsion tensor is crucial within the Finsler Geometry's framework. *A Finsler space is a Riemann space if and only if* $C_{ijk} = 0$ where C_{ijk} is the torsion tensor defined by E. Cartan as

$$C_{ijk} = \frac{1}{2}\frac{\partial f_{ij}}{\partial y^k} \tag{A.8}$$

Therefore a Finsler space can be treated as a natural generalization of a Riemann space.

B Appendix

For a Finslerian vector field $X^{\alpha}(x, y(x))$ the $\delta-$ covariant derivative has the form [1,31]

$$X^{\alpha}_{;\beta}(x, y(x)) = X^{\alpha}_{,\beta}(x) + \Gamma^{*\alpha}_{\rho\beta}(x, y(x))X^{\rho}(x) \tag{B.1}$$

and the Cartan's covariant derivative is given by

$$X^{\alpha}_{|\beta}(x, y(x)) = X^{\alpha}_{,\beta}(x) - \frac{\partial X^{\alpha}}{\partial y^{\rho}}(x, y(x))G^{\rho}_{\beta}(x, y(x)) + \Gamma^{*\alpha}_{\rho\beta}(x, y(x))X^{\rho}(x). \tag{B.2}$$

The $\Gamma^{*\kappa}_{\lambda\mu}$ are the Cartan's connection components defined as

$$\Gamma^{*\kappa}_{\lambda\mu}(x, y) = \left(\gamma^{\kappa}_{\lambda\mu} - C^{\kappa}_{\lambda\rho}G^{\rho}_{\mu} - C^{\kappa}_{\rho\mu}G^{\rho}_{\lambda} + C_{\lambda\mu\rho}G^{\rho}_{\nu}g^{\nu\kappa}\right)(x, y) \tag{B.3}$$

and the G^{μ}_{ν}, G^{μ} are

$$G^{\mu}_{\nu} = \frac{\partial G^{\mu}}{\partial y^{\nu}} \tag{B.4}$$

$$2G^{\mu} = \gamma^{\mu}_{\rho\sigma}y^{\rho}y^{\sigma}. \tag{B.5}$$

References

1. Asanov, G.S.: Finsler Geometry, Relativity and Gauge Theories. Reidel, Dordrecht (1985)
2. Asanov, G.S., Stavrinos, P.C.: Rep. Math. Phys. **30**, 63 (1991)
3. Beil, R.G.: Int. J. Thoery Phys **26**, 2 (1987)
4. Bekenstein, J.D.: Phys. Rev. D **48**(8), 3641 (1993)
5. Bennet, et al.: Astrophys. J. **583**, 1 (2003)
6. Bielewicz, P., et al.: Astrophys. Jour. **635**, 750 (2005) arXiv:astro-ph/0507186
7. Carol. S.M.: Spacetime and Geometry. Addison Wesley. (2004)
8. Copi, C.J., et al.: Mon. Not. Roy. Astron. Soc. **367**, 79 (2005) arXiv:astro-ph/0508047
9. D'Inverno, R.A.: Introducing Einstein's Relativity. Oxford University Press, Oxford (2005)
10. Eisenstein, D.J., et al.: Astrophys. J. **633**, 560 (2005) arXiv:astro-ph/0501171
11. Ellis, G.F.R., Maartens, R.: Class. Quant. Grav. **21**, 223 (2004) arXiv:gr-qc/0211082
12. Ellis, G.F., Hellaby, C., Matravers, D.R.: Astrophys. J. **364**, 400 (1990)
13. Fairbairn, M., Goobar, A.: Phys. Lett. B **642**, 432 (2006) arXiv:astro-ph/0511029
14. Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Space-Time. Cambridge University Press, Cambridge (1973)
15. Horvarth, J.I.: Acta Phys. Chem. Szeged **7**, 3 (1961)
16. Horvarth, J.I.: Nuovo Cimento **9**, 444 (1958)
17. Ikeda, S.: J. M. P. **22**(6), 12111 (1981)
18. Koyama, K., Maartens, R.: JCAP **0610**, 016 (2006) arXiv:astro-ph/0511634
19. Lampropoulos, P.G., Stavrinos, P.C.: A locally anisotropic geometrical model of space-time based on CMBR. In: 7th International Astronomy Conference of the Hellenic Astronomical Society (2006) arXiv:gr-qc/0509020
20. Lazkoz, R., Maartens, R., Majerotto, E.: Phys. Rev. D **74**, 083510 (2006) arXiv:astro-ph/0605701

21. Liddle, A.R.: An introduction to cosmological inflation (1999) arXiv:astro-ph/9901124
22. Mandl, F.: Statistical Physics J. Wiley, New York (1971)
23. Matzner, R.A., Misner, C.W.: Astrophys. J. **171**, 415 (1972)
24. Matravers, D.R., Tsagas, C.G.: Phys. Rev. D **62**, 103519 (2000) arXiv:gr-qc/0003181
25. Maartens, R.: J. Phys. Conf. **68**, 012046 (2007)
26. Maartens, R., Majerotto, E.: Phys. Rev. D **74** 023004 (2006) arXiv:astro-ph/0603353
27. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. Freeman, San Francisco (1973)
28. Ohanian, H.C., Ruffini, R.: Gravitation and Space-Time. Norton, New York (1994)
29. Peacock, J.A.: Cosmological Physics. Cambridge University Press, Cambridge (1999)
30. Peebles, P.J.E.: Principles of Physical Cosmology. Princeton Series in Physics (1993)
31. Rund, H.: The Differential Geometry of Finsler Spaces. Springer, Berlin (1959)
32. Spergel, D.N., et al.: Wilkinson Microwave Anisotropy Probe (WMAP) Three Year Observations: Implications to Cosmology. Astrophys. J. Suppl. **170**, 377 (2007) arXiv:astro-ph/0603449
33. Stavrinos, P.C., Nikolopoulos, D., Prezas, N.: Sectilor Stiintifice Sir.IV Tomul XX (Academia Rumana Bucharest 9 (2000))
34. Stavrinos, P.C.: J. Phys. Conf. **8**, 49 (2006)
35. Stavrinos, P.C.: Int. J. Theor. Phys. **44**(2), 245 (2005)
36. Stavrinos, P.C.: Remarks on the Applications of Finsler Geometry to Space-time. In: Conference in Proceedings of MSRI Berkeley, USA June 2002, p. 30 http://www.msri.org/publications
37. Stavrinos, P.C., Diakogiannis, F.I.: Gravitation and Cosmology **10**(4), 1 (2005)
38. Stavrinos, P.C., Ikeda, S.: Int. J. Theor. Phys. **43**(2), 437 (2004)
39. Tsagas, C.G.: Geometrical Aspects of Cosmic Magnetic Field, 2nd Hellenic Cosmology Workshop, Athens (2001) arXiv:gr-qc/0112077
40. Tsagas, C.G., Maartens, R.: Phys. Rev. D **61** 083519 (2000) arXiv:astro-ph/9904390
41. Vacaru, S.: Ann. Phys. **290**, 83 (2001)
42. Vacaru, S., Stavrinos, P.C., Gaburov, E., Gonta, D.: Clifford and Riemann-Finsler Structures in Geometric Mechanics and Gravity. Geometry Balcan University Press, Beijing (2005) arXiv:math-ph/0608024
43. Wang, Y., Mukherjee, P.: Astrophys. J. **606**, 654 (2004) arXiv:astro-ph/0510068
44. Wang, Y., Mukherjee, P.: Astrophys. J. **650**, 1 (2006) arXiv:astro-ph/0604051
45. Wald, R.M.: General Relativity. Chicago University Press, Chicago (1984)
46. Yu, G., Bogoslovsky, H.F.: Goenner Phys. Lett. A **244**, 222 (1998) arXiv:gr-qc/9804082